Chapter 3 Section 4

Find the general solution to:

$$x'' - 2x' + x = 0.$$

We begin by finding the roots of the characteristic polynomial:

$$0 = r^2 - 2r + 1 = (r-1)^2.$$

This means there is a repeated root and so we might guess that the general solution is:

$$x(t) = c_1 e^t + c_2 e^t.$$

BUT THIS IS WRONG!! Why? The two functions $x_1(t) = e^t$ and $x_2(t) = e^t$ are *NOT* linearly independent – indeed they are the same function. So they can't form a solution set. We know that e^t is a solution, we now need to find a different solution.
Let's consider a more general equation:

$$ax''(t) + bx'(t) + cx(t) = 0,$$

and let's assume that $ar^2 + br + c$ has one repeated root – r_1. Let $x_1(t) = e^{r_1 t}$ and define $x_2(t) = v(t)e^{r_1 t}$ for some unknown function $v(t)$. If $x_2(t)$ is a solution, then:

$$\begin{aligned}
0 &= a(v(t)x_1(t))'' + b(v(t)x_1(t))' + c(v(t)x_1(t)) \\
&= a(v'x_1 + vx_1')' + b(v'x_1 + vx_1') + cvx_1 \\
&= a(v''x_1 + v'x_1' + v'x_1' + vx_1'') + b(v'x_1 + vx_1') + cvx_1 \\
&= v(ax_1'' + bx_1' + cx_1) + a(v''x_1 + 2v'x_1') + bv'x_1 \\
&= a(v'' + 2r_1 v')e^{r_1 t} + bv'e^{r_1 t} \\
&= [av'' + (2r_1 a + b)v']e^{r_1 t}.
\end{aligned}$$

Now, $r_1 = -\frac{b}{2a}$. So

$$2r_1 a + b = 2\left(\frac{-b}{2a}\right)a + b = 0.$$

So then the above becomes:

$$\begin{aligned}
0 &= [av'' + (2r_1 a + b)v']e^{r_1 t} \\
&= av''e^{r_1 t},
\end{aligned}$$

whence $v''(t) = 0$ and so $v(t) = t + c$. Thus, $x_2(t) = (t + c)e^{r_1 t}$ and so the general solution is:

$$x(t) = c_1 e^{r_1 t} + c_2(t + c)e^{r_1 t}.$$

Note that we can write this as:

$$x(t) = (c_1 + c)e^{r_1 t} + c_2 t e^{r_1 t},$$

which can be written as:

$$x(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t},$$

and this is how we prefer to write it. So, re-defining $x_2(t) = te^{r_1 t}$, we see that x_1 and x_2 are not multiples of each other, and so they form a fundamental solution set.

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Returning to the previous problem: Find the general solution to:

$$x'' - 2x' + x = 0.$$

The general solution is:

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The same method can be used to find solutions of non–constant coefficient equations. Consider

$$x''(t) + p(t)x'(t) + q(t)x(t) = 0.$$

And, assume that $x_1(t)$ is a solution. If $v(t)x_1(t)$ is another solution, and we plug it into the equation, we get:

$$x_1v'' + (2x_1' + px_1)v' + (x_1'' + px_1' + qx_1)v = 0.$$

Since x_1 is a solution to the equation, the third term is zero, so this becomes:

$$x_1v'' + (2x_1' + px_1)v' = 0.$$

This is a *first order equation for* v':

$$\frac{v''}{v'} = -\frac{2x_1' + px_1}{x_1} = -(2\frac{x_1'}{x_1} + p).$$

By the chain rule, this is:

$$\frac{d}{dt}\ln v' = -(2\frac{d}{dt}\ln x_1 + p),$$

whence:

$$\ln v'(t) = -2\ln x_1(t) + \int p(t)dt + C.$$

Thus:

$$v'(t) = \frac{Ae^{\int p(t)dt}}{x_1^2}.$$

Using this formula isn't so great; you should probably just solve the equation:

