CONES ARISING FROM C^*-SUBALGEBRAS AND COMPLETE POSITIVITY

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Abstract

Let $B \subseteq A$ be an inclusion of C^*-algebras. Then B is said to norm A if, for each $X \in \mathbb{M}_n(A)$,

$$\|X\| = \sup\{\|RXC\| \colon R^*, C \in \mathrm{Col}_n(B),\ \|R\|, \|C\| \leq 1\}.$$

In this paper we introduce and study the cones

$$\{X \in \mathbb{M}_n(A)_{\mathrm{sa}} \colon C^*XC \geq 0 \text{ for } C \in \mathrm{Col}_n(B)\},\ n \geq 1.$$

These are shown to coincide with the standard positive cones precisely when B norms A, and we apply this to obtain automatic complete positivity of certain positive maps between C^*-algebras.

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1 Introduction

Two related problems in the theory of mappings between C^*-algebras are to determine when a given bounded map ϕ is completely bounded or completely positive. The transpose on $B(\ell^2)$ is the standard example of a positive isometric map which fails to be completely positive or completely bounded, so there is no implication of complete boundedness from boundedness, or of complete positivity from positivity. Additional hypotheses must be imposed before such deductions are valid. In [9], the first of these problems was addressed by introducing the notion of a norming subalgebra B of a C^*-algebra A, and establishing that B-modular maps ϕ automatically satisfy $\|\phi\| = \|\phi\|_{\mathrm{cb}}$. In this paper we investigate the second of these problems by studying certain cones which incorporate a given subalgebra B in their definitions. Before we describe these cones and the main results concerning them, we must establish some notation.

If A is a C^*-algebra, then $\mathbb{M}_{m,n}(A)$ will denote the space of rectangular $m \times n$ matrices over A, abbreviated to $\mathbb{M}_n(A)$ when $m = n$. The spaces $M_{n,1}(A)$ and $M_{1,n}(A)$ are viewed respectively as columns and rows over A, and for these we use the more descriptive notation $\mathrm{Col}_n(A)$ and $\mathrm{Row}_n(A)$. The positive cone in the C^*-algebra $\mathbb{M}_n(A)$ is denoted $\mathbb{M}_n(A)^+$, while $\mathbb{M}_n(A)_{\mathrm{sa}}$ is the self-adjoint part. When B is a C^*-subalgebra of A, we define a generalized system of cones in $\mathbb{M}_n(A)$, $n \geq 1$, as follows:

$$\mathbb{M}_n(A)^+_B = \{X \in \mathbb{M}_n(A)\colon\ C^*XC \geq 0 \quad \text{for all} \quad C \in \mathrm{Col}_n(B)\}, \tag{1.1}$$

where the symbol $\geq$ refers to the standard notion of positivity in $\mathbb{M}_n(A)$. It is clear from the definition that these are closed convex cones which satisfy $\mathbb{M}_n(A)^+ \subseteq \mathbb{M}_n(A)^+_B$. A cone K is called proper if $K \cap (-K) = \{0\}$, and we establish this property for the cones of (1.1) below. In the special case when $B = A$, we prove that the cones $\mathbb{M}_n(A)^+_A$ and $\mathbb{M}_n(A)^+$ coincide for $n \geq 1$, justifying the point of view that these cones generalize the standard positive cones $\mathbb{M}_n(A)^+$. To avoid technical complications, we will assume that A and B are unital with a common unit, although the results remain valid in general.

The main result of the paper characterizes when the cones $\mathbb{M}_n(A)^+_B$ and $\mathbb{M}_n(A)^+$ coincide for $n \geq 1$; this occurs precisely when B is a norming subalgebra of A. Recall from [9] that B is a norming subalgebra if

$$\|X\| = \sup\{\|RXC\|\colon\ C \in \mathrm{Col}_n(B),\ R \in \mathrm{Row}_n(B),\ \|C\| \leq 1, \|R\| \leq 1\} \tag{1.2}$$

for each $X \in \mathbb{M}_n(A)$ and each $n \geq 1$. If we observe that $RXC \in A$, then the purpose of this definition is to be able to determine norms of matrices over A in terms of norms of elements in A. One consequence of the definition is that bounded B-modular maps on A are completely bounded, [9], and we will show the companion result that positive B-modular maps are automatically completely positive, using the cones $\mathbb{M}_n(A)_B^+$.

