PERTURBATIONS OF C*-ALGEBRAIC INVARIANTS

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Abstract. Kadison and Kastler introduced a metric on the set of all C*-algebras on a fixed Hilbert space. In this paper structural properties of C*-algebras which are close in this metric are examined. Our main result is that the property of having a positive answer to Kadison's similarity problem transfers to close C*-algebras. In establishing this result we answer questions about closeness of commutants and tensor products when one algebra satisfies the similarity property. We also examine K-theory and traces of close C*-algebras, showing that sufficiently close algebras have isomorphic Elliott invariants when one algebra has the similarity property.

1. Introduction

In [23], Kadison and Kastler introduced the study of uniform perturbations of operator algebras. They considered a fixed C*-algebra C and equipped the set of all C*-subalgebras of C with a metric arising from Hausdorff distance between the unit balls of these subalgebras. In general terms, two C*-subalgebras A and B of C are close if elements from the unit ball of A can be approximated well in the unit ball of B, and vice versa. A precise definition will be given in Section 2 below. Kadison and Kastler conjectured that sufficiently close subalgebras must be isomorphic and that this isomorphism should be spatially implemented when C is faithfully represented on some Hilbert space. In the 1970's and 1980's various cases of this conjecture were established: [39] resolves the problem when one algebra is an injective von Neumann algebra (see also [9, 12]); [12] solves the problem when one algebra is separable and AF (see also [32]); [33] examines the situation for continuous trace algebras and [25] looks at extensions of some of the cases from [32, 12, 33]; and [21] examines sub-homogeneous C*-algebras. Recent progress has been made in [14] which gives a positive answer to the question when one algebra is separable and nuclear. In full generality [6] provides counterexamples to the conjecture. These counterexamples are non-separable C*-algebras and the problem remains open when A, B are von Neumann algebras or separable C*-algebras. In the absence of a general isomorphism result, a naturally arising question is whether close C*-algebras must share the same invariants. This will be a continuing theme of the paper. In this introduction we discuss our results in qualitative terms. Precise estimates will be given in the main text.

The principal objective of this article is to examine connections between the theory of perturbations and Kadison's similarity problem. Kadison's similarity problem was set out in [22] and asks whether every bounded unital representation from a unital C*-algebra A into $\mathbb{B}(\mathcal{H})$ is similar to a *-representation of A on $\mathcal{H}$. In [19], Haagerup gave a positive answer to this question for cyclic representations and showed that a bounded representation

Date: May 27, 2009.

π of a C*-algebra on $\mathbb{B}(\mathcal{H})$ is similar to a *-representation if and only if π is completely bounded. We say that A has the *similarity property* if the similarity problem has a positive answer for A. In [28], Kirchberg showed that A has the similarity property if and only if the *derivation problem* also has a positive answer for A, that is given a *-representation $\pi : A \to \mathbb{B}(\mathcal{H})$ and a bounded π-derivation $\delta : A \to \mathbb{B}(\mathcal{H})$, there is some $x \in \mathbb{B}(\mathcal{H})$ such that $\delta(a) = [x, \pi(a)] = x\pi(a) - \pi(a)x$ for all $a \in A$. Such derivations are called *inner*. There is another equivalent formulation that we now discuss.

Motivated by the similarity problem, Pisier introduced the notion of the *length* $\ell(A)$ of an operator algebra A in [34] and examined its properties in [35, 38]. This integer arises from the ability to write matrices over A as products of bounded length, where the constituent factors alternate between scalar matrices and diagonal matrices over A (the precise details are given in Definition 2.7). If such decompositions do not exist then $\ell(A) = \infty$, although no examples of this are currently known. An easy consequence of finite length is that all bounded homomorphisms of A into any $\mathbb{B}(\mathcal{H})$ are completely bounded, which solves the similarity problem for such algebras, and is indeed equivalent to it. Remarkably, nuclearity is characterised by $\ell(A) \leq 2$ [38], while all C*-algebras lacking tracial states have length at most 3. These results are surveyed in Pisier's monograph [36]. For our purposes, the finite length property will be a convenient formulation of the similarity problem, and we will be able to show that this property transfers to nearby C*-algebras. This also uses a more technical characterisation called the distance property, described below in Definition 2.4.

There are two open questions concerning the behaviour of the distance between algebras under standard constructions which arise from [12] and are connected to the similarity property. Given two C*-algebras A and B on some Hilbert space $\mathcal{H}$, with A and B close, must the commutants A' and B' be close? Under the same hypothesis, must the algebras $A \otimes E$ and $B \otimes E$ be close (as subalgebras of $\mathbb{B}(\mathcal{H}) \otimes E$) for any nuclear C*-algebra E? The work of [12] gives positive answers to these questions provided, in today's language, both A and B satisfy the similarity property. In section 4 we show that if A has the similarity property and B is sufficiently close to A, then B also has the similarity property (with constants depending on the similarity length and length constant). To do this, we initially answer the first question above regarding closeness of commutants when only one algebra has the similarity property. As a consequence, we also obtain a positive answer when one algebra has the similarity property.

Khoshkam examined the K-theory of close C*-algebras in [26], showing that there is a natural isomorphism between the ordered K-theories of sufficiently close nuclear C*-algebras. The key ingredient required for [26] was that if A and B are close and nuclear, then the matrix algebras $\mathbb{M}_n(A)$ and $\mathbb{M}_n(B)$ are uniformly close (so that the distance between these algebras is bounded independently of n). Khoshkam's isomorphism can be defined whenever this condition holds. In particular, we show in Corollary 5.3 that sufficiently close C*-algebras have isomorphic ordered K-theories provided that one algebra has the similarity property. The distance we require depends on the similarity length and constant of this algebra.

Khoshkam's work opens the possibility of using results from Elliott's classification programme to address perturbation questions. We discuss this topic in sections 5 and 6, with the objective of showing that invariants and properties used in the classification programme transfer to sufficiently close algebras. In Lemma 5.4 we construct an affine isomorphism between the traces on sufficiently close C*-algebras. When one algebra has the similarity property, this isomorphism and the isomorphism between K-theories from Corollary 5.3 respect the natural pairing between the K_0 and the traces. In particular there is an isomorphism between the Elliott invariants of sufficiently close nuclear C*-algebras.

Section 6 gives an example of how the classification programme can be used to quickly give perturbation results. We use Kirchberg and C. Phillips' classification of Kirchberg algebras (simple, separable, purely infinite and nuclear C*-algebras) [29, 27] to show that any C*-algebra satisfying the UCT which is sufficiently close to a Kirchberg algebra with the UCT is necessarily isomorphic to it. Given earlier results, it suffices to examine how the property of being purely infinite behaves under perturbations and we show that a C*-algebra that is close to a simple and purely infinite one is also purely infinite. We do this by showing that the property of being real rank zero also transfers to sufficiently close algebras. As in the previous section, we establish these results in as much generality as possible, not just in the nuclear setting.

The paper is structured as follows. In section 2 we recall the precise definition of the metric introduced by Kadison and Kastler in [23] and give a detailed account of how the similarity property gives rise to results in the theory of perturbations. In section 3 we establish some technical preliminaries required in our later work. In particular, we examine the behaviour of the centre valued trace and coupling constants in the context of close von Neumann algebras. These play important technical roles in section 4, where we establish our main result that algebras close to those of finite length again have finite length and discuss its consequences. Section 5 examines the K-theory and traces of close C*-algebras, while Section 6 contains our example of how the classification programme gives rise to perturbation results. The paper ends in Section 7 with a brief collection of open problems.

2. Similarity Length and Perturbations

This section fills in the quantitive versions of the definitions from the introduction and examines the connections between perturbation theory and the similarity problem from the literature. We begin by recalling the definition of the metric d on the collection of all C*-subalgebras of a fixed C*-algebra from [23] and the notion of a near inclusion from [12].

Definition 2.1. Let A and B be C*-subalgebras of some C*-algebra C. Define $d(A,B)$ to be the infimum of all $\gamma > 0$ with the property that given x in the unit ball of A or B, there exists y in the unit ball of the other algebra with $\|x-y\| < \gamma$.

