Math 172 Solutions Practice Final

Q1 Intersection Points: $x^3 - 4x^2 + 3x = 0$

$x(x^2 - 4x + 3) = 0, \quad x(x-1)(x-3) = 0$

$x = 0, 1, 3$

$$\int_0^1 x^3 - 4x^2 + 3x\,dx = \left[\frac{x^4}{4} - \frac{4}{3}x^3 + \frac{3x^2}{2}\right]_0^1$$

$$= \frac{1}{4} - \frac{4}{3} + \frac{3}{2} = \frac{3 - 16 + 18}{12} = \frac{5}{12}$$

$$\int_0^1 4x^2 - x^3 - 3x\,dx = \left[\frac{4}{3}x^3 - \frac{x^4}{4} - \frac{3}{2}x^2\right]_1^3$$

$$= 36 - \frac{81}{4} - \frac{27}{2} - \frac{4}{3} + \frac{1}{4} + \frac{3}{2}$$

$$= \frac{432 - 243 - 162 - 16 + 3 + 18}{12} = \frac{32}{12}$$

$$\text{Total} = \frac{37}{12}$$

Q2 $$\int \sin^4 x\,dx = \int \frac{1}{2}(1 - \cos 2x)\left(\frac{1 - \cos 2x}{2}\right)dx$$

$$= \int \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x\,dx$$

$$= \int \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x\,dx$$

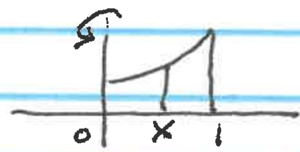
$$= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

Q3 $\int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx$

$= x^2 \sin x + 2x \cos x - \int 2 \cos x \, dx$

$= \underline{x^2 \sin x + 2x \cos x - 2 \sin x + C}$

Q4



Use shells.

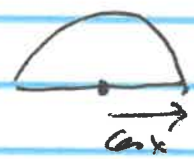
$$\int_0^1 2\pi x e^{3x} \, dx$$

$$= \left[\frac{2\pi x e^{3x}}{3} \right]_0^1 - \int_0^1 \frac{2\pi e^{3x}}{3} \, dx$$

$$= \left[\frac{2\pi x e^{3x}}{3} - \frac{2\pi e^{3x}}{9} \right]_0^1 = \frac{2\pi e^3}{3} - \frac{2\pi e^3}{9}$$

$$- 0 + \frac{2\pi}{9} = \underline{\frac{4\pi e^3}{9} + 2\pi/9}$$

Q5



Cross-section $= \frac{1}{2}\pi \cos^2 x$

$$V = \int_0^{\pi/2} \frac{1}{2}\pi \cos^2 x \, dx = \int_0^{\pi/2} \frac{1}{4}\pi (1 + \cos 2x) \, dx$$

$$= \left[\frac{1}{4}\pi x + \frac{1}{8}\pi \sin 2x \right]_0^{\pi/2} = \underline{\underline{\frac{\pi^2}{8}}}$$

Q6 $\int x^2(x+1)^{2018}\,dx \qquad u = x+1 \quad du = dx$

$$\int (u-1)^2 u^{2018}\,du = \int u^{2020} - 2u^{2019} + u^{2018}\,du$$

$$= \frac{u^{2021}}{2021} - \frac{2u^{2020}}{2020} + \frac{u^{2019}}{2019} + C$$

$$= \underline{\frac{(x+1)^{2021}}{2021} - \frac{2(x+1)^{2020}}{2020} + \frac{(x+1)^{2019}}{2019} + C}$$

Q7 $\frac{3x^2+x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$$3x^2+x+1 = A(x^2+1) + (Bx+C)x$$

$\underline{x=0}$ $\quad 1 = A$

$\underline{x^2 \,|}$ $\quad 3 = A+B, \quad B = 2$

$\underline{x \,|}$ $\quad 1 = C$

$$\int \frac{1}{x} + \frac{2x}{x^2+1} + \frac{1}{x^2+1}\,dx = \underline{\ln|x| + \ln|x^2+1| + \tan^{-1}x + C}$$

