Math 308 ODE **Name:** ______________________
Spring 2019
Exam 1
2/13/19
Time Limit: 75 Minutes

This exam contains 46 pages (including this cover page) and 34 questions. Total of points is 312.

Write your name on the line above. Each page contains one, two, or three questions. Write your solution on the pages that the problem is on. Use the back of the page if you need more room. Write clearly and neatly. Points may be deducted if your writing is unclear or not well organized. In each problem, you should provide enough justification for your answer; this means showing sufficient work to prove to that you know what you are doing or by providing an argument. Additionally, specific questions may require specific instructions that you must follow.

No calculators or notes are allowed.

Grade Table

Question	Points	Score
1	6	
2	9	
3	9	
4	9	
5	9	
6	9	
7	9	
8	9	
9	9	
10	9	
11	9	
12	9	
13	9	
14	9	
15	9	
16	9	
17	15	
18	15	
19	15	
20	15	
21	15	
22	15	
23	15	
24	15	
25	15	
26	15	
27	15	
28	0	
29	0	
30	0	

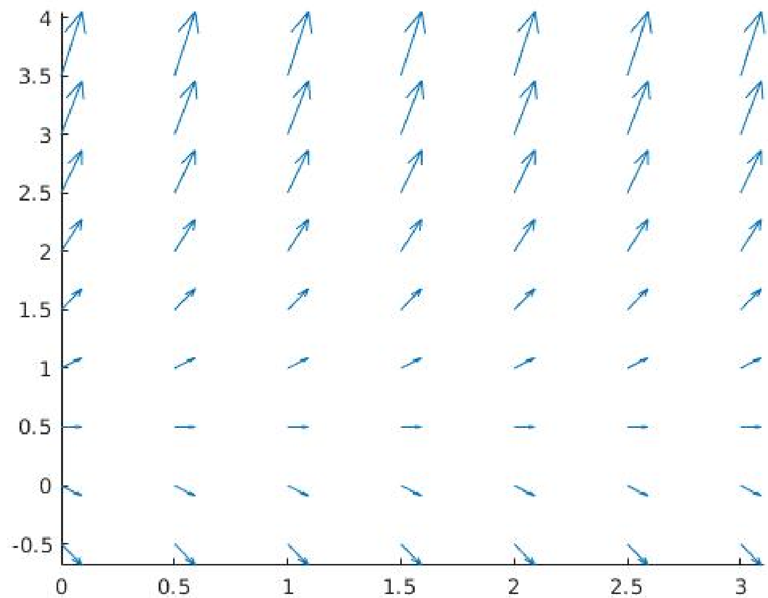
1. (6 points) (a) Consider the ordinary differential equation $\frac{dx}{dt} = f(t, x)$. What does it mean for a function $\varphi(t)$ to be a solution of this ODE?

Solution: It means that $\frac{d\varphi}{dt} = f(t, \varphi(t)$.

(b) Consider the ODE $\frac{dx}{dt} = 4t - \frac{2}{t}x$. Show that the function $\varphi(t) = t^2 + t^{-2}$ is a solution of this ODE.

Solution: $\varphi'(t) = 2t - 2t^{-3}$. And $4t - \frac{2}{t}\varphi(t) = 4t - \frac{2}{t}(t^2 + t^{-2} = 4t - (2t + \frac{2}{t^3} = 2t - \frac{2}{t^3} = \varphi'(t)$.

2. (9 points) Consider the direction field below:



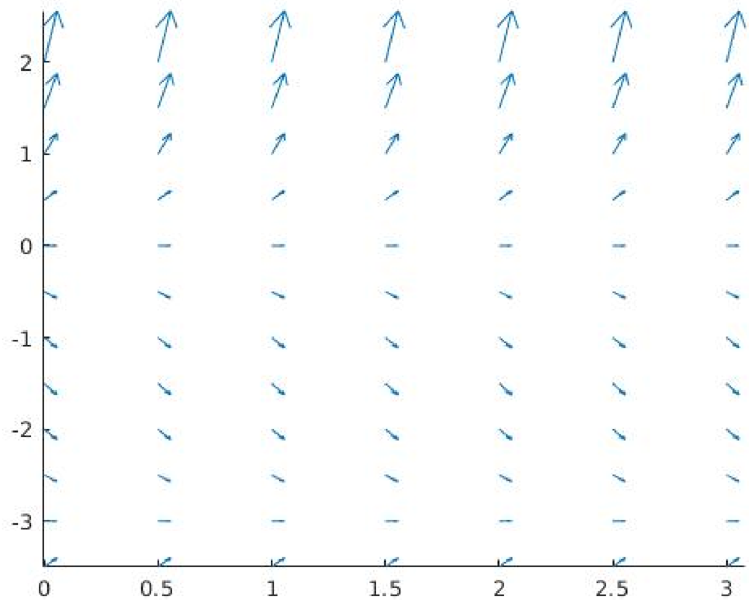
(a) What are the equilibrium solutions?