Definition 2.2. Let A and B be C*-subalgebras of some C*-algebra C and let $\gamma > 0$. Write $A \subseteq_\gamma B$ if for each x in the unit ball of A there is $y \in B$ with $\|x-y\| \leq \gamma$. Write $A \subset_\gamma B$ if $A \subseteq_{\gamma'} B$ for some $\gamma' < \gamma$.

Note that Definition 2.2 does not require that y lie in the unit ball of B. This means that the notion of distance between two C*-subalgebras A and B defined by considering the infimum of all γ for which $A \subseteq_\gamma B$ and $B \subseteq_\gamma A$ does not obviously satisfy the triangle inequality. The proposition below sets out the relationships between the concepts of Definitions 2.1 and 2.2. All are immediate consequences of the definitions and so we omit their proofs.

Proposition 2.3. *Let A, B and C be C*-subalgebras of some C*-algebra E.*

(i) *If $A \subseteq_\gamma B$ and $B \subseteq_\delta C$, then $A \subseteq_{\gamma+\delta(1+\gamma)} C$.*
(ii) *If $d(A,B) \leq \gamma$, then $A \subseteq_\gamma B$ and $B \subseteq_\gamma A$.*
(iii) *If $A \subseteq_\gamma B$ and $B \subseteq_\gamma A$, then $d(A,B) \leq 2\gamma$.*

In general it is unknown whether a near inclusion $A \subset_\gamma B$ of two C*-algebras on some Hilbert space $\mathcal{H}$ induces a near inclusion $B' \subset_{L\gamma} A'$ between the commutants for a suitably chosen constant L. Based on [10], a distance property D_k was introduced in [12, Definition 2.2] which allows such a deduction to be made. Subsequently it was shown in [11, 13] that a C*-algebra has such a distance property if, and only if, for every representation $\pi\colon A \to \mathbb{B}(\mathcal{H})$, every derivation from $\pi(A)$ into $\mathbb{B}(\mathcal{H})$ is inner. We now review this connection.

Let $A \subset \mathbb{B}(\mathcal{H})$ be a C*-algebra. Given $x \in \mathbb{B}(\mathcal{H})$ we can define a derivation $\mathrm{ad}(x)|_A$ on A by $\mathrm{ad}(x)|_A(a) = [x,a] = xa - ax$. In [1] Arveson computed the distance between an operator in $\mathbb{B}(\mathcal{H})$ to a nest-algebra. In our context, this described the distance to A' by

$$d(x, A') = \frac{1}{2}\|\mathrm{ad}(x)|_A\|_{\mathrm{cb}}, \quad x \in \mathbb{B}(\mathcal{H}), \tag{2.1}$$

see also [10, Proposition 2.1]. Theorem 3.2 of [13] shows that every derivation of A into $\mathbb{B}(\mathcal{H})$ is inner (i.e. of the form $\mathrm{ad}(x)|_A$ for some $x \in \mathbb{B}(\mathcal{H})$) if and only if there is some $k > 0$ such that

$$d(x, A') \leq k\|\mathrm{ad}(x)|_A\|, \quad x \in \mathbb{B}(\mathcal{H}). \tag{2.2}$$

Using the distance formula in (2.1), it follows that every derivation of A into $\mathbb{B}(\mathcal{H})$ is inner if and only if there is some $k > 0$ such that

$$\|\mathrm{ad}(x)|_A\|_{\mathrm{cb}} \leq 2k\|\mathrm{ad}(x)|_A\|, \quad x \in \mathbb{B}(\mathcal{H}). \tag{2.3}$$

We formalise these concepts in the following definitions, the latter being [12, Definition 2.2].

Definition 2.4. Let $k > 0$ and let A be a C*-algebra. A representation π of A on $\mathcal{H}$ has the *local distance property* LD_k if

$$d(x, \pi(A)') \leq k\|\mathrm{ad}(x)|_{\pi(A)}\|, \quad x \in \mathbb{B}(\mathcal{H}). \tag{2.4}$$

If every representation of A has the local distance property LD_k, then A has the *distance property* D_k. □

By the preceding discussion, A has the distance property D_k for some $k > 0$ if and only if the derivation problem has a positive answer for A. Furthermore a representation π of A on $\mathcal{H}$ has the local distance property LD_k for some $k > 0$ if and only if every π-derivation is inner. A near inclusion $A \subseteq_\gamma B$ of C*-algebras on a Hilbert space $\mathcal{H}$ induces a near inclusion of B' into A' when A has the local distance property on $\mathcal{H}$. This is easily established in the proposition below. The proof is extracted from the proof of [12, Theorem 3.1].

Proposition 2.5. *Let $A, B \subset \mathbb{B}(\mathcal{H})$ be C*-algebras with $A \subseteq_\gamma B$. If $A \subseteq \mathbb{B}(\mathcal{H})$ has the local distance property LD_k, then*

$$B' \subseteq_{2k\gamma} A'. \tag{2.5}$$

Proof. Fix $x \in B'$. For a in the unit ball of A, there is some $b \in B$ with $\|a-b\| \leq \gamma$. Then $\|[x,a]\| = \|[x,(a-b)]\| \leq 2\|x\|\gamma$. Property LD_k gives $d(x,A') \leq 2k\|x\|\gamma$ and hence the near inclusion (2.5). □

Corollary 5.4 of [13] shows that cyclic representations of C*-algebras have the local distance property LD_{12} and hence solves the derivation problem for cyclic representations. We need to consider representations with a finite set of cyclic vectors in Section 4. The next proposition is an easy extension of [13, Corollary 5.4] to this case.

Proposition 2.6. *Let π be a representation of a C*-algebra A on a Hilbert space $\mathcal{H}$. If $\pi(A)$ has a finite cyclic set of m vectors, then π has the local distance property LD_{12m}.*

Proof. Let $\xi_1,\dots,\xi_m$ in $\mathcal{H}$ be a cyclic set for $\pi(A)$. Then $(\xi_1,\dots,\xi_m)^T \in \mathcal{H}\otimes\mathbb{C}^m$ is a cyclic vector for $\pi(A)\otimes\mathbb{M}_m$. Fix $y \in \mathbb{B}(\mathcal{H})$ and let $\mathrm{ad}(y)|_{\pi(A)}$ be the associated derivation on $\pi(A)$. Then $\mathrm{ad}(y\otimes I_{\mathbb{M}_m})|_{\pi(A)\otimes\mathbb{M}_m}$ satisfies

$$d(y\otimes I_{\mathbb{M}_m}, \pi(A)'\otimes I_{\mathbb{M}_m}) \leq 12\,\|\mathrm{ad}(y\otimes I_{\mathbb{M}_m})|_{\pi(A)\otimes\mathbb{M}_m}\| \tag{2.6}$$

using [13, Cor. 5.4], which is valid for algebras with cyclic vectors. Since $\mathrm{ad}(y\otimes I_{\mathbb{M}_m}) = \mathrm{ad}(y)\otimes id_{\mathbb{M}_m}$, the estimate

$$d(y,\pi(A)') \leq 12\,m\,\|\mathrm{ad}(y)|_{\pi(A)}\| \tag{2.7}$$

follows from (2.6) and the general inequality $\|\phi\otimes id_{\mathbb{M}_m}\| \leq m\|\phi\|$ for bounded maps ϕ between C*-algebras, which is [30, Exercise 3.10]. This shows that we have property LD_{12m}. □

We now turn to Pisier's notion of the length of an operator algebra, [34].