There are two formally weaker notions of B being a norming subalgebra of A. For all $X \in \mathbb{M}_n(A)$, $n \geq 1$, the relations

$$\|X\| = \sup\{\|XC\|\colon\ C \in \mathrm{Col}_n(B),\ \|C\| \leq 1\} \tag{1.3}$$

and

$$\|X\| = \sup\{\|RX\|\colon\ R \in \mathrm{Row}_n(B),\ \|R\| \leq 1\} \tag{1.4}$$

are the definitions for B to be respectively column norming and row norming for A. These will be useful below, since they were both shown to be equivalent to (1.2) in [9, Lemma 2.4]. The original applications of norming were to the study of von Neumann algebra cohomology in [9], but this notion has also proved fruitful in other contexts, [3, 7, 8]. Many classes of norming subalgebras were discussed in [9], and we take the opportunity to add one more to this list in the final remarks.

For background material on operator systems, complete boundedness and complete positivity, we refer the reader to [2, 4, 6].

2 The main results

In (1.1), we introduced the cones $\mathbb{M}_n(A)^+_B$, $n \geq 1$. The first lemma establishes that these may be viewed as generalizations of the standard positive cones $\mathbb{M}_n(A)^+$, $n \geq 1$.

Lemma 2.1. *Let $n \geq 1$. A matrix $X \in \mathbb{M}_n(A)$ lies in $\mathbb{M}_n(A)^+$ if and only if $C^*XC \in A^+$ for every $C \in \mathrm{Col}_n(A)$. In particular, $\mathbb{M}_n(A)^+ = \mathbb{M}_n(A)^+_A$ for all $n \geq 1$.*

Proof. It is clear that $C^*XC \geq 0$ whenever $X \geq 0$, so we only consider the reverse implication. As a special case, we first suppose that A can be faithfully irreducibly represented on a Hilbert space H. The Kaplansky density theorem then shows that any nonzero vector $\xi \in H$ is cyclic for A. Then each vector $\eta \in H^n = H \oplus \ldots \oplus H$ is arbitrarily close to one of the form $C\xi$ for $C \in \mathrm{Col}_n(A)$, showing that $X \geq 0$ if and only if $C^*XC \geq 0$ for all $C \in \mathrm{Col}_n(A)$. Since in general A can be faithfully represented by a direct sum of irreducible representations, the argument for the special case above gives the result that $X \geq 0$ when $C^*XC \geq 0$ for all $C \in \mathrm{Col}_n(A)$. □

Let K be a proper convex cone in the self-adjoint part V_{sa} of a vector space V with involution. Then K induces an ordering $\leq_K$ on V_{sa} by $v_1 \leq_K v_2$ if and only if $v_2 - v_1 \in K$. An element $e \in K$ is called an order unit if each $v \in V_{\mathrm{sa}}$ satisfies $-\lambda e \leq_K v \leq_K \lambda e$ for some scalar $\lambda \in \mathbb{R}^+$. The cone is Archimedean if the inequalities $v \leq_K \varepsilon e$ for all $\varepsilon > 0$ imply that $v \leq_K 0$.

Proposition 2.2. *Let $B \subseteq A$ be an inclusion of unital C^*-algebras.*

(i) *If $X \in \mathbb{M}_n(A)^+_B$ and $S \in \mathbb{M}_{n,p}(B)$, then $S^*XS \in \mathbb{M}_p(A)^+_B$;*

(ii) *$\mathbb{M}_n(A)^+ \subseteq \mathbb{M}_n(A)^+_B \subseteq \mathbb{M}_n(A)_{\mathrm{sa}}$;*

(iii) *I_n is an order unit for $\mathbb{M}_n(A)^+_B$;*

(iv) *each $\mathbb{M}_n(A)^+_B$ is Archimedean;*

(v) *each $\mathbb{M}_n(A)^+_B$ is a proper convex cone.*

Proof. (i) This is immediate from the fact that $SC \in \mathrm{Col}_n(B)$ whenever $S \in \mathbb{M}_{n,p}(B)$ and $C \in \mathrm{Col}_p(B)$.