Solution: $x(t) = .5$ is the only one.

(b) Determine if each equilibrium solution is stable, unstable, or semi–stable.

Solution: Unstable since solutions that start a little above or a little below $x(t) = .5$ go away from it.

3. (9 points) Consider the direction field below:



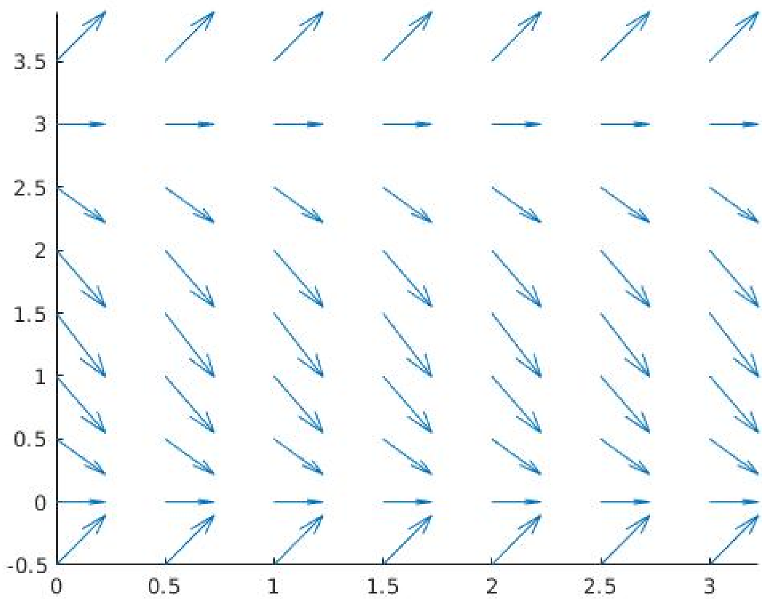
(a) What are the equilibrium solutions?

Solution: $x(t) = -3$ and $x(t) = 0$ are the equilibrium solutions.

(b) Determine if each equilibrium solution is stable, unstable, or semi–stable.

Solution: $x(t) \equiv -3$ is stable. $x(t) \equiv 0$ is unstable.

4. (9 points) Consider the direction field below:



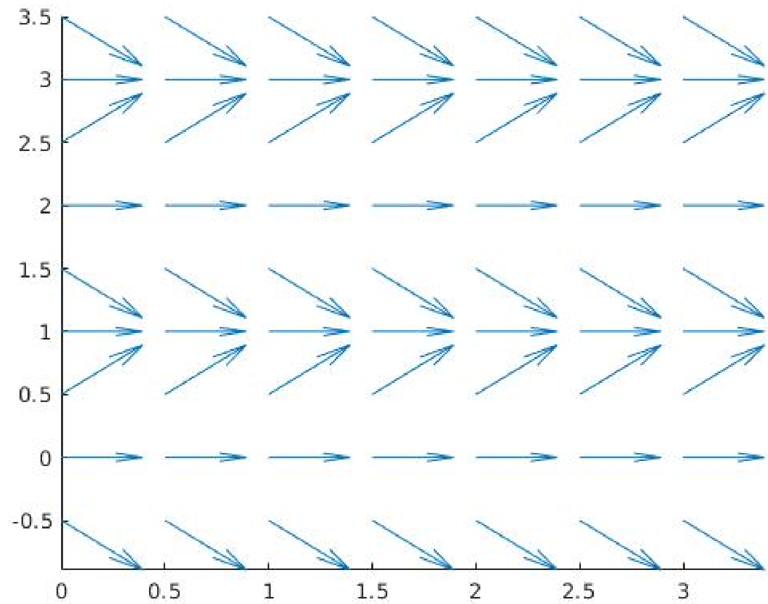
(a) What are the equilibrium solutions?

Solution: $x(t) \equiv 0$ and $x(t) \equiv 3$.

(b) Determine if each equilibrium solution is stable, unstable, or semi–stable.

Solution: $x(t) \equiv 0$ is stable; $x(t) \equiv 3$ is unstable.

5. (9 points) Consider the direction field below:



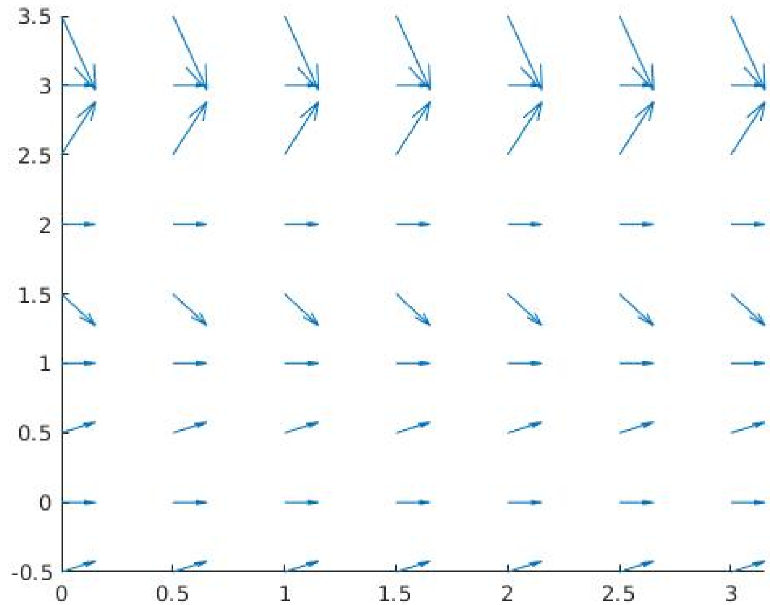
(a) What are the equilibrium solutions?

Solution: $x(t) \equiv 0$, $x(t) \equiv 1$, $x(t) \equiv 2$, and $x(t) \equiv 3$

(b) Determine if each equilibrium solution is stable, unstable, or semi–stable.

Solution: $x(t) \equiv 0$ is unstable; $x(t) \equiv 1$ is stable; $x(t) \equiv 2$ is unstable; $x(t) \equiv 3$ is stable.

6. (9 points) Consider the direction field below:



(a) What are the equilibrium solutions?

Solution: $x(t) \equiv 0$, $x(t) \equiv 1$, $x(t) \equiv 2$, and $x(t) \equiv 3$

(b) Determine if each equilibrium solution is stable, unstable, or semi–stable.

Solution: $x(t) \equiv 0$ is semi–stable; $x(t) \equiv 1$ is stable; $x(t) \equiv 2$ is unstable; $x(t) \equiv 3$ is stable.

7. (9 points) Solve the following initial value problem:

$$x' - x = 2te^{2t}, \qquad x(0) = 1.$$

(a) Find the integrating factor $\mu(t)$.

Solution: $\mu(t) = \exp\{\int -1\} = e^{-t}$.

(b) Find the general solution to the ODE.

Solution: $\frac{d}{dt}(x\mu) = 2te^{t}$. So that $x(t)e^{-t} = \int(2te^{t}dt) + C = 2e^{t}(t-1) + C$. So, $x(t) = 2e^{2t}(t-1) + Ce^{t}$.

(c) Find the solution to the initial value problem.

Solution: We have $1 = 2e^{2(0)}(0-1) + Ce^{0} = C - 2$. So, $C = 3$ and $x(t) = 2e^{2t}(t-1) + 3e^{t}$ is the solution.

8. (9 points) Solve the following initial value problem:

$$x' + 2x = te^{-2t}, \qquad x(1) = 0.$$

(a) Find the integrating factor $\mu(t)$.

Solution: $\mu(t) = \exp\{\int 2dt\} = e^{2t}$.

(b) Find the general solution to the ODE.

Solution: $x(t)e^{2t} = \int t dt = \frac{1}{2}t^2 + C$ so that $x(t) = e^{-2t}(\frac{1}{2}t^2 + C)$.

(c) Find the solution to the initial value problem.

Solution: $0 = x(1) = e^{-2}(\frac{1}{2} + C)$ so that $C = -\frac{1}{2}$. Then $x(t) = \frac{1}{2}e^{-2t}(t^2 + -1)$.

9. (9 points) Solve the following initial value problem:

$$tx' + 2x = t^2 - t + 1, \qquad x(1) = \frac{1}{2}.$$

(a) Find the integrating factor $\mu(t)$.

Solution: Re–write this as $x' + \frac{2}{t}x = t - 1 + t^{-1}$. $\mu(t) = \exp\{\int \frac{2}{t}\} = t^2$

(b) Find the general solution to the ODE.

Solution: Multiplying the ODE $x' + \frac{2}{t}x = t - 1 + t^{-1}$ by $\mu(t) = t^2$ gives $\frac{d}{dt}(x(t)t^2) = t^3 - t^2 + t$. Taking anti–derivatives of both sides gives $x(t)t^2 = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + C$ and so this gives $x(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + Ct^{-2}$.

(c) Find the solution to the initial value problem.

Solution: $\frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C$ so $C = 1/12$.

10. (9 points) Solve the following initial value problem:

$$x' + \frac{2}{t}x = \frac{\cos t}{t^2}, \qquad x(\pi) = 0.$$

(a) Find the integrating factor $\mu(t)$.

Solution: Same as above.

(b) Find the general solution to the ODE.

Solution: re-write this as $\frac{d}{dt}(x(t)t^2) = \cos t$. Taking anti–derivatives of both sides gives $x(t) = t^{-2}\sin t + Ct^{-2}$.

(c) Find the solution to the initial value problem.