$$x_1v'' + (2x_1' + px_1)v' = 0.$$

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Find a fundamental set of solutions to:

$$x'' + \frac{3}{2t}x' - \frac{1}{2t^2}x = 0,$$

given that $x_1(t) = \frac{1}{t}$ is a solution. We find v' using:

$$x_1 v'' + (2x_1' + px_1)v' = 0.$$

This is:

$$\begin{aligned}
0 &= \frac{1}{t}v'' + (-2\frac{1}{t^2} + \frac{3}{2t}\frac{1}{t})v' = \frac{1}{t}v'' - \frac{1}{2t^2}v' \\
\implies 0 &= 2tv'' - v' \\
\implies \frac{v''}{v'} &= \frac{1}{2t} \\
\implies v'(t) &= At^{\frac{1}{2}} \\
\implies v(t) &= At^{\frac{3}{2}} + C.
\end{aligned}$$

So:

$$x_2(t) = (At^{\frac{3}{2}} + C)\frac{1}{t} = At^{\frac{1}{2}} + C\frac{1}{t}.$$

So the general solution is:

$$x(t) = c_1\frac{1}{t} + c_2(At^{\frac{1}{2}} + C\frac{1}{t}).$$

We re–write this as:

$$x(t) = c_1\frac{1}{t} + c_2 t^{\frac{1}{2}}.$$

So, similar to above, if we re-define $x_2(t) = t^{\frac{1}{2}}$, we have a fundamental set of solutions.

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Consider the equation:

$$t^2x''(t) + atx'(t) + bx(t) = 0, \qquad t > 0.$$

An under appreciated technique is the "change of variables". Let $t = e^u$. Compute $\frac{d^2x}{dt^2}$ and $\frac{dx}{dt}$ in terms $\frac{d^2x}{du^2}$ and $\frac{dx}{du}$ and re-write the equation. This is a straight-forward application of the chain rule, but this doesn't mean it's easy.

$$\frac{dx}{du} = \frac{dx}{dt}\frac{du}{dt} = \frac{dx}{dt}\frac{d}{dt}(e^u) = \frac{dx}{dt}e^u.$$

Similarly:

$$\frac{d^2x}{du^2} = \frac{d}{du}\left(\frac{dx}{du}\right) = \frac{d}{du}\left(\frac{dx}{dt}e^u\right).$$

Now product rule:

$$\frac{d}{du}\left(\frac{dx}{dt}e^u\right) = \frac{dx'(t)}{du}e^u + \frac{dx}{dt}\frac{d}{du}e^u = x''(t)e^{2u} + x'(t)e^u.$$

Thus, we see:

$$\begin{aligned} tx'(t) &= \frac{dx}{du} \\ t^2x''(t) &= \frac{d^2x}{du^2} - tx'(t) = \frac{d^2x}{du^2} - \frac{dx}{du}. \end{aligned}$$

Plugging these into the original equation gives:

$$(\frac{d^2x}{du^2} - \frac{dx}{du}) + a(\frac{dx}{du}) + bx = 0.$$

Or, *LETTING PRIMES DENOTE DIFFERENTIATION WITH RESPECT TO* u:

$$x''(u) + (a-1)x'(u) + bx(u) = 0.$$

And we know how to solve an equation like this. Our solution will be $x_1(u)$ and $x_2(u)$ and so $x_1(\ln t)$ and $x_2(\ln t)$ are the solutions to the initial equation.

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Solve:

$$t^2x''(t) + tx'(t) + x(t) = 0.$$

In this problem, $a = b = 1$ and so using the method above, we transform it to:

$$x''(u) + x(u) = 0,$$

whence

$$x(u) = c_1 \cos u + c_2 \sin u$$

is the general solution. Now, using the fact that $e^u := t$ we have:

$$x(t) = c_1 \cos \ln t + c_2 \sin \ln t.$$

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