(ii) The first inclusion is clear, using Lemma 2.1 and the inclusion $\mathrm{Col}_n(B) \subseteq \mathrm{Col}_n(A)$. To prove the second inclusion, we first consider the case $n = 2$. Let $X = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{M}_2(A)^+_B$. We will show that X is self-adjoint. For arbitrary scalars $\alpha, \beta \in \mathbb{C}$, let $C = (\alpha, \beta)^t \in \mathrm{Col}_2(B)$. Then

$$|\alpha|^2 a + |\beta|^2 d + \bar{\alpha}\beta b + \alpha\bar{\beta} c = C^*XC \geq 0. \tag{2.1}$$

Taking (α, β) to be $(1, 0)$ or $(0, 1)$ shows that $a, d \geq 0$, so $\bar{\alpha}\beta b + \alpha\bar{\beta} c$ is self-adjoint. Take $\alpha = 1$ and $\beta = e^{i\theta}$ to obtain

$$e^{i\theta} b + e^{-i\theta} c = e^{-i\theta} b^* + e^{i\theta} c^* \tag{2.2}$$

for all $\theta \in [0, 2\pi]$. Multiply by $e^{i\theta}$ in (2.2) and then integrate θ over $[0, 2\pi]$ to obtain $c = b^*$. This shows that X is self-adjoint, establishing the inclusion $\mathbb{M}_2(A)^+_B \subseteq \mathbb{M}_2(A)_{\mathrm{sa}}$. The argument for $n \geq 3$ is essentially the same: reduce to the case $n = 2$ by compressing $X \in \mathbb{M}_n(A)^+_B$ to the i and j positions.

(iii) If $X \in \mathbb{M}_n(A)_{\mathrm{sa}}$ then $\|X\| I_n \pm X \in \mathbb{M}_n(A)^+$. Thus $\|X\| I_n \pm X \in \mathbb{M}_n(A)^+_B$ by the first inclusion of (ii), so I_n is an order unit for $\mathbb{M}_n(A)^+_B$.

(iv) Let $X \in \mathbb{M}_n(A)_{\mathrm{sa}}$ be such that $\varepsilon I_n - X \in \mathbb{M}_n(A)^+_B$ for all $\varepsilon > 0$. If $C \in \mathrm{Col}_n(B)$ is arbitrary then

$$\varepsilon C^* C - C^* X C \geq 0. \tag{2.3}$$

Let $\varepsilon \to 0$ in (2.3) to conclude that $C^* X C \leq 0$ for all $C \in \mathrm{Col}_n(B)$. This shows that $-X \in \mathbb{M}_n(A)^+_B$ as required.

(v) Convexity of $\mathbb{M}_n(A)^+_B$ is clear from the definition of this cone in (1.1), so we need only establish that this is a proper cone. When $n = 1$, $\mathbb{M}_1(A)^+_B$ and A^+ coincide, so this case is immediate. For $n \geq 2$, compression to the i and j positions reduces to the case $n = 2$, which is then the only one that we need consider. Let $X \in \mathbb{M}_2(A)_{\mathrm{sa}}$ be such that $\pm X \in \mathbb{M}_2(A)^+_B$, and write $X = \left(\begin{smallmatrix} a & b \\ b^* & d \end{smallmatrix}\right)$. The argument of (ii) shows that $\pm a, \pm d \in A^+$, and so $a = d = 0$. In (2.1), let $c = b^*$, $\alpha = 1$ and $\beta = e^{i\theta}$ to obtain $\pm(e^{i\theta} b + e^{-i\theta} b^*) \geq 0$. It follows that $e^{i\theta} b + e^{-i\theta} b^* = 0$ for all $\theta \in [0, 2\pi]$, showing that $b = 0$ and that $X = 0$. Thus each $\mathbb{M}_n(A)^+_B$ is a proper cone. $\square$

A concrete operator system is a self-adjoint subspace E of $B(H)$ which contains the identity operator. These were introduced by Choi and Effros, [1], who gave an abstract characterization as follows. A matrix ordered space is a complex vector space V with involution with a specified sequence of convex proper cones $V_n^+ \subseteq \mathbb{M}_n(V)_{\mathrm{sa}}$, $n \geq 1$, satisfying the compatibility requirement that $S^* V_m^+ S \subseteq V_n^+$ for all scalar $m \times n$ matrices S. Then V is an abstract operator system if V^+ also has an order unit e and the cones V_n^+, $n \geq 1$, are Archimedean. In [1, Theorem 4.4], it was shown that each abstract operator system is completely order isomorphic to a concrete operator system by a map Ψ which also takes the order unit to the identity operator. The end of the proof of this result in [1] shows that Ψ can be taken to be the direct sum of all completely positive maps $\omega\colon\ V \to \mathbb{M}_k$ satisfying $\omega(e) = I_k$, with k varying (see also the proof given in [10, Theorem 1.2.7]). When we refer to Ψ below, it is assumed to have this form. Proposition 2.2 shows that the cones $\mathbb{M}_n(A)^+_B$ give A the structure of an abstract operator system, and so the foregoing discussion applies in this case, giving a map $\Psi\colon\ A \to B(H)$ which is a complete order isomorphism between A and the operator system $\Psi(A) \subseteq B(H)$.