Solution: Now, using the condition $x(\pi) = 0$ we can solve for C: $0 = \pi^{-2}\sin(\pi) + C\pi^{-2}$ and this gives $C = 0$. So the solution is $x(t) = t^{-2}\sin t$.

11. (9 points) Solve the following initial value problem:

$$t^3 x' + 4t^2 x = e^{-t}, \qquad x(-1) = 0.$$

(a) Find the integrating factor $\mu(t)$.

(b) Find the general solution to the ODE.

(c) Find the solution to the initial value problem.

12. (9 points) Solve the following initial value problem:

$$x' = (1-2t)x^2, \qquad x(0) = -1/6.$$

Solution: Re–write it as $(2t-1) + \frac{1}{x^2}\frac{dx}{dt} = 0$. As above, we can write this as $\frac{d}{dt}\left(t^2 - t - \frac{1}{x}\right) = 0$ and so the solution is $t^2 - t - \frac{1}{x} = c$ for some constant c. We find the constant by setting $t = 0$ and $x = -1/6$ (since $x(0) = -1/6$). So $-\frac{1}{-1/6} = c$, that is $c = 6$. So the solution is $t^2 - t - \frac{1}{x} = 6$. Solving for x we get $x(t) = \frac{1}{t^2-t-6}$.

13. (9 points) Solve the following initial value problem:

$$x' = \frac{1 - 2t}{x}, \qquad x(1) = -2.$$

14. (9 points) Solve the following initial value problem:

$$x' = \frac{2t}{x + t^2 x}, \qquad x(0) = -2.$$

Solution: First, write it as $\frac{dx}{dt} = \frac{2t}{x(1+t^2)}$ and so we can write this as $-\frac{2t}{1+t^2} + x\frac{dx}{dt} = 0$. An anti–derivative of $-\frac{2t}{1+t^2}$ is $-\ln 1 + t^2$. And anti–derivative of x is $\frac{1}{2}x^2$. So the ODE can be written as $\frac{d}{dt}\left(-\ln 1 + t^2 + \frac{1}{2}x^2\right) = 0$. So the solution is $-\ln 1 + t^2 + \frac{1}{2}x^2 = c$. We can find the value of c by setting $t = 0$ and $x = -2$ (since $x(0) = -2$. This gives $\frac{1}{2}4 = c$ so $c = 2$. Solving this for x we get two solutions $x(t) = \pm\sqrt{4 + 2\ln(1 + t^2)}$. Now, if we pick the "+" solution, then $x(0) = \sqrt{4 + 2\ln 1} = 2$ and so the "+" solution isn't the right one (because we need to have $x(0) = -2$. Thus, the solution is $x(t) = -\sqrt{4 + 2\ln(1 + t^2)}$. This solution is valid for all values of t.

15. (9 points) Solve the following initial value problem:

$$x' = \frac{2t}{1+2x}, \qquad x(2) = 0.$$

Solution: $-2t+(1+2x)x' = 0$. So implicit solution is $-t^2+x+x^2 = C$. The initial condition gives $C = -4$. Solving for x we get $x(t) = -\frac{1}{2} \pm \sqrt{t^2 - 4 + \frac{1}{4}}$. The initial condition dictates that we take the plus solution: $x(t) = -\frac{1}{2} + \sqrt{t^2 - 4 + \frac{1}{4}}$.

16. (9 points) Solve the following initial value problem:

$$x' = \frac{3t^2 - e^t}{2x - 5}, \qquad x(0) = 1.$$

Solution: Re–write it as $e^t - 3t^2 + (2x-5)\frac{dx}{dt} = 0$. Following the same procedures as above, this can be re–written as $\frac{d}{dt}\left(e^t - t^3 + x^2 - 5x\right) = 0$. So the solution is $e^t - t^3 + x^2 - 5x = c$. We find c by setting $t = 0$ and $x = 1$ to get $c = -4$. So this gives $e^t - t^3 + x^2 - 5x = -3$. This gives two solutions $x(t) = \frac{5}{2} \pm \sqrt{t^3 - e^t + \frac{13}{4}}$. If we pick the "+" solution, then we get $x(0) = 4$, so we choose the "−" solution and get $x(t) = \frac{5}{2} - \sqrt{t^3 - e^t + \frac{13}{4}}$.

17. (15 points) Consider the following autonomous equation: $\frac{dx}{dt} = f(x) = ax + bx^2$, $a, b > 0$.

(a) Draw the phase diagram for this ODE.

(b) Use the phase portrait to fill out the table below indicating whether the given function is positive or negative on the interval (there are more rows than you will need). You may assume that $x(t) \geq 0$ (that is, you don't need to worry about the interval $(-\infty, 0)$).

Interval	$x' = f(x)$	$f'(x)$	$x'' = f'(x)f(x)$	Concavity

(c) Sketch some solution curves. Sketch at lest one curve with an initial value in each of the intervals from the table above. (As in class, you only need to sketch solutions in the quadrant $t \geq 0$ and $x \geq 0$.