Definition 2.7. Let A be a C*-algebra faithfully represented on $\mathbb{B}(\mathcal{H})$. Say that A has length at most ℓ if there exists a constant $K>0$ such that for each $n\in\mathbb{N}$ and $x\in\mathbb{M}_n(A)$, there is an integer N, diagonal matrices $d_1,\dots,d_\ell \in \mathbb{M}_N(A)$ and scalar matrices $\lambda_0 \in \mathbb{M}_{n,N}, \lambda_1,\dots,\lambda_{\ell-1}\in\mathbb{M}_N, \lambda_\ell\in\mathbb{M}_{N,n}$ such that

$$x = \lambda_0 d_1 \lambda_1 d_1 \lambda_2 \dots \lambda_{\ell-1} d_\ell \lambda_\ell \tag{2.8}$$

and

$$\prod_{i=0}^{\ell}\|\lambda_i\|\prod_{i=1}^{\ell}\|d_i\| \leq K\|x\|. \tag{2.9}$$

In this case we say that A has *length constant at most* K. □

It is easy to see that this definition does not depend on the choice of the faithful representation of A, but phrasing it in this fashion ensures that we do not have to distinguish between the unital and non-unital cases. In [19], it is shown that a unital C*-algebra A has the similarity property if and only if there exists some $d\geq 1$ and positive constant K' such that

$$\|u\|_{\mathrm{cb}} \leq K'\|u\|^d, \tag{2.10}$$

for all bounded unital homomorphisms $u: A\to\mathbb{B}(\mathcal{H})$. In [34], Pisier shows that this happens if and only if A has finite length. Furthermore, the infimum over all d for which there is a constant K' so that (2.10) holds is precisely the length of A. One direction is easy to see: if A has length at most ℓ and length constant K, then (2.10) holds with $K' = K$ and $d = \ell$. Note too that while (2.10) implies that A has length at most $\lfloor d \rfloor$, it does not give us information about the length constant of A. For more information on this topic we refer the reader to Pisier's monograph on similarity problems [36] and his operator space text [37, Chapter 27].

The next two propositions give quantified versions of the equivalence between the properties of satisfying the derivation problem and having finite length. The first can be found in [34, Section 4 (in particular Remark 4.7)], while the second is the derivation version of the calculation [36, Proposition 10.6]. This is well known but we include the proof for completeness.

Proposition 2.8. *Let A have property D_k for some k. Then the length of A is at most $\lfloor 2k \rfloor$.*

Proposition 2.9. *Let A be a C^*-algebra with length at most ℓ and length constant at most K. Then A has property D_k for $k = K\ell/2$.*

Proof. Suppose that we are given a representation $\pi : A \to \mathbb{B}(\mathcal{H})$. Fix $y \in \mathbb{B}(\mathcal{H})$. Given $n \in \mathbb{N}$ and an operator $x \in \mathbb{M}_n(\pi(A))$, let N, $\lambda_0, \dots, \lambda_\ell$ and $d_1, \dots, d_\ell$ be as in Definition 2.7. Then $\mathrm{ad}(y) \otimes \mathrm{id}_{\mathbb{M}_n} = \mathrm{ad}(y \otimes I_{\mathbb{M}_n})$. Using the facts that $(y \otimes I_{\mathbb{M}_n})\lambda_0 = \lambda_0(y \otimes I_{\mathbb{M}_N})$, $(y \otimes I_{\mathbb{M}_N})\lambda_\ell = \lambda_\ell(y \otimes I_{\mathbb{M}_n})$ and that $y \otimes I_{\mathbb{M}_N}$ commutes with each $\lambda_1, \dots, \lambda_{\ell-1}$, we can apply Leibnitz's rule to obtain

$$\mathrm{ad}(y \otimes I_{\mathbb{M}_n})(x) = \sum_{i=1}^{l} \lambda_0 d_1 \lambda_1 \dots \lambda_i [(y \otimes I_{\mathbb{M}_N}), d_i] \lambda_{i+1} d_{i+1} \dots \lambda_{\ell-1} d_\ell \lambda_\ell. \tag{2.11}$$

Therefore

$$\begin{aligned} &\|\mathrm{ad}(y) \otimes \mathrm{id}_{\mathbb{M}_n}(x)\| \\ &\qquad \leq \sum_{i=1}^{\ell} \|\lambda_0\| \|d_1\| \|\lambda_1\| \dots \|\lambda_i\| \|[(y \otimes I_{\mathbb{M}_N}), d_i]\| \|\lambda_{i+1}\| \|d_{i+1}\| \dots \|\lambda_{\ell-1}\| \|d_\ell\| \|\lambda_\ell\| \\ &\qquad \leq \ell \|\mathrm{ad}(y)|_{\pi(A)}\| \prod_{i=0}^{\ell} \|\lambda_i\| \prod_{i=1}^{\ell} \|d_i\| \leq K\ell \|\mathrm{ad}(y)|_{\pi(A)}\| \|x\|. \qquad (2.12) \end{aligned}$$

The result follows from (2.1). □

We can use the factorisations of Definition 2.7 to lift near inclusions $A \subseteq_\gamma B$ to near inclusions $A \otimes \mathbb{M}_n \subseteq_{L\gamma} A \otimes \mathbb{M}_n$ when A has finite length. The next proposition has been known to Pisier for some time and is the similarity length version of [12, Theorem 3.1] which obtains an analogous result for algebras using property D_k.

Proposition 2.10. *Let $A, B \subset \mathbb{B}(\mathcal{H})$ be C^*-algebras with $A \subseteq_\gamma B$ for some $\gamma > 0$. Suppose that A has length at most ℓ and length constant at most K. Then $A \otimes \mathbb{M}_n \subseteq_\mu B \otimes \mathbb{M}_n$ for all $n \in \mathbb{N}$, where μ is given by*

$$\mu = K((1+\gamma)^\ell - 1). \tag{2.13}$$

Proof. Fix $n \in \mathbb{N}$ and identify $A \otimes \mathbb{M}_n$ and $B \otimes \mathbb{M}_n$ with $\mathbb{M}_n(A)$ and $\mathbb{M}_n(B)$ respectively. Take x in the unit ball of $\mathbb{M}_n(A)$ and find scalar matrices $\lambda_0, \ldots, \lambda_\ell$ and diagonal matrices $d_1, \ldots, d_\ell$ as in Definition 2.7. For each diagonal matrix $d_i \in \mathbb{M}_N(A)$, we can apply the near inclusion $A \subseteq_\gamma B$ to each entry to produce a diagonal matrix $e_i \in \mathbb{M}_N(B)$ with $\|d_i - e_i\| \leq \gamma\|d_i\|$. Then

$$y = \lambda_0 e_1 \lambda_1 e_1 \lambda_2 \ldots \lambda_{\ell-1} e_\ell \lambda_\ell \tag{2.14}$$

defines an element of $\mathbb{M}_n(B)$ and an inductive calculation gives

$$\|x-y\| \leq K\Big(\gamma + \gamma(1+\gamma) + \gamma(1+\gamma)^2 + \cdots + \gamma(1+\gamma)^{\ell-1}\Big) = K\Big((1+\gamma)^\ell - 1\Big) = \mu, \tag{2.15}$$

which completes the proof. □

Remark 2.11. There is also a version of Proposition 2.10 for finite sets which we state here for use in [14]. Suppose that $A, B \subset \mathbb{B}(\mathcal{H})$ are C*-algebras and that A has length at most ℓ and length constant K. Given any $n \in \mathbb{N}$ and finite set X in the unit ball of $A \otimes \mathbb{M}_n$, there exists a finite set Y in the unit ball of A such that if $Y \subseteq_\gamma B$ for some $\gamma > 0$ (by which we mean that for each $y \in Y$, there is some $b \in B$ with $\|y - b\| \leq \gamma$), then $X \subseteq_\mu B \otimes \mathbb{M}_n$, where $\mu = K((1+\gamma)^\ell - 1)$. Note that the set Y consists of all the entries of the diagonal matrices d_i in the proof of Proposition 2.10 and so depends only on X (and not on B or the value of γ).

The next corollary follows from Proposition 2.10 using the completely positive approximation property for nuclear C*-algebras [7]. The proof is identical to the deduction of Theorem 3.1 of [12] from equation (3) on page 253 of [12] and so is omitted.

Corollary 2.12. *Let $A, B \subset \mathbb{B}(\mathcal{H})$ be C*-algebras with $A \subseteq_\gamma B$ for some $\gamma > 0$. Suppose that A has length at most ℓ and length constant at most K. Given any nuclear C*-algebra E, we have $A \otimes E \subseteq_\mu B \otimes E$ inside $\mathbb{B}(\mathcal{H}) \otimes E$, where $\mu = K\Big((1+\gamma)^\ell - 1\Big)$.*

Every nuclear C*-algebra has length 2 with length constant 1 (the similarity property for nuclear C*-algebras can be found in [5]) and property D_1 [10]. In this case the μ of Proposition 2.10 and Corollary 2.12 is given by $\mu = 2\gamma + \gamma^2$ and the corollary gives better estimates than the original version [12, Theorem 3.1], which uses property D_1 to lift near inclusions $A \subset_\gamma B$ to inclusions $A \otimes E \subset_{6\gamma} B \otimes E$, when A and E are nuclear.

