If \( \sum_{n=1}^{\infty} a_n \) is a convergent series with sum \( S \) and \( n \)th partial sum \( S_n \), we can write
\[
S = S_n + T_n
\]
where \( T_n = a_{n+1} + a_{n+2} + \ldots \)
and is called the tail.

Since \( S_n \to S \), we get
\[
T_n = S - S_n \to 0 \text{ as } n \to \infty.
\]
Power Series

A power series is one of the form \[ \sum_{n=0}^{\infty} a_n z^n \], or more generally \[ \sum_{n=0}^{\infty} a_n (z-z_0)^n \], where the \( a_n \)'s are complex numbers. The substitution \( w = z - z_0 \) takes \( \sum a_n (z-z_0)^n \) to \( \sum a_n w^n \), so we concentrate on the first type.
\[ \text{Ex (Geometric series). For a fixed } z, \text{ we consider } \sum_{n=0}^{\infty} z^n \]

\[ S_n = 1 + z + \ldots + z^{n-1} \]
\[ zS_n = z + z^2 + \ldots + z^n + z^n \]
\[ (1-z)S_n = 1 - z^n. \]

For \( |z| < 1 \), \( S_n = \frac{1 - z^n}{1 - z} \to \frac{1}{1-z} \) as \( n \to \infty \).

For \( |z| \geq 1 \), \( z^n \to 0 \) as \( n \to \infty \)

so we have convergence for \( |z| < 1 \)

and divergence for \( |z| \geq 1 \).

We have a number \( R (=1 \text{ in this case}) \)
so that the series converges for

\[ |z| < R \]

and diverges for \( |z| > R \).

**Theorem** If \( w \) is a number so that

\[
\sum_{n=0}^{\infty} a_n w^n \text{ converges, then}
\]

\[
\sum_{n=0}^{\infty} a_n z^n \text{ converges absolutely for}
\]

\[ |z| < |w| \]

**Pf.** If \( w = 0 \), nothing to prove, so

assume \( w \neq 0 \).

Convergence of \( \sum_{n=0}^{\infty} a_n w^n \) \( \Rightarrow \) \( \lim_{n \to \infty} a_n w^n = 0 \)

so there is a constant \( M > 0 \) so that
\[ |a_n w^n| \leq M \text{ for } n \geq 1. \]

If \( |z| < |w| \) then \( \frac{|z|}{|w|} < 1 \) and

\[ \sum |a_n z^n| = \sum |a_n w^n \left( \frac{z}{w} \right)^n| \]
\[ \leq \sum M |\frac{z}{w}|^n = \frac{M}{1 - |\frac{z}{w}|} < \infty. \]

\[ \therefore \sum a_n z^n \text{ converges absolutely.} \]

If we let \( R = \sup \{ |w| : \sum a_n w^n \text{ converges} \} \),

then \( \sum a_n z^n \text{ converges absolutely for } |z| < R \), and diverges for \( |z| > R \).
A \[ R \] is called the radius of convergence.

**Example:**
(i) \[ \sum_{n=0}^{\infty} \frac{z^n}{n!} \] converges absolutely for all \( z \) by the ratio test since

\[
\left| \frac{z^{n+1}}{(n+1)!} \frac{n!}{z^n} \right| = \frac{|z|}{n+1} \to 0 \text{ as } n \to \infty
\]

Here \( R = \infty \).

(ii) \[ \sum_{n=0}^{\infty} \frac{n! z^n}{n^n} \]

If \( z \neq 0 \) then

\[
\left| \frac{(n+1)! z^{n+1}}{n! z^n} \right| = (n+1) |z|
\]

\( \to \infty \) as \( n \to \infty \)

Here \( R = 0 \).

(iii) \[ \sum \frac{z^n}{R^n} \] converges for \( |\frac{z}{R}| < 1 \)

so here the radius of convergence is \( R \).
Ex. Anything can happen on the boundary.
\[ \sum z^n \text{ diverges for } |z| = 1, \]
\[ \sum \frac{z^n}{n} \text{ converges absolutely for } |z| = 1 \]
since \[ \sum \frac{1}{n^2} \text{ converges.} \]

Theorem If \[ \sum_{n=0}^{\infty} a_n z^n \] has radius of convergence \( R > 0 \), then define
\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ for } |z| < R. \]
Then \( f(z) \) is analytic on \( |z| < R \).
Proof. Fix $z_0$ with $|z_0| < R$ and choose $r$ so that $|z_0| < r < R$.

Then choose $p$ so that $p < r < R$.

Then $\sum a_n z^n$ converges absolutely when $z = r$ so there is an $M$ with $|a_n r^n| \leq M$. For $121 \leq p$

$$|T_n(z)| \leq \sum_{k=n+1}^{\infty} |a_k z^k| \leq \sum_{k=n+1}^{\infty} |a_k r^k| p^k/r^k$$

$$\leq M \sum_{k=n+1}^{\infty} |(p/r)^k| = M (p/r)^{n+1}/(1 - p/r)$$
For a given $\epsilon > 0$, choose $N$ so large that $M(\gamma_0^n) / (1 - \rho N) < \epsilon$ for $n > N$. If $C$ is a closed curve in the disc $|z| < \rho$ then

$$\oint_C f(z) \, dz = \oint_C \sum_{n=0}^N f_n(z) \, dz + \oint_C T_N(z) \, dz$$

$$= 0 + \oint_C T_N(z) \, dz$$

so

$$|\oint_C f(z) \, dz| \leq \epsilon \text{ length } C.$$ 

$\epsilon$ may be arbitrary so $\oint_C f(z) \, dz = 0$. By Mera, $f(z)$ is analytic in $|z| < \rho$ and in particular at $z_0$. 