(d) Determine the equilibrium solutions.

(e) Determine if each equilibrium solution is stable, unstable, or semi–stable.

18. (15 points) Consider the following autonomous equation: $\frac{dx}{dt} = f(x) = e^x - 1$.

(a) Draw the phase diagram for this ODE.

(b) Use the phase portrait to fill out the table below indicating whether the given function is positive or negative on the interval (there are more rows than you will need). You may assume that $x(t) \geq 0$ (that is, you don't need to worry about the interval $(-\infty, 0)$.

Interval	$x' = f(x)$	$f'(x)$	$x'' = f'(x)f(x)$	Concavity

(c) Sketch some solution curves. Sketch at lest one curve with an initial value in each of the intervals from the table above. (As in class, you only need to sketch solutions in the quadrant $t \geq 0$ and $x \geq 0$.

(d) Determine the equilibrium solutions.

(e) Determine if each equilibrium solution is stable, unstable, or semi–stable.

19. (15 points) Consider the following autonomous equation: $\frac{dx}{dt} = f(x) = -\frac{\arctan x}{1+x^2}$.

(a) Draw the phase diagram for this ODE.

(b) Use the phase portrait to fill out the table below indicating whether the given function is positive or negative on the interval (there are more rows than you will need). You may assume that $x(t) \geq 0$ (that is, you don't need to worry about the interval $(-\infty, 0)$.

Interval	$x' = f(x)$	$f'(x)$	$x'' = f'(x)f(x)$	Concavity

(c) Sketch some solution curves. Sketch at lest one curve with an initial value in each of the intervals from the table above. (As in class, you only need to sketch solutions in the quadrant $t \geq 0$ and $x \geq 0$.

(d) Determine the equilibrium solutions.

(e) Determine if each equilibrium solution is stable, unstable, or semi–stable.

20. (15 points) Consider the following autonomous equation: $\frac{dx}{dt} = f(x) = (x-1)^2$.

(a) Draw the phase diagram for this ODE.

(b) Use the phase portrait to fill out the table below indicating whether the given function is positive or negative on the interval (there are more rows than you will need). You may assume that $x(t) \geq 0$ (that is, you don't need to worry about the interval $(-\infty, 0)$.

Interval	$x' = f(x)$	$f'(x)$	$x'' = f'(x)f(x)$	Concavity

(c) Sketch some solution curves. Sketch at lest one curve with an initial value in each of the intervals from the table above. (As in class, you only need to sketch solutions in the quadrant $t \geq 0$ and $x \geq 0$.

(d) Determine the equilibrium solutions.

(e) Determine if each equilibrium solution is stable, unstable, or semi–stable.

21. (15 points) Consider the following autonomous equation: $\frac{dx}{dt} = f(x) = x^2(x^2 - 1)$.

(a) Draw the phase diagram for this ODE.

(b) Use the phase portrait to fill out the table below indicating whether the given function is positive or negative on the interval (there are more rows than you will need). You may assume that $x(t) \geq 0$ (that is, you don't need to worry about the interval $(-\infty, 0)$.

Interval	$x' = f(x)$	$f'(x)$	$x'' = f'(x)f(x)$	Concavity

(c) Sketch some solution curves. Sketch at lest one curve with an initial value in each of the intervals from the table above. (As in class, you only need to sketch solutions in the quadrant $t \geq 0$ and $x \geq 0$.

(d) Determine the equilibrium solutions.

(e) Determine if each equilibrium solution is stable, unstable, or semi–stable.

22. (15 points) Consider the following autonomous equation: $\frac{dx}{dt} = f(x) = x(1 - x^2)$.

(a) Draw the phase diagram for this ODE.

(b) Use the phase portrait to fill out the table below indicating whether the given function is positive or negative on the interval (there are more rows than you will need). You may assume that $x(t) \geq 0$ (that is, you don't need to worry about the interval $(-\infty, 0)$.

Interval	$x' = f(x)$	$f'(x)$	$x'' = f'(x)f(x)$	Concavity

(c) Sketch some solution curves. Sketch at lest one curve with an initial value in each of the intervals from the table above. (As in class, you only need to sketch solutions in the quadrant $t \geq 0$ and $x \geq 0$.

(d) Determine the equilibrium solutions.

(e) Determine if each equilibrium solution is stable, unstable, or semi–stable.

23. (15 points) Consider the following autonomous equation: $\frac{dx}{dt} = f(x) = x^2(1-x)^2$.

(a) Draw the phase diagram for this ODE.

(b) Use the phase portrait to fill out the table below indicating whether the given function is positive or negative on the interval (there are more rows than you will need). You may assume that $x(t) \geq 0$ (that is, you don't need to worry about the interval $(-\infty, 0)$.