We now come to our main result.

Theorem 2.3. *Let $B \subseteq A$ be an inclusion of unital C^*-algebras. Then the following statements are equivalent:*

(i) *for all $n \geq 1$, $\mathbb{M}_n(A)^+_B = \mathbb{M}_n(A)^+$;*

(ii) *the map Ψ for the operator system A with cones $\mathbb{M}_n(A)^+_B$, $n \geq 1$, is a complete isometry;*

(iii) *B is norming for A.*

Proof. We will show that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii).
(ii) $\Rightarrow$ (iii). For each $X \in \mathbb{M}_n(A)_{\mathrm{sa}}$, define

$$\rho_n(X) = \sup\{\|C^*XC\|\colon\ C \in \mathrm{Col}_n(B),\ \|C\| \leq 1\}. \tag{2.4}$$

If $X_0 = \left(\begin{smallmatrix} X & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathbb{M}_{n+1}(A)_{\mathrm{sa}}$ is obtained from X by adding in zero matrices of appropriate sizes, then it is easy to see that $\rho_n(X) = \rho_{n+1}(X_0)$. We will now show that

$$\rho_n(X)I_n - X \in \mathbb{M}_n(A)^+_B. \tag{2.5}$$

To this end, take $C = (b_1, \ldots, b_n)^t \in \mathrm{Col}_n(B)$, and modify it, for each $\varepsilon > 0$, to $C_\varepsilon = (b_1(\varepsilon), \ldots, b_{n+1}(\varepsilon))^t \in \mathrm{Col}_{n+1}(B)$ by starting with $(b_1, \ldots, b_n, \varepsilon 1)^t$ and multiplying on the right by the invertible element $b(\varepsilon) = \left(\varepsilon^2 1 + \sum\limits_{i=1}^n b_i^* b_i\right)^{-1/2} \in B$. This last factor ensures that $C_\varepsilon^* C_\varepsilon = 1$, and in particular $\|C_\varepsilon\| \leq 1$. By definition of $\rho_n(X)$, we have

$$C_\varepsilon^* X_0 C_\varepsilon \leq \rho_{n+1}(X_0)1 = \rho_n(X)1, \tag{2.6}$$

and so

$$C_\varepsilon^*(\rho_n(X)I_{n+1} - X_0)C_\varepsilon = \rho_n(X)1 - C_\varepsilon^* X_0 C_\varepsilon \geq 0. \tag{2.7}$$

If we multiply on the left and right in (2.7) by $b(\varepsilon)^{-1}$, the result is

$$(C^*, \varepsilon 1)\begin{pmatrix} \rho_n(X)I_n - X & 0 \\ 0 & \rho_n(X)1 \end{pmatrix}\begin{pmatrix} C \\ \varepsilon 1 \end{pmatrix} \geq 0. \tag{2.8}$$

Letting $\varepsilon \to 0$ in (2.8) leads to

$$C^*(\rho_n(X)I_n - X)C \geq 0, \tag{2.9}$$

and this proves (2.5) since $C \in \mathrm{Col}_n(B)$ was arbitrary. The argument to obtain (2.9) applies equally to $-X$, and so we have the more general inequality

$$C^*(\rho_n(X)I_n \pm X)C \geq 0 \tag{2.10}$$

for all $C \in \mathrm{Col}_n(B)$ and $X \in \mathbb{M}_n(A)_{\mathrm{sa}}$. Since Ψ is a complete order isomorphism, we may apply Ψ to (2.5), yielding

$$\rho_n(X)I_n \pm \Psi_n(X) \geq 0 \tag{2.11}$$

for $X \in \mathbb{M}_n(A)_{\mathrm{sa}}$, from which it follows that $\|\Psi_n(X)\| \leq \rho_n(X)$. The assumption that (ii) holds allows us to conclude that $\|X\| \leq \rho_n(X)$. It then follows that $\|X\| = \rho_n(X)$, since the reverse inequality is obvious from (2.4).