Q8 $dy/dx = 2x, \quad S = \int_0^1 2\pi x\sqrt{1+4x^2}\,dx$

$u = 1+4x^2, \; du = 8x\,dx$

$$S = \int_1^5 \frac{\pi}{4} u^{1/2}\,du = \left[\frac{\pi}{6}u^{3/2}\right]_1^5$$

$$= \underline{\pi/6\left[5\sqrt{5}-1\right].}$$

Q9 $\int x^3 \ln x \, dx$ $\quad u = \ln x \quad \frac{dv}{dx} = x^3$

$\frac{du}{dx} = \frac{1}{x} \quad v = x^4/4$

$$I = x^4/4 \ln x - \int x^3/4 \, dx = \underline{\frac{x^4}{4} \ln x - \frac{x^4}{16} + C}$$

Q10 $\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x \quad \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \tan^2 x} = \sec x$

$$L = \int_0^{\pi/4} \sec x \, dx = \left[\ln |\sec x + \tan x|\right]_0^{\pi/4}$$

$$= \ln |\sqrt{2} + 1| - \ln |1| = \underline{\ln(\sqrt{2} + 1)}$$

Q11 Compare to $\sum \frac{1}{n}$.

$$\lim_{n \to \infty} \sqrt{\frac{n^4 + 2}{n^6 + n}} \, n = \lim_{n \to \infty} \sqrt{\frac{n^6 + 2n^2}{n^6 + n}}$$

$$= \lim_{n \to \infty} \sqrt{\frac{1 + 2/n^4}{1 + 1/n^5}} = 1.$$

By limit comparison, both converge or both diverge.

$\sum \frac{1}{n}$ diverges, p-series $p = 1$, so

$\sum \sqrt{\frac{n^4 + 2}{n^6 + n}}$ <u>diverges.</u>

Q12 Ratio test. $\lim_{n\to\infty} \frac{(n+1)!^4}{(4n+4)!} \frac{(4n)!}{(n!)^4} =$

$\lim_{n\to\infty} \frac{(n+1)(n+1)(n+1)(n+1)}{(4n+4)(4n+3)(4n+2)(4n+1)} = \frac{1}{4^4} < 1.$

The series converges by the ratio test.

Q13 $\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{3^{n+1}(n+1)\ln(n+1)} \frac{n \ln n \, 3^n}{(x-2)^n} \right|$ (Typo in the problem) start at $n=2$.

$= \frac{|x-2|}{3} \lim_{n\to\infty} \frac{\ln n}{\ln n+1} = \frac{|x-2|}{3} \lim_{n\to\infty} \frac{\frac{1}{n}}{1/n+1} = \frac{|x-2|}{3} < 1.$

$-3 < (x-2) < 3, \quad -1 < x-2 < 5.$

Check the end points.

$\underline{x=5.}$ $\sum \frac{1}{n \ln n}$. $\frac{1}{x \ln x}$ is decreasing and $\to 0$ as

$x \to \infty$. The series diverges by the integral test

since $\int_2^t \frac{1}{x \ln x} dx = [\ln(\ln x)]_2^t \to \infty$ as $t \to \infty$.

$\underline{x=-1}$ $\sum \frac{(-1)^n}{n \ln n}$. $\frac{1}{n \ln n}$ is decreasing and

$\to 0$ as $n \to \infty$ so the series converges by the AST.

Interval of convergence is $[-1, 5)$.

Q14 The problem is at 0. The minimum value of $\cos^2 x$ on $[0,1]$ is at $x=1$, so $\cos^2(1)$.

$$\frac{\cos^2 x}{x} \geq \frac{\cos^2(1)}{x}.$$

$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0} -\ln t = \infty.$$

$\therefore \int_0^1 \frac{\cos^2 x}{x}\,dx$ diverges by comparison to $\frac{1}{x}$.

Q15. $$\frac{4}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}, \quad 4 = A(x-2) + Bx$$

$\underline{x=0}$ $A=-2$, $\underline{x=2}$ $B=2$

$$2\sum_{n=4}^{\infty} \frac{1}{n-2} - \frac{1}{n} = 2\left(\frac{1}{2} - \cancel{\frac{1}{4}} + \frac{1}{3} - \cancel{\frac{1}{5}} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} \cdots\right)$$

$$= 2\left(\frac{1}{2} + \frac{1}{3}\right)$$

$$= \underline{\underline{5/3}}$$