Interval	$x' = f(x)$	$f'(x)$	$x'' = f'(x)f(x)$	Concavity

(c) Sketch some solution curves. Sketch at lest one curve with an initial value in each of the intervals from the table above. (As in class, you only need to sketch solutions in the quadrant $t \geq 0$ and $x \geq 0$.

(d) Determine the equilibrium solutions.

(e) Determine if each equilibrium solution is stable, unstable, or semi–stable.

24. (15 points) Find the general solution to the following ODE (remember, it will not always be possible to find x explicitly as a function of t; in those cases, the solution will be a relation between x and t that gives x implicitly as a function of t.

$$(2t+3)+(2x-2)\frac{dx}{dt}=0.$$

25. (15 points) Find the general solution to the following ODE (remember, it will not always be possible to find x explicitly as a function of t; in those cases, the solution will be a relation between x and t that gives x implicitly as a function of t.

$$(3t^2 - 2tx + 2) + (6x^2 - t^2 + 3)\frac{dx}{dt} = 0.$$

26. (15 points) Find the solution to the following IVP (remember, it will not always be possible to find x explicitly as a function of t; in those cases, the solution will be a relation between x and t that gives x implicitly as a function of t. Determine at least approximately where the solution is valid. Use desmos (or something) to plot the solution and sketch the part of the curve that corresponds to the solution to the given IVP.

$$(2t - x) + (2x - t)\frac{dx}{dt} = 0 \qquad\qquad x(1) = 3.$$

27. (15 points) Find the solution to the following IVP (remember, it will not always be possible to find x explicitly as a function of t; in those cases, the solution will be a relation between x and t that gives x implicitly as a function of t. Determine at least approximately where the solution is valid. Use desmos (or something) to plot the solution and sketch the part of the curve that corresponds to the solution to the given IVP.

$$(9t^2 + x - 1) - (4x - t)\frac{dx}{dt} = 0 \qquad\qquad x(1) = 0.$$

28. For the following initial value problems, use the method of exact equations to find an implicit solution. If you can find an explicit solution (that is, if you can solve for x, then do so).

(1) $(2t + 3) + (2x - 2)\frac{dx}{dt} = 0$, $x(1) = 2$.

Solution: $M(t, x) = 2t+3$ and $N(t, x) = 2x-2$. Since $\partial_x M = 0 = \partial_t N$, this is exact. So we want to find a function $F(t, x)$ such that:

$$\frac{dF}{dt} = \partial_t F + \frac{dx}{dt}\partial_x F = M(t, x) + N(t, x)\frac{dx}{dt}.$$

So $F(t, x) = \int(2t + 3)dt + h(x) = t^2 + 3t + h(x)$. Now to find $h(x)$, note that $\partial_x F = (2x-2) = h'(x)$. So $h(x) = x^2-2x+C$. So the implicit solution is $F(t, x) = t^2 + 3t + x^2 - 2x = C$. From the initial condition, we have that $1+3(1)+2^2-2(2) = 4 = C$ and so $C = 4$. So we have $x^2-2x = 4-t^2-3t$. Thus, $(x - 1)^2 = t^2 + 3t - 5$. So, $x(t) = 1 \pm \sqrt{5 - t^2 - 3t}$.

(2) $(3t^2 - 2tx + 2) + (6x^2 - t^2 + 3)\frac{dx}{dt} = 0$, $x(0) = 1$.

Solution: $M(t, x) = 3t^2 - 2tx + 2$ and $M(t, x) = 6x^2 - t^2 + 3$. Since $\partial_x M = \partial_t N = -2t$ this is exact.

$$F(t, x) = \int(3t^2 - 2tx + 2)dt + h(x) = t^3 - t^2x + 2t + h(x).$$

To find $h(x)$, note that:

$$\partial_x F = 6x^2 - t^2 + 3 = \partial_x(t^2 - t^2x + 2t + h(x)) = -t^2 + h'(x).$$

This gives $6x^2 + 3 = h'(x)$ and so $h(x) = 2x^3 + 3x + C$. So that:

$$F(t, x) = t^3 - t^2x + 2t + 2x^3 + 3x = C.$$

Using the initial condition, $2(1^2) + 3(1) = 5 = C$. So this gives:

$$F(t, x) = t^3 - t^2x + 2t + 2x^3 + 3x = 5.$$

In theory, we could solve this for x since it is only a cubic in x. But I won't make you do that.

(3) $(e^t \sin x - 2x \sin t) + (e^t \cos x + 2 \cos t)\frac{dx}{dt} = 0$, $x(-1) = 2$.

Solution:

$$\begin{aligned} F(t, x) &= \int e^t \sin x - 2x \sin t dt \\ &= e^t \sin x + 2x \cos t + h(x). \end{aligned}$$

So:

$$\frac{\partial F}{\partial x} = e^t \cos x + 2 \cos t + h'(x) = e^t \cos x + 2 \cos t$$

So, $h'(x) = 0$ so $h(x) = c$. So the solution is $e^t \sin x + 2x \cos t = c$. Now with the initial condition: $e^{-1} \sin 2 + 2(2) \cos -1 = c$.