3. Technical Preliminaries

In this section we collect various technical results from the literature as well as establish some further preliminaries. We start with some standard estimates which we will use repeatedly.

Proposition 3.1. *Let A and B be C*-subalgebras of a C*-algebra C.*

- (i) *Suppose that $A \subset_\gamma B$ for some $\gamma < 1/2$. Given a projection $p \in A$, there exists a projection $q \in B$ with $\|p - q\| < 2\gamma$.*
- (ii) *Suppose that A and B are unital and share the same unit. Suppose that $\gamma < 1$ and that $A \subset_\gamma B$. Then the following hold.*
  - (a) *Given a unitary $u \in A$, there exists a unitary $v \in B$ with $\|u - v\| < \sqrt{2}\gamma$.*
  - (b) *Given a projection $p \in A$, there exists a projection $q \in B$ with $\|p - q\| < \gamma/\sqrt{2}$.*

Proof. (i) This is Lemma 2.1 of [8]. Although the result in [8] is stated for von Neumann algebras, the proof works for C*-algebras.
(ii) Both (a) and (b) are slightly weaker statements than those in [26, Lemma 1.10]. They follow from noting that the $\alpha(t)$ of [26, 1.9] has $\alpha(t) \leq \sqrt{2}t$ for $0 \leq t < 1$. □

As seen in the previous proposition, better constants are often obtained when the C*-algebras we consider are both unital and share the same unit. One way of reducing to this case is to simultaneously unitise all the algebras involved. The next proposition offers another solution to this problem when one algebra is already unital.

Proposition 3.2. *Let A and B be C*-subalgebras of a unital C*-algebra C, and fix γ satisfying $d(A,B) < \gamma < 1/4$. Then A is unital if and only if B is unital. Furthermore, in this case there exists a unitary $u \in C$ with $\|u - 1_C\| < 2\sqrt{2}\gamma$ and $u1_Au^* = 1_B$.*

Proof. Suppose that A is unital, so that its unit, 1_A, is a projection in C. By Proposition 3.1 (i) there exists a projection $q \in B$ with $\|1_A - q\| < 2\gamma$. We will show that q is the unit of B. Take b in the unit ball of B and find a in the unit ball of A with $\|a - b\| < \gamma$. Then

$$\begin{aligned} \|qb - b\| &\leq \|q(b-a)\| + \|(q - 1_A)a\| + \|a - b\| \\ &\leq \gamma + 2\gamma + \gamma = 4\gamma < 1. \end{aligned} \tag{3.1}$$

Now let (e_α) be an approximate identity for B. Working in B^{**}, we have $e_\alpha \nearrow 1_{B^{**}}$ so that taking a weak*-limit in the previous estimate gives

$$\|1_{B^{**}} - q\| = \|1_{B^{**}}q - 1_{B^{**}}\| \leq 4\gamma < 1. \tag{3.2}$$

It follows that the projection $1_{B^{**}} - q$ is zero and so $q = 1_{B^{**}}$. Accordingly B is unital with unit $q = 1_B$. Since $\|1_A - q\| < 2\gamma < 1$, there exists a unitary $u \in C$ with $\|u - 1_C\| < \sqrt{2}\|1_A - 1_B\| = 2\sqrt{2}\gamma$ and $u1_Au^* = 1_B$. □

Section 5 of [12] shows that, given a sufficiently close inclusion $Q \subseteq_\gamma B$ of C*-algebras with Q finite dimensional, there exists a partial isometry close to I_Q with $vQv^* \subseteq B$ and with all the constants independent of the structure of Q. When Q has small dimension, better constants can be achieved using elementary techniques going back to the work of Murray and von Neumann on hyperfinite factors, subsequently employed by Glimm [18] and Bratteli [4]. The proposition below records the constants required when Q is a copy of the 2×2 matrices. The proof is omitted.

Proposition 3.3. *Let Q, B be C*-subalgebras of a unital C*-algebra C which contain I_C. Suppose that Q is *-isomorphic to a copy of the 2×2 matrices and $Q \subset_\gamma B$ for some $\gamma < 1/3\sqrt{2}$. Then there exists a unitary $v \in C^*(B,Q)$ with $vQv^* \subseteq B$ and*

$$\|v - I_C\| < (3\sqrt{2} + 1)\gamma. \tag{3.3}$$

In their pioneering article [23], Kadison and Kastler showed that the type decomposition of a von Neumann algebra is stable under small perturbations. Many of our subsequent arguments use a type decomposition approach, as we handle the finite type I, the type II_1 and the infinite type cases separately. The constants we can achieve will depend on the constants appearing in the stability of the type decomposition. These constants can now be improved using techniques which were not available in [23]. We shall demonstrate this below in the cases we need. We will also collect some reduction arguments for later use.

Our first lemma uses results from [8] and shows that, when considering close von Neumann algebras, we can reduce to the case where they have common centres.

Lemma 3.4. *Let $M, N \subset \mathbb{B}(\mathcal{H})$ be von Neumann algebras whose centres are denoted $Z(M)$ and $Z(N)$ respectively. Suppose that $d(M,N) \leq \gamma$ for some $\gamma < 1/6$. Then there exists a unitary $u \in (Z(M) \cup Z(N))''$ such that $uZ(M)u^* = Z(uMu^*) = Z(N)$ and*

$$\|u - I_{\mathcal{H}}\| \leq 2^{5/2}\gamma(1 + (1 - 16\gamma^2)^{1/2})^{-1/2} \leq 5\gamma. \tag{3.4}$$

In particular $d(uMu^*, N) \leq 11\gamma$ and uMu^* and N have common centre.

Proof. As $\gamma < 1/6$, Lemma 2.2 of [8] shows that the Hausdorff distance between the projections in $Z(M)$ and the projections in $Z(N)$ is at most 2γ. As $2\gamma < 1/2$, the result follows from Theorem 3.2 of [8]. For $\gamma < 1/6$, direct computation gives the second inequality of (3.4). The estimate

$$d(uMu^*, N) \leq d(uMu^*, M) + d(M, N) \leq 2\|u - I_{\mathcal{H}}\| + \gamma \leq 11\gamma \tag{3.5}$$

follows. □

Once two von Neumann algebras have the same centre, we can directly compare their type decompositions.

Lemma 3.5. *Let $M, N \subseteq \mathbb{B}(\mathcal{H})$ be von Neumann algebras with a common centre Z, and suppose that $d(M,N) < 1/10$. If $z_1, z_2, z_3 \in Z$ are central projections so that*

$$M = Mz_1 \oplus Mz_2 \oplus Mz_3 \tag{3.6}$$

is the decomposition of M into respectively the finite type I*, type* II_1 *and infinite parts, then*

$$N = Nz_1 \oplus Nz_2 \oplus Nz_3 \tag{3.7}$$

is the corresponding decomposition for N.

Proof. Let $N = N\tilde{z}_1 \oplus N\tilde{z}_2 \oplus N\tilde{z}_3$ be the corresponding decomposition for N. We first show that $z_3 = \tilde{z}_3$. If this is not the case then, without loss of generality, there is a non-zero central projection z such that Mz is finite and Nz is infinite. By cutting by z, we may then assume that M is finite and N is infinite. Let $v \in N$ be an isometry which is not a unitary, and choose $x \in M$ with $\|x - v\| < 10^{-1}$ and $\|x\| \leq 1$. For each $\xi \in \mathcal{H}$,

$$(1 - 10^{-1})\|\xi\| \leq \|x\xi\| \leq \|\xi\| \tag{3.8}$$

and so

$$(1 - 10^{-1})^2 I \leq x^*x \leq I. \tag{3.9}$$

Thus $|x|$ is invertible, so $u = x|x|^{-1} \in M$ satisfies $u^*u = I$. By finiteness of M, u is a unitary, and so x is invertible with $\|x^{-1}\| \leq (1 - 10^{-1})^{-1}$ from (3.8). Then

$$\|I - x^{-1}v\| = \|x^{-1}(x - v)\| \leq (1 - 10^{-1})^{-1}10^{-1} < 1, \tag{3.10}$$

showing that $x^{-1}v$ and hence v are invertible. This contradicts the assumption that v is not a unitary and establishes that $z_3 = \tilde{z}_3$.

