Math 407 First Test Solution

Q1 \[
\frac{1+i}{-2i} = \frac{(1+i)(1+2i)}{(-2i)(1+2i)} = \frac{1+i+2i-2}{5} = -\frac{1}{5} + \frac{3}{5}i
\]

Q2 \[
i = e^{i\left(\frac{\pi}{2} + 2\pi n\right)} \text{ so the 4th roots are}
\]
\[
e^{i\left(\frac{\pi}{8} + \frac{n\pi}{2}\right)} \text{ so the 4 distinct ones are}
\]
\[
e^{\frac{i\pi}{8}}, e^{\frac{i5\pi}{8}}, e^{\frac{i9\pi}{8}}, e^{\frac{i13\pi}{8}}
\]

Q3 \[
|z| = |z-2i| \text{ is the same as}
\]
\[
|z|^2 = (z-2i)^2 = 0
\]
\[
x^2 + y^2 = x^2 + (y-2)^2 = x^2 + y^2 - 4y + 4
\]
\[
\text{so } 4y = 4, \quad y = 1.
\]

The solution set is \[
\{z : \text{Im} z = 1\}
\]
\[ f(z) = (x-y)^3 + i(x-y)^2 \]

Check C-R equations.

\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} : 3(x-y)^2 = -2(x-y) \]

\[ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} : -3(x-y)^2 = -2(x-y) \]

Add to get \(-4(x-y) = 0 \Rightarrow x = y\).

The partials are continuous so \(f''(z)\) exists at every point on the line \(y = x\), nowhere else.

None of these points have a neighborhood where \(f'(z)\) exists, so \(f(z)\) is not analytic at any point.
Q5 \[ \frac{1+i}{\sqrt{2}} = e^{i \frac{\pi}{4} + 2\pi n} \text{ for } n = 0, \pm 1, \pm 2, \ldots \]

Thus \( (\frac{1+i}{\sqrt{2}})^i \) has possible values
\[ e^{i \frac{\pi}{4} + 2\pi n} = e^{-i \frac{\pi}{4} + 2\pi n}, \quad n = 0, \pm 1, \pm 2, \ldots \]

Q6 \[ u(x, y) = (2x+1) y \]
\[ \frac{\partial u}{\partial x} = 2y, \quad \frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial u}{\partial y} = 2x+1, \quad \frac{\partial^2 u}{\partial y^2} = 0 \]

\[ \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad u \text{ is harmonic.} \]

First solve \( \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \), \( \frac{\partial v}{\partial x} = 2y \), \( v = y^2 + g(x) \)

\[ \frac{\partial v}{\partial y} = 2x+1 = -\frac{\partial v}{\partial x} = -g'(x), \quad \Rightarrow \]

\[ g(x) = -x^2 - x + C \]

Thus \( v = y^2 - x^2 - x + C \)
\[ f(z) = \overline{z}^2 = (x - iy)^2 = x^2 - y^2 - 2ixy \]

Check C-R eqns:

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} : 2x = -2x, \quad x = 0 \\
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} : -2y = 2y, \quad y = 0.
\]

The C-R eqns only hold at \( z = 0 \), and the partials are continuous there so

\[ f'(0) \text{ exists, but } f'(z) \text{ does not exist for } z \neq 0. \]

\[ \text{Q8} \]

Since \( f(z) \) is analytic, it has:

(i) C-R equations for \( f(z) = u + iv \)

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} / \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]

(ii) Since \( v = 0 \), the C-R eqns are

\[
\frac{\partial u}{\partial x} = 0 / \frac{\partial u}{\partial y} = 0.
\]

Then \( u = g(y) \) and \( g'(y) = 0 \) so \( g(y) \) is a constant. Thus

\[ f(z) = c \text{ where } c \text{ is a real constant.} \]