29. (1) $x'' - x' - 2x = 0$, $x(0) = 1$, $x'(0) = 1$.

Solution: $x(t) = \frac{2}{3}e^{2t} + \frac{1}{3}e^{-t}$.

(2) $x'' + 4x' + 3x = 0, x(0) = 2, x'(0) = -1$.

Solution: $x(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$.

(3) $6x'' - 5x' + x = 0, x(0) = 4.x'(0) = 0$.

Solution: $x(t) = 12e^{\frac{t}{3}} - 8e^{\frac{t}{2}}$.

30. Solve the following initial value problems:

- $y'' - 2y' + 2y = 0$, $y(0) = 1$, $y'(0) = 0$.

Solution: The characteristic polynomial is $r^2 - 2r + 2 = 0$ and the roots are $r_{1,2} = 1 \pm i$. So the general solution is $y(t) = c_1 e^t \cos t + c_2 e^t \sin t$. Using the initial conditions, we get that $y(t) = e^t \cos t - e^t \sin t$.

- $y'' - 2y' + 6y = 0$, $y(0) = 0$, $y'(0) = 1$.

Solution: $x(t) = \frac{e^t \sin \sqrt{5}t}{\sqrt{5}}$.

- $y'' + 2y' - 8y = 0$, $y(0) = 2$, $y'(0) = 0$.

Solution: $x(t) = \frac{4}{3}e^{2t} + \frac{2}{3}e^{-4t}$.

- $y'' + 2y' + 2y = 0$, $y(0) = 1$, $y'(0) = 2$.

Solution: The characteristic polynomial is $r^2 + 2r + 2 = (r+1)^2 + 1$ and so the roots are $r_{1,2} = -1 \pm i$. Thus, the general solution is $y(t) = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$. To find the constants, use the initial condition, $1 = y(0) = c_1$ and since $2 = y'(t) = -c_1 e^{-t} \cos t - c_2 e^{-t} \sin t - c_2 e^{-t} \sin t + c_2 e^{-t} \cos t$, and $c_1 = 1$ we have $2 = y'(0) = -1 + c_2$ and so $c_2 = 3$. So the solution is $y(t) = e^{-t} \cos t + 3e^{-t} \sin t$.

- $y'' + 6y' + 13y = 0$, $y(0) = 0$, $y'(0) = 1$.

Solution: $y(t) = \frac{1}{2}e^{-3t} \sin 2t$.

- $4y'' + 9y = 0$, $y(0) = 1$, $y'(0) = 0$.

Solution: $y(t) = \cos \sqrt{\frac{3}{2}}t$.

31. Solve the following initial value problems:

- $y'' - 2y' + y = 0$, $y(0) = 1$, $y'(0) = 0$.

> **Solution:** The characteristic polynomial is $r^2 - 2r + 1 = (r-1)^2$. There are repeated roots so the general solution is $y(t) = c_1e^t + c_2te^t$. Using the initial condition let's us solve for c_1, c_2: $y(t) = e^t - te^t$.

- $4y'' - 4y' - 3y = 0$, $y(0) = 0$, $y'(0) = 1$.

> **Solution:** $y(t) = \frac{1}{2}e^{\frac{3}{2}t} - \frac{1}{2}e^{-\frac{1}{2}t}$.

- $y'' - 2y' + 10y = 0$, $y(0) = 2$, $y'(0) = 0$.

> **Solution:** The characteristic polynomial is $r^2 - 2r + 10$ and the roots are $r_{1,2} = 1 \pm 3i$ and so the general solution is $x(t) = c_1e^t \cos 3t + c_2e^t \sin 3t$. Using the initial conditions gives $x(t) = 2e^t \cos 3t - \frac{2}{3}e^t \sin 3t$.

- $9y'' + 6y' + y = 0$, $y(0) = 1$, $y'(0) = 2$.
- $4y'' + 12y' + 9y = 0$, $y(0) = 1$, $y'(0) = 2$.
- $4y'' + 12y' + 9y = 0$, $y(0) = 0$, $y'(0) = 1$.
- $y'' - 6y' + 9y = 0$, $y(0) = 1$, $y'(0) = 0$.

32. Solve the following initial value problems.

- $x'' - 5x' + 6x = 2e^t$, $x(0) = 1$, $x'(0) = 0$.