Now consider a general element $Y \in \mathbb{M}_n(A)$, and let X be $Y^*Y \in \mathbb{M}_n(A)_{\mathrm{sa}}$. Then

$$\begin{aligned} \|Y\|^2 = \|X\| = \rho_n(X) &= \sup\{\|CY^*YC\|\colon\ C \in \mathrm{Col}_n(B),\ \|C\| \leq 1\} \\ &= \sup\{\|YC\|^2\colon\ C \in \mathrm{Col}_n(B),\ \|C\| \leq 1\}. \end{aligned} \tag{2.12}$$

Then (2.12) says that B column norms A (see (1.3) or [9, Definition 2.1]), and it follows from [9, Lemma 2.4] that B norms A.
(iii) $\Rightarrow$ (i). We now assume that B norms A. We also assume that A is faithfully represented on a Hilbert space H, since we will require spectral projections for self-adjoint elements of $\mathbb{M}_n(A)$ and these can be obtained in $\mathbb{M}_n(B(H))$.

For a fixed positive integer n, it suffices to prove that $\mathbb{M}_n(A)_B^+ \subseteq \mathbb{M}_n(A)^+$, since the reverse inclusion is Proposition 2.2 (ii). To derive a contradiction, let X lie in $\mathbb{M}_n(A)_B^+ \backslash \mathbb{M}_n(A)^+$. Then X is self-adjoint, and has canonical decomposition $X = X_+ - X_-$ where $X_\pm \geq 0$, $X_+X_- = 0$ and $X_- \neq 0$. Without loss of generality, we assume that $\|X_-\| = 1$. Let

$$f(t) = \begin{cases} 0, & 0 \leq t \leq 1/2, \\ 4t-2, & 1/2 < t < 3/4, \\ 1, & 3/4 \leq t \leq 1. \end{cases}$$

Then f is a continuous function on $[0,1]$, so the functional calculus allows us to define $Y = f(X_-) \in \mathbb{M}_n(A)^+$ and $\|Y\| = 1$. If $P \in \mathbb{M}_n(B(H))$ is the spectral projection of

X_- corresponding to the interval $[1/2,1]$, then P satisfies $P \geq Y$. Given $\varepsilon > 0$, there exists $C \in \mathrm{Col}_n(B)$, $\|C\| \leq 1$, such that $\|Y^{1/2}C\| > (1-\varepsilon^2)^{1/2}$ by [9, Lemma 2.4]. Then $\|C^*YC\| > 1-\varepsilon^2$, and the inequality $C^*PC \geq C^*YC$ implies that $\|C^*PC\| > 1-\varepsilon^2$. Note that the introduction of Y to obtain this last inequality was necessary, since norming subalgebras of a C^*-algebra A need not norm a containing von Neumann algebra.

Choose a unit vector $\xi \in H$ such that

$$\langle PC\xi, C\xi\rangle = \langle C^*PC\xi, \xi\rangle > 1-\varepsilon^2, \tag{2.13}$$

possible because of the norm estimate on C^*PC. Then define η to be $C\xi$ in H^n, the direct sum of n copies of H. The inequality $\|\eta\| \leq 1$ follows from $\|C\| \leq 1$, while $\|P\eta\| > (1-\varepsilon^2)^{1/2}$ is a consequence of (2.13). Orthogonality of $P\eta$ and $(I_n - P)\eta$ then gives

$$\|\eta - P\eta\|^2 = \|\eta\|^2 - \|P\eta\|^2 < \varepsilon^2, \tag{2.14}$$

and so $\|\eta - P\eta\| < \varepsilon$. The functional calculus shows that $X_- \geq P/2$ and so

$$\langle X_-P\eta, P\eta\rangle \geq \langle P\eta, P\eta\rangle/2 > (1-\varepsilon^2)/2. \tag{2.15}$$

Since $X_+P = 0$, (2.15) leads to

$$\langle XP\eta, P\eta\rangle = -\langle X_-P\eta, P\eta\rangle < -(1-\varepsilon^2)/2. \tag{2.16}$$

The operators X, X_+, X_- and P all lie in the abelian von Neumann algebra generated by X, and so $X = PXP + (I_n - P)X(I_n - P)$. From this it follows that

$$\begin{aligned}\langle C^*XC\xi, \xi\rangle = \langle X\eta, \eta\rangle &= \langle XP\eta, P\eta\rangle + \langle X(\eta - P\eta), \eta - P\eta\rangle \\ &< -(1-\varepsilon^2)/2 + \|X\|\|\eta - P\eta\|^2 \\ &< -(1-\varepsilon^2)/2 + \|X\|\varepsilon^2, \end{aligned} \tag{2.17}$$

using (2.14) and (2.16). This last inequality contradicts $C^*XC \geq 0$ when ε is sufficiently small, and so contradicts the definition of X lying in $\mathbb{M}_n(A)^+_B$. Thus (iii) $\Rightarrow$ (i) is proved.