After cutting by $(I - z_3)$ we may now assume that both M and N are direct sums of finite type I parts and type II_1 parts, so that $z_1 + z_2 = \tilde{z}_1 + \tilde{z}_2 = I$. To establish that $z_1 = \tilde{z}_1$ we again argue by contradiction by assuming that there is a central projection z

so that Mz is finite type I and Nz is type II_1, and after cutting by z we can make these assumptions on M and N. Let $p \in M$ be a non-zero abelian projection and choose, by Proposition 3.1 (ii b), a projection $q \in N$ with $\|p-q\| < 1/(10\sqrt{2})$. Thus $d(pMp, qNq) \leq d(M,N) + 2\|p-q\| \leq (1+\sqrt{2})/10 < 1/4$. By [8, Lemma 2.3], qNq is abelian and so $q \in N$ is a non-zero abelian projection. This contradiction proves the result. □

Given a finite von Neumann algebra M, we write $\mathbb{T}_M$ for the centre valued trace on M. The next lemma examines the behaviour of centre valued traces on close projections. We need it both in Section 4 for our analysis of C*-algebras close to those of finite length and in Section 5 to examine traces of close C*-algebras. The next result and some succeeding ones are phrased in terms of near containments rather than distances in order to obtain better estimates.

Lemma 3.6. *Let M and N be finite von Neumann algebras acting on a Hilbert space $\mathcal{H}$ with common centre $Z = Z(M) = Z(N)$. Suppose that $M \subseteq_\gamma N$ and $N \subseteq_\gamma M$ for some constant $\gamma < 1/200$. If $p \in M$ and $q \in N$ are projections with $\|p-q\| < 1/2$, then $\mathbb{T}_M(p) = \mathbb{T}_N(q)$.*

Proof. By Lemma 3.5, there is a central projection z such that Mz and Nz are finite and of type I while $M(1-z)$ and $N(1-z)$ are type II_1. It suffices to consider these parts separately, so we initially assume that M and N are finite type I, and thus injective.

Since both algebras are injective, the bound on γ allows us to apply [12, Corollary 4.4] to conclude that there is a surjective isomorphism $\phi : M \to N$ satisfying $\|\phi(x) - x\| \leq 100\gamma\|x\| < (1/2)\|x\|$. Accordingly

$$\|\phi(p) - q\| < 100/200 + 1/2 = 1 \tag{3.11}$$

so $\phi(p)$ and q are equivalent projections in N. Thus $\mathbb{T}_N(\phi(p)) = \mathbb{T}_N(q)$. Now ϕ maps Z to Z and also fixes the elements of Z pointwise because central projections $z \in Z$ satisfy $\|\phi(z) - z\| \leq 1/2$. Thus

$$\mathbb{T}_M(x) = \phi(\mathbb{T}_M(x)) = \mathbb{T}_{\phi(M)}(\phi(x)) = \mathbb{T}_N(\phi(x)), \quad x \in M. \tag{3.12}$$

Then $\mathbb{T}_M(p) = \mathbb{T}_N(\phi(p))$ and the result is proved in this case.

Now assume that M and N are both type II_1 and, to derive a contradiction, suppose that $\mathbb{T}_M(p) \neq \mathbb{T}_N(q)$. By cutting by a suitable central projection, we may assume without loss of generality that there exist constants $0 \leq c < d$ such that

$$\mathbb{T}_N(q) \leq cI < dI \leq \mathbb{T}_M(p). \tag{3.13}$$

Choose an integer n satisfying $1/n < d - c$. Then $[d, 1]$ is covered by the collection of intervals $[j/n, (j+1)/n)$, $1 \leq j \leq n$, so the spectral projections of $\mathbb{T}_M(p)$ for these intervals cannot all be 0. Choose one that is non-zero and cut by this central projection. This allows us to make the further assumption that

$$(j/n)I \leq \mathbb{T}_M(p) < ((j+1)/n)I \tag{3.14}$$

for some integer $j \in \{1, 2, \ldots, n\}$. The case $j = n$ implies that $p = 1$, whereupon $q = 1$ follows from $\|p-q\| < 1/2$, and a contradiction is reached. Thus we can assume $j < n$. We may then choose orthogonal projections $e_1, \ldots, e_j \in M$ satisfying $e_i \leq p$ and $\mathbb{T}_M(e_i) = I/n$, $1 \leq i \leq j$. Since $\mathbb{T}_M(I-p) > ((n-j-1)/n)I$, we may also choose orthogonal projections

$f_i \in M$, $1 \le i \le n-j-1$, satisfying $\mathbb{T}_M(f_i) = I/n$ and $f_i \le I - p$. Note that there may be no f_i's if $j = n-1$. Let $h = I - \sum_{i=1}^{j} e_i - \sum_{i=1}^{n-j-1} f_i$, which also has centre valued trace I/n. Then $\{e_1, \ldots, e_j, h, f_1, \ldots, f_{n-j-1}\}$ is a set of n equivalent projections in M with sum I, so lie in a matrix subalgebra $F \subset M$ as the minimal diagonal projections. Let $h_1 = p - \sum_{i=1}^{j} e_i$ and $h_2 = (1-p) - \sum_{i=1}^{n-j-1} f_i$. Then $h_1 + h_2 = h$ and $h_1, h_2 \le h$. Thus the algebra Q generated by h_1, h_2 and F is finite dimensional, so injective, and contains p. By [12, Theorem 4.3] there is a $*$-isomorphism ϕ of Q into N satisfying $\|\phi(x) - x\| \le 100\gamma\|x\|$, for $x \in Q$. Again $\|\phi(p) - q\| < 100/200 + 1/2 = 1$, so $\phi(p)$ and q are equivalent in N which yields $\mathbb{T}_N(\phi(p)) = \mathbb{T}_N(q)$. The projections $\{\phi(e_1), \ldots, \phi(e_j), \phi(h), \phi(f_1), \ldots, \phi(f_{n-j-1})\}$ are equivalent in N and sum to I. Thus each has centre valued trace I/n. It follows that

$$\mathbb{T}_N(q) = \mathbb{T}_N(\phi(p)) \ge \sum_{i=1}^{j} \mathbb{T}_N(\phi(e_i)) \ge j/nI > \mathbb{T}_M(p) - 1/nI \ge (d - 1/n)I. \tag{3.15}$$

This implies $d - c \le 1/n$, contradicting the choice of n, and proving the result. □

The final result in this section examines von Neumann algebras close to those in standard position. Recall from [16, I §6.1] or [24, p. 691] that the coupling function $\Gamma(M, M')$ for a finite von Neumann algebra M with finite commutant M' is a possibly unbounded positive operator affiliated to the centre Z, having the following property. For each vector ξ in the underlying Hilbert space $\mathcal{H}$

$$\mathbb{T}_M(e_\xi^{M'}) = \Gamma(M, M')\mathbb{T}_{M'}(e_\xi^{M}) \tag{3.16}$$

where $e_\xi^{M'} \in M$ is the projection onto the cyclic subspace $\overline{M'\xi}$, while $e_\xi^{M} \in M'$ projects onto $\overline{M\xi}$. Recall too that a finite von Neumann algebra M is in standard position on a Hilbert space $\mathcal{H}$ if and only if M' is finite and $\Gamma(M, M') = I$. From this point of view, the next lemma shows that a von Neumann algebra which is close to an algebra in standard position is approximately in standard position.

