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A *second order, linear, homogeneous equation* is an equation of the form:

$$P(t)x''(t) + Q(t)x'(t) + R(t)x(t) = 0,$$

when $p(t) \neq 0$, we can divide both sides by $P(t)$ to put it in the form:

$$x''(t) + q(t)x'(t) + r(t)x(t) = 0.$$

We will be dealing – almost exclusively – with equations that have *constant coefficients*:

$$ax''(t) + bx'(t) + cx(t) = 0,$$

and $A \neq 0$. Dividing by A, we can put the equation in the form:

$$x''(t) + bx'(t) + cx(t) = 0.$$

Observe that a *first order, linear, constant–coefficient, homogeneous equation* is an equation of the form:

$$x'(t) + bx(t) = 0.$$

We know the general solution is given by $x(t) = Ce^{-bt}$.

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We want to find solutions to second order equations like this. First, for which values of r is the function $x(t) = e^{rt}$ a solution to

$$ax''(t) + bx'(t) + cx(t) = 0.$$

Plugging this function into the equation we get:

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0,$$

dividing both sides by e^{rt} gives:

$$ar^2 + br + c = 0.$$

This is a second degree polynomial in the variable r. There are two roots $-r_1, r_2$ to this equation and these roots can be:

(1) real and distinct (i.e. r_1, r_2 are real and $r_1 \neq r_2$,

(2) complex conjugates (i.e. r_1, r_2 have non–zero imaginary parts and $r_2 = \overline{r_2}$,

(3) real and the same (i.e. r_1, r_2 are real and $r_1 = r_2$).

This indicates that there are at most two solutions of the form e^{rt}. The polynomial from above is called the *characteristic polynomial* of the equation.

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We want to find solutions to second order equations like this. For which values of r is the function $x(t) = e^{rt}$ a solution to

$$x''(t) - x'(t) - 2x(t) = 0.$$

Plugging the function into the equation gives:

$$r^2e^{rt} - re^{rt} - 2e^{rt} = 0,$$

dividing both sides by e^{rt} gives:

$$0 = r^2 - r - 2 = (r-2)(r+1).$$

So, the only solutions to this equation of the form e^{rt} are e^{-t} and e^{2t}.

Observe that the function $x(t) = C_1e^{-t} + C_2e^{2t}$ is also a solution to this equation:

$$(C_1e^{-t} + C_2e^{2t})'' - (C_1e^{-t} + C_2e^{2t})' - 2(C_1e^{-t} + C_2e^{2t})(C_1(e^{-t})'' - C_1(e^{-t})' - 2e^{-t}) + (C_2(e^{2t})'' - C_2(e^{2t})' - 2e^{2t})$$
$$= 0 + 0.$$

As we will see below, the function $x(t) = C_1e^{-t} + C_2e^{2t}$ is called the *general solution* to this equation. We need *two* initial conditions to determine these constants.

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An initial value problem for the equations above, is:

$$ax''(t) + bx'(t) + cx(t) = 0, \qquad x(t_0) = x_0, x'(t_0) = x'_0,$$

here x_0 and x'_0 are two known constants.
Two functions $x_1(t), x_2(t)$ are called *linearly independent* if they are *not* multiples of each other. A theorem (that we will not prove) says:

Theorem. Let $x_1(t), x_2(t)$ be two linearly independent solutions to:

$$ax''(t) + bx'(t) + cx(t) = 0.$$

Then the IVP

$$ax''(t) + bx'(t) + cx(t) = 0, \qquad x(t_0) = x_0, x'(t_0) = x'_0,$$

has a unique solution and that solution can be written in the form

$$x(t) = c_1x_1(t) + c_2x_2(t),$$

for two constants c_1, c_2.

The proof of this – as well as some additional reasoning – will come when we study systems of equations in chapter 7. When the constants c_1, c_2 aren't specified, the function $x(t) = c_1x_1(t) + c_2x_2(t)$ is called the (a) general solution to the differential equation.