(i) $\Rightarrow$ (ii). We now assume that $\mathbb{M}_n(A)^+_B = \mathbb{M}_n(A)^+$ for all $n \geq 1$. Then Ψ is a unital completely positive map in the usual sense (relative to the cones $\mathbb{M}_n(A)^+$, $n \geq 1$), and is thus completely contractive. Hence it is sufficient to prove that $\|\Psi_n(X)\| \geq \|X\|$ for any given $X \in \mathbb{M}_n(A)$. Let $\pi\colon\ A \to B(H)$ be a faithful $*$-representation of A, so that $\pi_n\colon\ \mathbb{M}_n(A) \to B(H^n)$ is also faithful. Given $\varepsilon > 0$, choose a unit vector $\xi = (\xi_1, \ldots, \xi_n)^t \in H^n$ such that $\|\pi_n(X)\xi\| > \|X\| - \varepsilon$ and let $\pi_n(X)\xi$ be written as $(\eta_1, \ldots, \eta_n)^t$. Then let p be the finite rank projection onto $\mathrm{span}\{\xi_i, \eta_j\colon\ 1 \le i, j \le n\} \subseteq H$. We identify $pB(H)p$ with a finite dimensional matrix algebra $\mathbb{M}_k$, and define a completely positive unital map $\omega\colon\ A \to \mathbb{M}_k$ by $\omega(a) = p\pi(a)p$ for $a \in A$. It is clear from the choice of p that $\omega_n(X)\xi = \pi_n(X)\xi$ and, since ω is one of the summands in Ψ, we obtain $\|\Psi_n(X)\| > \|X\| - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the proof is complete. □

Corollary 2.4. *Let A and D be unital C^*-algebras with a common unital C^*-subalgebra B. If $\phi\colon\ D \to A$ is a positive B-bimodule map and B norms A, then ϕ is completely positive.*

Proof. Let $X \in \mathbb{M}_n(D)^+$. Since B norms A, the cones $\mathbb{M}_n(A)^+_B$ and $\mathbb{M}_n(A)^+$ coincide, by Theorem 2.3. For each $C \in \mathrm{Col}_n(B)$, the B-bimodularity of ϕ implies that

$$C^*\phi_n(X)C = \phi(C^*XC) \ge 0. \tag{2.18}$$

Thus $\phi_n(X) \in \mathbb{M}_n(A)^+_B$, so $\phi_n(X) \ge 0$ and ϕ is completely positive. □

Remark 2.5. (i) We commented above that our results do not require A and B to be unital. This is justified by the following two observations. The first is that if B norms A then it also norms the unitization $\tilde{A}$. Suppose that $X \in \mathbb{M}_n(A)$, $\Lambda \in \mathbb{M}_n(\mathbb{C})$ and $Y = X + \Lambda \in \mathbb{M}_n(\tilde{A})$ has norm 1. Fix a norm 1 approximate identity (e_α) for A. Then $Y_\alpha := (e_\alpha \otimes I_n)Y \in \mathbb{M}_n(A)$ and $\|Y_\alpha\| \to 1$ as $\alpha \to \infty$. Since B column norms A, by [9, Lemma 2.4], there exist $C_\alpha \in \mathrm{Col}_n(B)$, $\|C_\alpha\| \le 1$, such that $\lim\limits_\alpha \|Y_\alpha C_\alpha\| = 1$. But then $\lim\limits_\alpha \|YC_\alpha\| = 1$, so B norms $\tilde{A}$. Moreover $\widetilde{B}$ then norms $\tilde{A}$. The second observation is that a bounded positive map $\phi\colon\ D \to A$ extends to a positive map on the unitizations by letting $\phi(1)$ be $\|\phi\|1$. If ϕ is B-bimodular then it is also $\widetilde{B}$-bimodular, so that Corollary 2.4 can be applied.

