Math 407 2nd Test Solutions

Q1. (i) $C \ni z = 1 + t(i-t)$ so $dz = (i-t)dt$

$$\int_C \bar{z} \, dz = \int_0^1 (1-t-i)t(i-t) \, dt$$

$$= (i-t) \left[ t - \frac{t^2}{2} - i \frac{t^2}{2} \right]_0^1 = (i-t)\left( \frac{1}{2} - \frac{i}{2} \right)$$

$$= -\frac{1}{2} + \frac{1}{2} + \frac{i}{2} + \frac{i}{2} = i$$

(ii) $C \ni z=e^{i\theta}$ so $dz = i e^{i\theta} d\theta$

$$\int_0^{\pi/2} e^{-i\theta} i e^{i\theta} \, d\theta = \frac{i\pi}{2}$$

Q2.

Adding the 2 circuits gives $\oint_{C_1+C_2} \, dz = \oint_{C_1} \, dz + \oint_{C_2} \, dz$.

$$C_1 \oint_{C_1} \frac{\cos \pi z}{(z-1)^2} \, dz = 2\pi i \left. \frac{\cos \pi z}{(z-1)^2} \right|_{z=0} = 2\pi i$$

$$C_2 \oint_{C_2} \frac{\cos \pi z}{(z-1)^2} \, dz = 2\pi i \left. \frac{d}{dz} \left( \frac{\cos \pi z}{z} \right) \right|_{z=1}$$

$$= 2\pi i \left[ -\pi \sin \pi z \frac{1}{z} - \frac{\cos \pi z}{z^2} \right]_{z=1} = 2\pi i$$

Ans = $4\pi i$
Q3. \( |z| = 2 \) contains only the singularity \( z = 0 \)

So the integral is 
\[
2\pi i \left. \frac{e^z}{4-z} \right|_{z=0}
\]

\[
= 2\pi i \left[ \frac{e^z (4-z) + e^z}{(4-z)^2} \right] \bigg|_{z=0} = 2\pi i \cdot 5 = \frac{5\pi i}{16} = \frac{5}{8}
\]

Q4. (i) \( f^{(n)}(z_0) = n! \int_{\gamma} \frac{f(z)}{(z-z_0)^{n+1}} \) for \( C \) containing \( z_0 \) and \( f(z) \) is analytic on and inside \( C \).

(ii) \( f^{(n)}(0) = n! \int_{|z|=R} \frac{f(z)}{z^{n+1}} \) when \( C \) is

\[
\left| \frac{f(z)}{z^{n+1}} \right| \leq \left| \frac{2}{z^{n+1}} \right| = \frac{1}{R^n} \quad \text{and the}
\]

length is 
\[
2\pi R \times \int_{|z|=R} \left( \left| \frac{f(z)}{z^{n+1}} \right| \right) \leq 2\pi R \cdot \frac{1}{R^n+1} \to 0 \quad \text{as} \quad R \to \infty \quad \text{when} \quad n > 2.
\]

Thus, 
\[
f^{(n)}(0) = 0 \quad \text{for} \quad n \geq 2.
\]
Q5. (i) If \( f(z) \) is analytic on and inside a closed contour \( C \) then the maximum value of \( |f(z)| \) occurs on the boundary.

(ii) By (i) we only have to check the maximum over the boundary.

On \( L_1 \), \( e^z = e^x e^{-i} \), \( \max |e^z| = e \) at \( x = 1 \).

On \( L_2 \), \( e^z = e^i e^{it} \), \( \max |e^z| = e \).

On \( L_3 \), \( e^z = e^x e^i \), \( \max |e^z| = e \).

On \( L_4 \), \( e^z = e^{-1} e^{iy} \), \( \max |e^z| = e^{-1} \).

Max of \( |e^z| \) is \( e \).
Q6. (i) If \( f(z) = u + iv \) then
\[
e^{f(z)} = e^{u+iv} = e^u.
\]

(ii) The max of \( |e^{f(z)}| \) occurs on \( |z| = R \) and \( e^u \) peaks when \( u \) peaks, so the
max value of \( u \) occurs on the boundary.

Q7. (i) In a region \( R \), if \( \oint_C f(z)\,dz = 0 \) for all
closed contours \( C \) in \( R \) then \( f(z) \) is analytic.

(ii) If \( C \) is a closed contour then
\[
\oint_C \int e^{z^2}\,dz\,dz = \oint_C \int e^{z^2}\,dz
dt = 0 \text{ since } e^{z^2} \text{ is analytic in } z.
\]

By Morera, \( \oint_C \int e^{z^2}\,dz \) is analytic in \( z \).

Q8. \( z = 0 \) is the singularity, so
\[
\oint e^{\frac{1}{2}z}\,dz = 0. \text{ This is } \oint e^{\frac{1}{2}z}\,dz - \oint e^{\frac{1}{2}z}\,dz = 0,
\]
\( 121 = \mathbb{R} \) and \( 121 = \mathbb{R} \).