Lemma 3.7. *Let M and N be finite von Neumann algebras on a Hilbert space $\mathcal{H}$ with common centre Z. Let γ, $\delta < 1/200$ be constants such that the near inclusions*

$$M \subseteq_\gamma N, \quad N \subseteq_\gamma M, \quad M' \subseteq_\delta N', \quad \text{and} \quad N' \subseteq_\delta M' \tag{3.17}$$

hold. If M is in standard position on $\mathcal{H}$, then N' is finite and $\Gamma(N, N')$ satisfies

$$0.99\ I < (1 - \gamma/\sqrt{2})I \le \Gamma(N, N') \le \frac{1}{1 - \delta/\sqrt{2}} I < 1.01\ I. \tag{3.18}$$

Proof. Since $d(M', N') \le 2\delta < 1/100 < 1/10$ and M' is finite, Lemma 3.5 shows that N' is also finite. We are not requiring that $\mathcal{H}$ be separable, so M need not have a faithful trace. However, as M certainly has a separating family of normal tracial states, a maximality argument gives a set $\{z_j\}_{j \in J}$ of orthogonal central projections summing to I so that each Mz_j has a faithful normal trace. By proving the result for each of the pairs (Mz_j, Nz_j) separately, we may then assume that M has a faithful normal trace τ. Let $t > 0$ be fixed but arbitrary in the spectrum of $\Gamma(N, N')$. It suffices to demonstrate the inequalities

$$1 - \gamma/\sqrt{2} \le t \le \frac{1}{1 - \delta/\sqrt{2}}. \tag{3.19}$$

Given $\varepsilon > 0$, let $e \in Z$ be the non-zero spectral projection of $\Gamma(N, N')$ for $(t-\varepsilon, t+\varepsilon)$. We may cut by this projection, which allows us to assume that

$$(t-\varepsilon)I \leq \Gamma(N, N') \leq (t+\varepsilon)I. \tag{3.20}$$

Since M is in standard position on $\mathcal{H}$, there is a unit vector $\xi \in \mathcal{H}$ so that the vector state $\langle \cdot\, \xi, \xi\rangle$ defines a faithful tracial state both on M and on M'. Define two cyclic projections $p \in N$ and $q \in N'$ with range spaces $\overline{N'\xi}$ and $\overline{N\xi}$ respectively. By Proposition 3.1 (ii b), we may choose projections $r \in M$ and $s \in M'$ so that $\|p-r\| \leq \gamma/\sqrt{2}$ and $\|q-s\| \leq \delta/\sqrt{2}$. The hypotheses of Lemma 3.6 are satisfied and so $\mathbb{T}_N(p) = \mathbb{T}_M(r)$ and $\mathbb{T}_{N'}(q) = \mathbb{T}_{M'}(s)$. The centre valued traces $\mathbb{T}_M$ and $\mathbb{T}_{M'}$ preserve the trace $\langle \cdot\, \xi, \xi\rangle$ on M and M' and so

$$\langle \mathbb{T}_{N'}(q)\xi, \xi\rangle = \langle \mathbb{T}_{M'}(s)\xi, \xi\rangle = \langle s\xi, \xi\rangle \tag{3.21}$$

and

$$\langle \mathbb{T}_{N}(p)\xi, \xi\rangle = \langle \mathbb{T}_{M}(r)\xi, \xi\rangle = \langle r\xi, \xi\rangle. \tag{3.22}$$

Define α to be $\langle \mathbb{T}_{N'}(q)\xi, \xi\rangle > 0$, and β to be such that

$$\alpha\beta = \langle \mathbb{T}_{N}(p)\xi, \xi\rangle = \langle \Gamma(N, N')\mathbb{T}_{N'}(q)\xi, \xi\rangle. \tag{3.23}$$

The relations (3.20) and (3.23) imply that

$$(t-\varepsilon)\alpha \leq \alpha\beta \leq (t+\varepsilon)\alpha \tag{3.24}$$

and so

$$\beta - \varepsilon \leq t \leq \beta + \varepsilon. \tag{3.25}$$

Since $p\xi = q\xi = \xi$, the choices of r and s imply that

$$1 - \delta/\sqrt{2} \leq \langle s\xi, \xi\rangle \leq 1 \tag{3.26}$$

and

$$1 - \gamma/\sqrt{2} \leq \langle r\xi, \xi\rangle \leq 1. \tag{3.27}$$

The definitions of α and $\alpha\beta$, together with (3.21) and (3.22), allow us to rewrite these inequalities as

$$1 - \delta/\sqrt{2} \leq \alpha \leq 1 \tag{3.28}$$

and

$$1 - \gamma/\sqrt{2} \leq \alpha\beta \leq 1, \tag{3.29}$$

after which division yields

$$1 - \gamma/\sqrt{2} \leq \beta \leq \frac{1}{1-\delta/\sqrt{2}}. \tag{3.30}$$

From (3.25), we now have the inequalities

$$1 - \gamma/\sqrt{2} - \varepsilon \leq t \leq \frac{1}{1-\delta/\sqrt{2}} + \varepsilon. \tag{3.31}$$

Now let $\varepsilon \to 0$, and we have proved (3.19) as required. □

4. Stability of finite length

The main result of this section is Theorem 4.4 which shows that a C*-algebra B which is close to a C*-algebra A of finite length must also have finite length and obtains a bound on the length of B in terms of the length and the length constant of A. When A has finite length, Proposition 2.5 gives a near inclusion of B' inside A' (with constants depending on $d(A, B)$, the length of A, and its associated length constant). The key step in Theorem 4.4 is to obtain a reverse near inclusion of A' inside B' which we achieve in Theorem 4.2. This in turn is established by a type decomposition argument, handling the finite type I, the type II_1, and the infinite cases separately. Existing results enable us to deal with the first and last cases quickly so the heart of the matter is the II_1 case.

Lemma 4.1. *Let M and N be von Neumann algebras of type II_1 faithfully and non-degenerately represented on $\mathcal{H}$. Suppose further that M and N have common centre Z which admits a faithful state. Suppose that $d(M, N) = \alpha$ and M contains an ultraweakly dense C*-algebra A of length at most ℓ and length constant at most K. Write $k = K\ell/2$. If α satisfies the inequality*

$$24(12\sqrt{2}k + 4k + 1)\alpha < 1/200, \tag{4.1}$$

then

$$d(M', N') \leq 2\beta + 1200k\alpha(1 + \beta), \tag{4.2}$$

where $\beta = K((1 + 28800k\alpha + 48\alpha)^{\ell} - 1)$.

The proof of this result is long and intricate, so it will be helpful to give a brief summary before embarking on it. Our objective is to reduce to the following situation:

(i) $\mathcal{H}$ decomposes as $\mathcal{H}_0 \otimes \ell^2(\Lambda)$;
(ii) the von Neumann algebras M and N simultaneously decompose as $M \cong M_0 \otimes I_{\ell^2(\Lambda)}$ and $N \cong N_0 \otimes I_{\ell^2(\Lambda)}$;
(iii) M_0 is in standard position on $\mathcal{H}_0$;
(iv) N_0 has the local distance property LD_{24} on $\mathcal{H}_0$.

Once (i)-(iv) have been achieved, the proof is completed by the following steps. The local distance property immediately gives a near inclusion

$$M_0' \subseteq_{\alpha'} N_0', \quad \text{on } \mathcal{H}_0 \tag{4.3}$$

for a suitable constant α'. Since M_0 is in standard position on $\mathcal{H}_0$, it is anti-isomorphic to its commutant. In particular, this commutant has a weakly dense C*-algebra of finite length so we can use results from Section 2 to lift the near inclusion (4.3) to obtain a near inclusion of the form

$$M' = M_0' \mathbin{\overline{\otimes}} \mathbb{B}(\ell^2(\Lambda)) \subseteq_{\alpha''} N_0' \mathbin{\overline{\otimes}} \mathbb{B}(\ell^2(\Lambda)) = N' \tag{4.4}$$

for a suitable constant α''. Since a reverse near inclusion is immediate from the hypotheses of the lemma, this establishes the result.

To reach the situation detailed in (i)-(iv) above, a number of further reductions are necessary. We first adjust the Hilbert space and arrange for the representation of N to be an amplification of its standard position. We then find non-zero close projections $e \in M'$ and $d \in N'$ of full central support so that Me is in standard position on $e(\mathcal{H})$ and Nd

has the local distance property LD_{24} on $d(\mathcal{H})$. This is the main technical step in the proof, requiring our earlier results regarding the behaviour of the centre valued trace and coupling function under small perturbations. This enables us to transfer the property that some cut down of N is in standard position to the same property for M. We then use the perturbation theory for injective von Neumann algebras from [12] to obtain the situation of (i)-(iv) above.