Solution: First find the solution to the corresponding homogeneous system. So find the roots of $r^2 - 5r + 6 = (r-3)(r-2) = 0$ so the homogeneous solution is $x_H(t) = c_1 e^{2t} + c_2 e^{3t}$. Now we can find the particular solution using (for example) variation of parameters. First, $W(x_1, x_2)(t) = \det \begin{pmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{pmatrix} = e^{5t}$. So then:

$$\begin{aligned} x_P(t) &= -e^{2t} \int \frac{e^{3t} 2e^t}{e^{5t}} dt + e^{3t} \int \frac{e^{2t} 2e^t}{e^{5t}} dt \\ &= 2e^t - e^t \\ &= e^t. \end{aligned}$$

So then $x(t) = c_1 e^{2t} + c_2 e^{3t} + e^t$. Using the initial conditions we get:

$$1 = x(0) = c_1 + c_2 + 10 = 2c_1 + 3c_2 + 1,$$

So then $c_1 = 1$ and $c_2 = -1$ so that $x(t) = e^{2t} - e^{3t} + e^t$.

- $x'' - x' - 2x = 2e^{-t}$, $x(0) = 0$, $x'(0) = 2$.

Solution: The solution to the corresponding homogeneous equation is $x_H(t) = c_1 e^{-t} + c_2 e^{2t}$. To find a particular solution, note that $W(x_1, x_2)(t) = \begin{pmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{pmatrix} = 3e^t$. So then:

$$\begin{aligned} x_P(t) &= -e^{-t} \int \frac{e^{2t} 2e^{-t}}{3e^t} dt + e^{2t} \int \frac{e^{-t} 2e^{-t}}{3e^t} dt \\ &= -\frac{2}{3} t e^{-t} - \frac{2}{9} e^{-t}, \end{aligned}$$

Since e^{-t} is a solution to the corresponding homogeneous equation, we take $x_P(t) = -\frac{2}{3} t e^{-t}$. So the general solution is $x(t) = x_H(t) + x_P(t) = c_1 e^{-t} + c_2 e^{2t} - \frac{2}{3} t e^{-t}$. Using the initial conditions we get $x(t) = -\frac{8}{9} e^{-t} + \frac{4}{9} e^{3t} - \frac{2}{3} t e^{-t}$.

- $x'' + 2x' + x = 3e^{-t}$, $x(0) = 1$, $x'(0) = 1$.

Solution: $x(t) = \frac{1}{2}e^{-t}(3t^2 + 4t + 2)$.

- $4x'' - 4x' + x = 16e^{\frac{t}{2}}$, $x(0) = 0$, $x'(0) = 1$.

Solution: The characteristic polynomial is $4r^2 - 4r + 1 = (2r - 1)^2$. So $r = \frac{1}{2}$ is a double root. This means the guess for $x_p(t)$ should be:

$$x_p(t) = At^2e^{\frac{t}{2}}$$
$$x_p'(t) = A2te^{\frac{t}{2}} + A\frac{1}{2}t^2e^{\frac{t}{2}}$$
$$x_p''(t) = A2e^{\frac{t}{2}} + 2Ate^{\frac{t}{2}} + A\frac{1}{4}t^2e^{\frac{t}{2}}.$$

Plugging this into the equation, we get the equations:

$$t^2e^{\frac{t}{2}} : 0 = 4A\frac{1}{4} - 4A\frac{1}{2} + A$$
$$te^{\frac{t}{2}} : 0 = 8A - 8A$$
$$e^{\frac{t}{2}} : 16 = 8A.$$

So, $A = 2$. And so $x_p = 2t^2e^{\frac{t}{2}}$ so the general solution is:

$$x(t) = c_1e^{\frac{t}{2}} + c_2te^{\frac{t}{2}} + 2t^2e^{\frac{t}{2}}.$$

Finding the solution to the IVP:

$$x(t) = e^{\frac{t}{2}} + 2t^2e^{\frac{t}{2}}.$$

- $x'' + 4x' + 4x = t^2e^{-2t}$, $x(0) = 1$, $x'(0) = 0$.

Solution: The characteristic polynomial is $r^2+4r+4 = (r+2)^2$ so $r = -2$ is the only root. So the solution to the CHE is $x_H(t) = c_1e^{-2t} + c_2te^{-2t}$. Our guess for $x_p(t)$ is:

$$x_p(t) = (At^4 + Bt^3 + Ct^2)e^{-2t}.$$

This gives the general solution (via Undetermined Coefficients):

$$x(t) = c_1 e^{-2t} + c_2 t e^{-2t} + \frac{1}{12} t^4 e^{-2t}.$$

(You won't have to find the coefficients on such a long problem on a test).

- $x'' - 2x' + x = \frac{e^t}{1+t^2}$, $x(0) = -1$, $x'(0) = 1$.

33. (3 points) Consider the ODE:

$$x'(t) + p(t)x(t) = g(t).$$

What does it mean for $\mu(t)$ to be an integrating factor for this equation? Using that, derive (similar to what we did in class) an expression for $\mu(t)$.

34. (3 points) What does it mean for the equation $M(t, x) + N(t, x)\frac{dx}{dt} = 0$ to be exact? What does it mean for $\mu(t)$ to be an integrating factor of this equation? Derive an expression for $\mu(t)$ (here, I am assuming that there is an integrating factor that is a function of t only).