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We are now going to look at second order, linear, constant coefficient *non–homogeneous* equations. That is, equations of the form:

$$ax''(t) + bx'(t) + cx(t) = g(t).$$

Here, x is the unknown function and a, b, c and $g(t)$ are all known (a, b, c are constants).
There are (typically) two parts in the strategy to solve such an equation:

(1) Find the general solution of the corresponding homogeneous equation: $ax'' + bx' + cx = 0$. Call this $x_H(t)$

(2) find one solution to the non-homogeneous equation $ax'' + bx' + cx = g$. Call this $x_P(t)$. The general solution is then $x(t) = x_H(t) + x_P(t)$.

In other words, if $x_H(t) = c_1x_1(t) + c_2x_2(t)$ is the general solution to $ax'' + bx' + cx = 0$ and $x_P(t)$ is any solution to $ax'' + bx' + cx = g$, then any solution to the IVP

$$ax''(t) + bx'(t) + cx(t) = g(t), \qquad x(t_0) = x_0, x'(t_0) = x'_0$$

can be written in the form:

$$x(t) = c_1x_1(t) + c_2x_2(t) + x_P(t),$$

for a unique choice of c_1 and c_2.
In general, finding $x_P(t)$ is the "hard part" of the process.

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Find the general solution to:

$$x''(t) - 3x'(t) - 4x(t) = 3e^{2t}.$$

First, find the general solution to

$$x''(t) - 3x'(t) - 4x(t) = 0.$$

The characteristic polynomial is $r^2 - 3r - 4 = (r-4)(r+1)$ and so:

$$x_H(t) = c_1e^{-t} + c_2e^{4t}.$$

If $x_P(t)$ is a solution to the non–homogeneous equation, it means that:

$$x_P''(t) - 3x_P(t) - 4x_P(t) = 3e^{2t}.$$

So, in other words, the LHS is a linear combination of derivatives of x_P and it is equal to $3e^{2t}$. This means a good guess for $x_P(t)$ is a function of the form $x_P(t) = Ae^{2t}$ for some to-be-determined constant A. To determine this constant, just plug Ae^{2t} into the equation:

$$\begin{aligned} 3e^{2t} &= (Ae^{2t}) - 3(Ae^{2t})' - 4(Ae^{2t}) \\ &= 4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t}, \end{aligned}$$

whence:

$$3 = 4A - 6A - 4A = -6A \Longrightarrow A = -\frac{1}{2}.$$

So:

$$x_P(t) = -\frac{1}{2}e^{2t}.$$

So the general solution to the problem is:

$$x(t) = x_H(t) + x_P(t) = c_1e^{-t} + c_2e^{4t} - \frac{1}{2}e^{2t}.$$

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Find the general solution to:

$$x''(t) - 3x'(t) - 4x(t) = \cos t.$$

Notice the LHS is the same as above, so the CHE (corresponding homogeneous equation) is the same so

$$x_H(t) = c_1 e^{-t} + c_2 e^{4t}.$$

Now, if we try to guess a solution $x_P(t)$ we know that if $x_P(t)$ is a solution to the non–homogeneous equation, it means that:

$$x_P''(t) - 3x_P(t) - 4x_P(t) = \cos t.$$

So, in other words, the LHS is a linear combination of derivatives of x_P and it is equal to $\cos(t)$. So, a good guess for $x_P(t)$ is going to be functions whose first and second derivatives are "close" to $\cos t$. For example, we guess:

$$x_P(t) = A\cos t + B\sin t,$$

for some to be determined constants A and B. Plugging this into the equation gives:

$$\begin{aligned}\cos t &= (A\cos t + B\sin t)'' - 3(A\cos t + B\sin t)' - 4(A\cos t + B\sin t)\\ &= -A\cos t - B\sin t + 3A\sin t - 3B\cos t - 4A\cos t - 4B\sin t\\ &= (-5A - 3B)\cos t + (3A - 5B)\sin t.\end{aligned}$$

This gives the equations:

$$\begin{aligned}1 &= -5A - 3B\\ 0 &= 3A - 5B.\end{aligned}$$

The second equation says that $B = \frac{3}{5}A$. Plugging this into the first equation:

$$1 = -5A - 3\frac{3}{5}A = \frac{-25-9}{5}A = -\frac{34}{5}A$$

whence:

$$A = -\frac{5}{34} \qquad B = \frac{3}{5}A = \frac{3}{5}\frac{-5}{34} = -\frac{3}{34}.$$

So:

$$x_P(t) = -\frac{5}{34}\cos t - \frac{3}{34}\sin t.$$

So the general solution is:

$$x(t) = c_1 e^{-t} + c_2 e^{4t} - \frac{5}{34}\cos t - \frac{3}{34}\sin t.$$

Try this same procedure, but guess the solution has the form $A\cos t$. Similar calculations lead to:

$$\begin{aligned}1 &= -5A\\ 0 &= 3A,\end{aligned}$$

and there is no solution to this system.