Proof of Lemma 4.1. Let S be an isomorphic copy of N, acting in standard position on a Hilbert space $\mathcal{K}$. The general theory of isomorphisms of von Neumann algebras [16, I §4 Theorem 3] allows us to choose a sufficiently large set Ω (which we insist has dimension at least 2) so that the amplifications $\widetilde{N}$ of N to $\mathcal{H}\otimes\ell^2(\Omega)$ and $\widetilde{S}$ of S to $\mathcal{K}\otimes\ell^2(\Omega)$ are spatially isomorphic. Amplification increases the distance between commutants, so if the result is true in this context then it is true generally. Thus we can assume that $\mathcal{H}$ decomposes as $\mathcal{K}\otimes\ell^2(\Omega)$ and that $N = S\otimes I_{\ell^2(\Omega)}$. Then $N' = S'\ \overline{\otimes}\ \mathbb{B}(\ell^2(\Omega))$.

Proposition 2.9 shows that the C*-algebra A has property D_k, so Proposition 2.5 gives

$$N' \subseteq_{2k\alpha} M'. \tag{4.5}$$

Choose a copy Q_0 of the 2×2 matrices in $\mathbb{B}(\ell^2(\Omega))$ such that the minimal projections of Q_0 are rank one projections in $\mathbb{B}(\ell^2(\Omega))$ and let $Q = I_{\mathcal{K}}\otimes Q_0 \subset N'$. The near inclusion (4.5) gives

$$Q \subseteq_{2k\alpha} M', \tag{4.6}$$

and note that Q and M' both lie in the algebra Z'. The inequality (4.1) implies $2k\alpha < 1/(3\sqrt{2})$, so Proposition 3.3 gives us a unitary $u_1\in Z'$ with

$$\|u_1 - I_{\mathcal{K}}\| \le (3\sqrt{2}+1)2k\alpha, \tag{4.7}$$

such that $u_1Qu_1^*\subset M'$.

Define $N_1 = u_1Nu_1^*$. Since $u_1\in Z'$ it follows that N_1 has centre Z. Let $Q_1 = u_1Qu_1^*$ so that $Q_1\subset M'\cap N_1'$. The estimate (4.7) gives the distance estimate

$$d(M,N_1)\le 2\|u_1 - I_{\mathcal{K}}\| + d(M,N) \le (12\sqrt{2}k+4k+1)\alpha. \tag{4.8}$$

Similarly, the near inclusion (4.5) induces the near inclusion

$$N_1' \subseteq_{6(2\sqrt{2}+1)k\alpha} M'. \tag{4.9}$$

The construction of Q ensures that every non-zero projection in Q_1 has central support I in N_1' and hence central support I in M'. Fix a minimal projection $f\in Q_1$. By choice of Q_1, the algebra N_1f is in standard position on $f(\mathcal{H})$. Then N_1f has a cyclic vector and so has the local distance property LD_{12} on this space by Proposition 2.6. The distance estimate (4.8) compresses to $f(\mathcal{H})$ to give the near inclusion

$$N_1f \subseteq_{(12\sqrt{2}k+4k+1)\alpha} Mf. \tag{4.10}$$

Applying Proposition 2.5 then gives

$$(M')_f \subseteq_{24(12\sqrt{2}k+4k+1)\alpha} (N_1')_f. \tag{4.11}$$

Since f lies in $M'\cap N_1'$, we can also compress (4.9) by f to obtain

$$(N_1')_f \subseteq_{6(2\sqrt{2}+1)k\alpha} (M')_f. \tag{4.12}$$

Now N_1f is in standard position on $f(\mathcal{H})$. The inequalities (4.1) and (4.8) ensure that $d(Mf, N_1f) < 1/200$. Moreover,

$$6(\sqrt{2}+1)k\alpha < 1/200 \quad \text{and} \quad 24(12\sqrt{2}k + 4k + 1)\alpha < 1/200, \tag{4.13}$$

so the hypotheses of Lemma 3.7 are met for the algebras Mf and N_1f. Writing $\Gamma(Mf,(M')_f)$ for the coupling function of Mf on $f(\mathcal{H})$, we obtain

$$0.99\ f \leq \Gamma(Mf,(M')_f) \leq 1.01\ f. \tag{4.14}$$

Let I_{Q_1} denote the unit of Q_1 and $\Gamma(MI_{Q_1},(M')_{I_{Q_1}})$ denote the coupling function of MI_{Q_1} on $I_{Q_1}(\mathcal{H})$. As MI_{Q_1} is a two-fold amplification of Mf, it follows that

$$1.98\ I_{Q_1} \leq \Gamma(MI_{Q_1},(M')_{I_{Q_1}}) \leq 2.02\ I_{Q_1}. \tag{4.15}$$

In particular $\Gamma(MI_{Q_1},(M')_{I_{Q_1}}) \geq I_{Q_1}$ and so any state on MI_{Q_1} is a vector state (see [16, III §.1 Proposition 3], for example).

Let τ be a faithful tracial state on M, the existence of which is guaranteed by our hypothesis that Z admits a faithful state. As I_{Q_1} has central support I in M', the representation $m \mapsto mI_{Q_1}$ of M on $I_{Q_1}(\mathcal{H})$ is faithful. Therefore the previous paragraph gives us a unit vector $\xi \in I_{Q_1}(\mathcal{H})$ with

$$\tau(m) = \langle m\xi, \xi\rangle, \quad m \in M. \tag{4.16}$$

Let $e_0 \in M'$ be the projection onto $\overline{M\xi}$. Then Me_0 is in standard position on $e_0(\mathcal{H})$ and $e_0 \leq I_{Q_1}$. Since the range of e_0 contains a trace vector for the faithful trace τ on M, it follows that e_0 has central support I for M'. Indeed, given a non-zero projection $z \in Z$, $\tau(z) = \langle z\xi,\xi\rangle = \langle ze_0\xi,\xi\rangle \neq 0$, so that $ze_0 \neq 0$.

By construction $(N_1)_{I_{Q_1}}$ has a 2-cyclic set and so property LD_{24} by Proposition 2.6. Accordingly, putting the distance estimate (4.8) into Proposition 2.5 gives the near inclusion

$$(M')_{I_{Q_1}} \subset_{48(12\sqrt{2}k+4k+1)\alpha} (N_1')_{I_{Q_1}}. \tag{4.17}$$

Proposition 3.1 (ii b) then allows us to find a projection $d_0 \in (N_1')_{I_{Q_1}}$ with

$$\|e_0 - d_0\| \leq 48(12\sqrt{2}k + 4k + 1)\alpha/\sqrt{2}. \tag{4.18}$$

Since $N_1I_{Q_1}$ has a 2-cyclic set, so too does N_1d_0. In particular Proposition 2.6 shows that the algebra N_1d_0 on $d_0(\mathcal{H})$ retains property LD_{24}. Since $\|e_0 - d_0\| < 1$ (this follows from the inequality (4.1)) and M' and N_1' have common centres, d_0 has central support I in N_1'.

Define $d = u_1^*d_0u_1$. This lies in N' and has the same properties there that d_0 has in N_1' Thus the algebra Nd on $d(\mathcal{H})$ has the local distance property LD_{24} and d has central support I in N'. It is convenient to adjust e_0 as this improves the estimates obtained in the lemma. Since $N' \subset_{2k\alpha} M'$, applying Proposition 3.1 (ii b) again gives us a projection $e \in M'$ with

$$\|d - e\| < 2k\alpha/\sqrt{2} = \sqrt{2}k\alpha. \tag{4.19}$$

It follows that

$$\|e - e_0\| \leq \|e - d\| + \|d - d_0\| + \|e_0 - d_0\| \tag{4.20}$$

$$\leq 2k\alpha/\sqrt{2} + 2\|u_1 - 1_{\mathcal{H}}\| + 48(12\sqrt{2}k + 4k + 1)\alpha/\sqrt{2} < 1, \tag{4.21}$$

where we obtain the bound of 1 from the inequality (4.1). Then e and e_0 are unitarily equivalent in M' and in particular e has central support I and Me is in standard position on $e(\mathcal{H})$. This completes the first stage of the proof.