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Find the solution to:

$$x'' + 5x' + 6x = 0, \qquad x(0) = 1, x'(0) = 0.$$

First, we find the general solution. The characteristic polynomial is:

$$0 = r^2 + 5r + 6 = (r+2)(r+3).$$

This means the functions $x_1(t) = e^{-3t}$ and $x_2(t) = e^{-2t}$ are solutions to the ODE. And, since they are linearly independent (WHY?!) this means that $x(t) = c_1 e^{-3t} + c_2 e^{-2t}$ is the general solution. We can now use the initial condition to solve for c_1, c_2:

$$\begin{aligned} 1 &= x(0) = c_1 + c_2 \\ 0 &= x'(0) = -3c_1 - 2c_2. \end{aligned}$$

The second equation says that $c_2 = -\frac{3}{2}c_1$. Plugging this into the first equation:

$$1 = c_1 - \frac{3}{2}c_1 = -\frac{1}{2}c_1$$

whence $c_1 = -2$ and thus $c_2 = 3$. So the solution is:

$$x(t) = -2e^{-3t} + 3e^{-2t}.$$

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Find the solution to:

$$x'' - 2x' = 0 \qquad x(0) = 1, x'(0) = -1.$$

The characteristic polynomial is:

$$0 = r^2 - 2r = r(r-2).$$

So the functions $x_1(t) = e^{0t} = 1$ and $x_2(t) = e^{2t}$ form a solution set. That is, the solution to the IVP has the form $x(t) = c_1 x_1(t) + c_2 x_2(t)$. The initial condition allows us to find the constants c_1 and c_2:

$$\begin{aligned} 1 &= x(0) = c_1 + c_2 \\ -1 &= x'(0) = 2c_2. \end{aligned}$$

The bottom equation says that $c_2 = -\frac{1}{2}$. Plugging this into the top equation gives:

$$1 = c_1 - \frac{1}{2},$$

whence $c_1 = \frac{3}{2}$. So, the solution is:

$$x(t) = \frac{3}{2} - \frac{1}{2}e^{2t}.$$

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Find the general solution to:

$$x'' - 4x' + 5x = 0.$$

The characteristic polynomial is:

$$r^2 - 4r + 5 = (r-2)^2 - 4 + 5 = (r-2)^2 + 1 = 0.$$

Solving for r gives:

$$r = 2 \pm \sqrt{-1}.$$

In other words, the two roots are complex conjugates. It is true that:

$$x(t) = c_1 e^{(2+i))t} + c_2 e^{(2-i)t}$$

is one way to write the general solution. However, we want to write the solution in a way that doesn't (explicitly) include imaginary numbers.
First, recall that if $z = x + iy$ is a complex number, then x is called the real part and is denoted $\Re z$ and y is called the imaginary part and is denoted $\Im z$. Note also that:

$$\Re z = \frac{1}{2}(z + \overline{z})$$
$$\Im z = \frac{1}{2i}(z - \overline{z}).$$

This tells us that the real and imaginary parts of a solution are also solutions. So, if we can show that the real and imaginary parts are linearly independent, then the general solution can be written as a linear combination of these solutions.
Next, recall Euler's formula:

$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

Applying this to solution gives:

$$e^{(2+i)t} = e^{2t} e^{it} = e^{2t}(\cos t + i \sin t),$$

and so:

$$\Re e^{(2+i)t} = e^{2t} \cos t \qquad \Im e^{(2+i)t} = e^{2t} \sin t.$$

These two functions are solutions and are linearly independent. So, the general solution is:

$$x(t) = c_1 e^{2t} \cos t + c_2 e^{2t} \sin t.$$

In general, if the roots of the characteristic equation are $r_{1,2} = \lambda \pm \mu i$, the general solution is:

$$x(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t.$$

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Find the solution to the IVP

$$16x'' - 8x' + 145x = 0, \qquad x(0) = -2, x'(0) = 1.$$

The characteristic equation:

$$0 = 16r^2 - 8r + 145,$$

has roots $r_{1,2} = \frac{1}{4} \pm 3i$. So, the general solution is:

$$x(t) = c_1 e^{\frac{1}{4}t} \cos 3t + c_2 e^{\frac{1}{4}t} \sin 3t.$$

The initial condition let's us find the coefficients. The condition $x(0) = -2$ gives implies that $c_1 = -2$. For the second equation, we have:

$$1 = x'(0) = \frac{1}{4}c_1 + 3c_2 = -\frac{1}{2} + 3c_2,$$

whence $c_2 = \frac{1}{2}$. So the solution is:

$$x(t) = -2e^{\frac{1}{4}t} \cos 3t + \frac{1}{2}e^{\frac{1}{4}t} \sin 3t.$$

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