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Find the general solution to:

$$x''(t) - 3x'(t) - 4x(t) = t^2 + 1.$$

Similar to the previous examples:

$$x_H(t) = c_1 e^{-t} + c_2 e^{4t}.$$

For $x_P(t)$, a guess is any second degree polynomial – so $At^2 + Bt + C$. Plugging this into the ODE gives:

$$\begin{aligned} t^2 + 1 &= (At^2 + Bt + C)'' - 3(At^2 + Bt + C)' - 4(At^2 + Bt + C) \\ &= 2A - 6At - 3B - 4At^2 - 4Bt - 4C. \end{aligned}$$

This gives:

$$\begin{aligned} 1 &= -4A \\ 0 &= -6A - 4B \\ 1 &= 2A - 3B - 4C. \end{aligned}$$

The first equation says $A = -\frac{1}{4}$. Plugging this into the second equation gives $-6(-\frac{1}{4}) - 4B = \frac{3}{2} - 4B = 0$ whence $B = \frac{3}{8}$. Plugging these into the third equation gives:

$$1 = 2\frac{-1}{4} - 3\frac{3}{8} - 4C = -\frac{4}{8} - \frac{9}{8} - 4C = -\frac{13}{8} - 4C,$$

whence $4C = -\frac{21}{8}$ whence $C = -\frac{21}{32}$. Thus:

$$x_P(t) = -\frac{1}{4}t^2 + \frac{3}{8}t - \frac{21}{32}.$$

And so:

$$x(t) = c_1 e^{-t} + c_2 e^{4t} - \frac{1}{4}t^2 + \frac{3}{8}t - \frac{21}{32}.$$

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Consider an equation like:

$$ax''(t) + bx'(t) + cx(t) = g_1(t) + g_2(t).$$

Suppose that it is known that $x_{P1}(t)$ and $x_{P2}(t)$ are particular solutions to:

$$ax''(t) + bx'(t) + cx(t) = g_1(t) \qquad \text{and} \qquad ax''(t) + bx'(t) + cx(t) = g_2(t),$$

respectively. Then if $x_H(t)$ is the general solution to the CHE, the function:

$$x(t) = x_H(t) + x_{P1}(t) + x_{P2}(t)$$

is the general solution to

$$ax''(t) + bx'(t) + cx(t) = g_1(t) + g_2(t).$$

To see why, observe:

$$a(x_H(t) + x_{P1}(t) + x_{P2}(t))'' + b(x_H(t) + x_{P1}(t) + x_{P2}(t))' + c(x_H(t) + x_{P1}(t) + x_{P2}(t))$$

equals

$$[ax_H''(t) + bx_H'(t) + cx_H(t)] + [ax_{P1}''(t) + bx_{P1}'(t) + cx_{P1}'(t)] + [ax_{P2}''(t) + bx_{P2}'(t) + cx_{P2}'(t)] = 0 + g_1(t) + g_2(t).$$

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Find the general solution to:

$$x''(t) - 3x'(t) - 4x(t) = 3e^{2t} + \cos t + t^2 + 1.$$

Using the last four pages:

$$x(t) = c_1 e^{-t} + c_2 e^{4t} - \frac{1}{2}e^{2t} - \frac{5}{34}\cos t - \frac{3}{34}\sin t - \frac{1}{4}t^2 + \frac{3}{8}t - \frac{21}{32}.$$

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Find the general solution to:

$$x''(t) - 3x'(t) - 4x(t) = 2e^{4t}.$$

Our guess is $x_P(t) = Ae^{4t}$. Plugging this into the equation gives:

$$2e^{4t} = 16Ae^{4t} - 12Ae^{4t} - 4Ae^{4t} = 0.$$

So, this function $x_P(t) = Ae^{4t}$ can't be a particular solution. Why? Because it is a solution to the CHE. In this case, we guess that the solution is Ate^{4t}. We use this:

$$\begin{aligned} 2e^{4t} &= (Ate^{4t})'' - 3(Ate^{4t})' - 4(Ate^{4t}) \\ &= A(4e^{4t} + 4e^{4t} + 16te^{4t}) - 3A(e^{4t} + 4te^{4t}) - 4Ate^{4t} \\ &= 5Ae^{4t}, \end{aligned}$$

whence $A = \frac{2}{5}$ and so the solution is:

$$x(t) = c_1e^{-t} + c_2e^{4t} + \frac{2}{5}te^{4t}.$$

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Find the general solution to:

$$x''(t) - 2x'(t) + 2x(t) = e^t \cos t.$$

The characterstic polynomial is $(r-1)^2 + 1$ and so:

$$x_H(t) = c_1 e^t \cos t + c_2 e^t \sin t.$$

Our guess for $x_P(t)$ should be something like $Ae^t \cos t + Be^t \sin t$, but since this is the solution to the corresponding homogeneous equation we multiply it by t:

$$\begin{aligned}
x_P(t) &= t(Ae^t \cos t + Be^t \sin t) \\
x_P'(t) &= (Ae^t \cos t + Be^t \sin t) + t(Ae^t \cos t - Ae^t \sin t + Be^t \sin t + Be^t \cos t) \\
&= (Ae^t \cos t + Be^t \sin t) + t((A+B)e^t \cos t + (B-A)e^t \sin t) \\
x_P''(t) &= (Ae^t \cos t - Ae^t \sin t + Be^t \sin t + Be^t \cos t) + (Ae^t \cos t - Ae^t \sin t + Be^t \sin t + Be^t \cos t) \\
&\quad + t(Ae^t \cos t - Ae^t \sin t - Ae^t \sin t - Ae^t \cos t + Be^t \sin t + Be^t \cos t + Be^t \cos t - Be^t \sin t) \\
&= 2((A+B)e^t \cos t + (B-A)e^t \sin t) + t(2Be^t \cos t - 2Ae^t \sin t).
\end{aligned}$$

Now we plug this into the LHS of the ODE. Yikes! It's best to just organize the terms to get the equations:

$$\begin{aligned}
te^t \cos t(2B - 2(A+B) + 2A) &= 0te^t \cos t \\
te^t \sin t(-2A - 2(B-A) + 2B) &= te^t \sin t \\
e^t \cos t(2(A+B) - 2A) &= e^t \cos t \\
e^t \sin t(2(B-A) - 2B) &= 0e^t \sin t.
\end{aligned}$$

The first two equations are trivial and just say $0 = 0$ (do you see why??) The second two equations say:

$$\begin{aligned}
2B &= 1 \\
-2A &= 0
\end{aligned}$$

whence $B = \frac{1}{2}$ and $A = 0$. Together this implies that $x_P(t) = \frac{1}{2}e^t \sin t$. Thus the general solution is:

$$x(t) = c_1 e^t \cos t + c_2 e^t \sin t + \frac{1}{2} e^t \sin t.$$

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Find the general solution to:

$$x''(t) - 2x'(t) + x(t) = te^{2t}.$$

The characteristic polynomial is $(r-1)^2$ and so the solution to the CHE is $x_H(t) = c_1e^t + c_2te^t$. The guess at a solution should be a general first degree polynomial times e^{2t}: $x_P(t) = (At+B)e^{2t}$:

$$\begin{aligned} x_P(t) &= (At+B)e^{2t} \\ x_P'(t) &= Ae^{2t} + 2(At+B)e^{2t} \\ x_P''(t) &= 2Ae^{2t} + 2Ae^{2t} + 4(At+B)e^{2t}. \end{aligned}$$

Plugging this into the LHS and equating coefficients:

$$\begin{aligned} te^{2t}(4A - 4A + A) &= te^{2t} \\ e^{2t}(4A + 4B - 2A - 4B + B) &= 0e^t. \end{aligned}$$

This gives the system:

$$\begin{aligned} A &= 1 \\ 2A + B &= 0. \end{aligned}$$

Whence $A = 1$ and $B = -2$. So:

$$x(t) = c_1e^t + c_2te^t + (t-2)e^{2t}.$$

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Find the general solution to:

$$x''(t) - 2x'(t) + x(t) = te^t.$$

The characteristic polynomial is $(r-1)^2$ and so the solution to the CHE is $x_H(t) = c_1e^t + c_2te^t$. Our initial guess at $x_P(t)$ would be $(At+B)e^t$ but that's a solution to the CHE. So we guess $t^2(At+B)e^t$ (the square is so that no term in the guess is a solution to the CHE):

$$\begin{aligned}
x_P(t) &= (At^3 + Bt^2)e^t \\
x_P'(t) &= (3At^2 + 2Bt)e^t + (At^3 + Bt^2)e^t \\
x_P''(t) &= (6At + 2B)e^t + 2(3At^2 + 2Bt)e^t + (At^3 + Bt^2)e^t.
\end{aligned}$$

Plugging this into the ODE gives:

$$\begin{aligned}
t^3e^t(A - 2A + A) &= 0t^3e^t \\
t^2e^t(6A + B - 6A - 2B + B) &= 0t^2e^t \\
te^t(6A + 4B - 4B) &= 1e^t \\
e^t(2B) &= 0e^t.
\end{aligned}$$

This leads to the system:

$$\begin{aligned}
6A &= 1 \\
2B &= 0
\end{aligned}$$

whence $A = \frac{1}{6}$ and $B = 0$. So:

$$x(t) = c_1e^t + c_2te^t + \frac{1}{6}t^3e^t.$$

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