(ii) If B is a non-norming subalgebra of A, then a consequence of Theorem 2.3 is that there must exist elements of $\mathbb{M}_n(A)^+_B$ which are not positive in the usual sense. A concrete example can be obtained from [9, Proposition 5.2] as follows. To fit the notation of this paper, we let A be $L(\mathbb{F}_2)$ and B be the abelian von Neumann subalgebra generated by a, where a and b denote the generators of the free group $\mathbb{F}_2$. For each $n \ge 1$, let

$$Y_n = (bab/\sqrt{n}, b^2ab^2/\sqrt{n}, \ldots, b^nab^n/\sqrt{n})^t \in \mathrm{Col}_n(A).$$

It was shown in [9, Proposition 5.2] that $\|Y_n^*C\| \leq \sqrt{12/n}$ for any $C \in \mathrm{Col}_n(B)$ with $\|C\| \leq 1$. Then $\rho_n(Y_nY_n^*) \leq 12/n$, and so $12I_n - nY_nY_n^* \in \mathbb{M}_n(A)_B^+$. However, $Y_nY_n^*$ is a projection since $Y_n^*Y_n = 1$, and so the inequality $12I_n - nY_nY_n^* \geq 0$ fails for $n \geq 13$.

(iii) A new example of a norming subalgebra is the following. Let N be a II_1 factor and let G be a discrete group acting on N by outer automorphisms, the action denoted by α_g. Then N norms the crossed product $N \rtimes_\alpha G$, as we now explain. The standard basis for $\ell^2(G)$ is $\{\delta_g \colon g \in G\}$, and we denote by $e_{g,h}$ the rank one matrix units in $B(\ell^2(G))$ which satisfy $e_{g,h}\delta_h = \delta_g$ for $g, h \in G$. When the crossed product is represented on $L^2(N) \otimes \ell^2(G)$, N is represented by diagonal matrices of the form

$$\pi(x) = \sum_{g \in G} \alpha_{g^{-1}}(x) \otimes e_{g,g},$$

where $x \in N$ and the sum converges strongly to a bounded operator (see the discussion of crossed products in [11]). Then $N \rtimes_\alpha G \subseteq N \overline{\otimes} B(\ell^2(G))$, so it suffices to show that $\pi(N)$ norms this latter algebra. For each finite subset F of G, let p_F be the projection onto $\mathrm{span}\{\delta_g \colon g \in F\}$. Note that the compression of $N \overline{\otimes} B(\ell^2(G))$ by $1 \otimes p_F$ has the form $\mathbb{M}_k(N)$ for some integer k equal to the cardinality of F, while the corresponding compression of each operator $\pi(x) \in \pi(N)$ is a diagonal matrix of the form $\sum_{g \in F} \alpha_{g^{-1}}(x) \otimes e_{g,g}$. It is easy to see that $\mathbb{M}_k(N)$ is a finitely generated left module over $(1 \otimes p_F)\pi(N)(1 \otimes p_F)$, giving us a finite index inclusion of II_1 factors, by [5]. Thus the smaller algebra norms the larger one, by [9, Corollary 3.3]. Now consider $X \in \mathbb{M}_n(N \overline{\otimes} B(\ell^2(G)))$. The net of projections $\{1 \otimes p_F \otimes I_n\}_F$ from $\mathbb{M}_n(N \overline{\otimes} B(\ell^2(G))) \cong N \overline{\otimes} B(\ell^2(G)) \otimes \mathbb{M}_n$ commutes with $\pi(N) \otimes I_n$ and also converges strongly to the identity. Given $\varepsilon > 0$, first choose F so that $\|(1 \otimes p_F \otimes I_n)X(1 \otimes p_F \otimes I_n)\| > \|X\| - \varepsilon$. Using the finite index inclusion above, we can then choose $C \in \mathrm{Col}_n((1 \otimes p_F)\pi(N))$ such that $\|(1 \otimes p_F \otimes I_n)X(1 \otimes p_F \otimes I_n)C\| > \|X\| - \varepsilon$. Since $C(1 \otimes p_F) = (1 \otimes p_F \otimes I_n)C$, the last inequality implies that $\|XC\| > \|X\| - \varepsilon$, showing that N norms the crossed product $N \rtimes_\alpha G$. It should be noted that this result was proved in [9] for the special case when the factor N is hyperfinite. $\square$

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