Since d has central support I, the von Neumann algebras N and Nd are isomorphic. The general theory of isomorphisms between von Neumann algebras again shows that N and Nd are spatially isomorphic after a further suitably large amplification. Just as at the beginning of the proof such an amplification only increases the distances between commutants so we may assume that N is actually an amplification of Nd. Thus we can find a family of pairwise orthogonal equivalent projections $(d_{i,i})_{i\in\Lambda}$ in N' with sum I so that $d = d_{i_0,i_0}$. These projections may be extended to a set of matrix units $(d_{i,j})_{i,j\in\Lambda}$ in N'. Let P be the injective von Neumann subalgebra of $N' \subseteq Z'$ generated by these matrix units. The near inclusion (4.5) gives

$$P \subset_{2k\alpha} M', \tag{4.22}$$

so by [12, Theorem 4.3], there is a unitary $u_2 \in (M' \cup N')'' \subseteq Z'$ such that $\|u_2 - I_{\mathcal{H}}\| \leq 300k\alpha$ and $u_2 P u_2^* \subset M'$. Again the required hypothesis that $2k\alpha < 1/100$ to use [12, Theorem 4.3] is immediate from our initial inequality (4.1).

Define $N_2 = u_2 N u_2^*$. This algebra also has centre Z as $u_2 \in Z'$. Define matrix units by $e_{i,j} = u_2 d_{i,j} u_2^*$ and note that these matrix units lie in $M' \cap N_2'$. The projection e_{i_0,i_0} has

$$\|e_{i_0,i_0} - e\| \leq \|e_{i_0,i_0} - d_{i_0,i_0}\| + \|e - d\| \leq 2\|u_2 - I_{\mathcal{H}}\| + \|e - d\| < 1, \tag{4.23}$$

where again we collect our previous estimates and apply (4.1) to achieve this estimate. Therefore e_{i_0,i_0} and e are unitarily equivalent in M' and so Me_{i_0,i_0} is in standard position on $e_{i_0,i_0}(\mathcal{H})$. Using these matrix units we see that M' and N_2' are simultaneously spatially isomorphic to $(M')_{e_{i_0,i_0}} \overline{\otimes} \mathbb{B}(\ell^2(\Lambda))$ and $(N_2')_{e_{i_0,i_0}} \overline{\otimes} \mathbb{B}(\ell^2(\Lambda))$. The algebras M and N_2 are now in the position described by conditions (i)-(iv) in the discussion preceding the proof. To ease notation write T_M for the von Neumann algebra Me_{i_0,i_0} acting on $e_{i_0,i_0}(\mathcal{H}) = \mathcal{H}_0$ and T_{N_2} for $N_2 e_{i_0,i_0}$ acting on the same space. We have the distance estimate

$$d(T_M, T_{N_2}) \leq d(M, N_2) \leq d(M, N) + 2\|u_2 - I_{\mathcal{H}}\| \leq 600k\alpha + \alpha. \tag{4.24}$$

By construction $T_{N_2} = N_2 e_{i_0,i_0}$ has property LD_{24} on $\mathcal{H}_0$ so Proposition 2.5 gives

$$T_M' \subset_{48(600k\alpha+\alpha)} T_{N_2}'. \tag{4.25}$$

Since T_M lies in standard position, there is a conjugate linear isometry J on $\mathcal{H}_0 = e_{i_0,i_0}(\mathcal{H})$ with $JT_MJ = T_M'$. Now T_M, as a cut down of M, has a weak*-dense C*-algebra with length at most ℓ with length constant at most K. Write T_A for this C*-algebra and note that JT_AJ is weak*-dense in T_M' and also has length at most ℓ and length constant at most K. Since

$$JT_AJ \subset_{48(600k\alpha+\alpha)} T_{N_2}', \tag{4.26}$$

Corollary 2.12 gives

$$JT_AJ \otimes \mathbb{K}(\ell^2(\Lambda)) \subset_\beta T_{N_2}' \otimes \mathbb{K}(\ell^2(\Lambda)), \tag{4.27}$$

where $\beta = K((1 + 28800k\alpha + 48\alpha)^\ell - 1)$. Lemma 5 of [23] allows us to take the weak operator closure of this near inclusion (note that although the statement is only given for the two-sided notion of closeness, the proof works in the one-sided context we need). This gives a near inclusion

$$M' \subset_\beta N_2'. \tag{4.28}$$

Since $d(N_2', N') \leq 2\|u_2 - I\| \leq 600k\alpha$, Proposition 2.3 (i) gives

$$M' \subset_{\beta+600k\alpha(1+\beta)} N'. \tag{4.29}$$

Combining this with the initial near inclusion (4.5) and using Proposition 2.3 (iii) gives the estimate

$$d(M', N') \leq 2\beta + 1200k\alpha(1+\beta), \tag{4.30}$$

which completes the proof. □

The next theorem combines the previous lemma with results from [10] to show that sufficiently close algebras have close commutants if one algebra has finite length. We do not assume that A and B are represented non-degenerately and so we use the notation $\overline{A}^w$ rather than A'' to denote the von Neumann algebra generated by A.

Theorem 4.2. *Let A and B be C^*-algebras acting on a Hilbert space $\mathcal{H}$. Let γ denote $d(A, B)$. Suppose that A has finite length at most ℓ with length constant at most K, and suppose that γ satisfies*

$$24(12\sqrt{2}k + 4k + 1)\gamma < 1/2200, \tag{4.31}$$

where $k = K\ell/2$. Then

$$d(A', B') \leq 10\gamma + 2\beta + 13200k\gamma(1+\beta), \tag{4.32}$$

where $\beta = K((1 + 316800k\gamma)^\ell - 1)$.

Proof. Let $M = \overline{A}^w$ and $N = \overline{B}^w$ and write $Z(M)$ and $Z(N)$ for the centres of M and N respectively. Lemma 5 of [23] gives $d(M, N) \leq d(A, B) = \gamma$. By Lemma 3.4, there is a unitary $u \in (Z(M) \cup Z(N))''$ such that $uZ(M)u^* = Z(uMu^*) = Z(N)$ and

$$\|u_1 - I_{\mathcal{H}}\| \leq 5\gamma. \tag{4.33}$$

Write $M_0 = uMu^*$. Then

$$d(M_0, N) \leq 2\|u_1 - I_{\mathcal{H}}\| + d(M, N) \leq 11\gamma \tag{4.34}$$

Since $11\gamma < 1/10$, Lemma 3.5 applies. Thus we can find orthogonal projections $z_{\mathrm{I_{fin}}}, z_{\mathrm{II}_1}, z_\infty$ in $Z(M_0)$ which sum to $I_{\mathcal{H}}$ such that:

(i) $M_0 z_{\mathrm{I_{fin}}}$ and $N z_{\mathrm{I_{fin}}}$ are finite type I;
(ii) $M_0 z_\infty$ and $N z_\infty$ are properly infinite;
(iii) $M_0 z_{\mathrm{II}_1}$ and $N z_{\mathrm{II}_1}$ are type II_1.

Finite type I von Neumann algebras are injective, so have property D_1 ([10, Theorem 2.3]) while properly infinite algebras have property $D_{3/2}$ ([10, Theorem 2.4]). Applying Proposition 2.5 and Proposition 2.3 yields

$$d(M_0' z_{\mathrm{I_{fin}}}, N' z_{\mathrm{I_{fin}}}) \leq 2 \cdot 2d(M_0 z_{\mathrm{I_{fin}}}, N z_{\mathrm{I_{fin}}}) \leq 4d(M_0, N) \leq 44\gamma, \tag{4.35}$$

and

$$d(M_0' z_\infty, N' z_\infty) \leq 2 \cdot 3d(M_0 z_\infty, N z_\infty) \leq 6d(M_0, N) \leq 66\gamma. \tag{4.36}$$

Choose a maximal family of projections $(z_i)_{i\in\Lambda}$ in Zz_{II_1} so that each Zz_i has a faithful state. For $\alpha \leq 11\gamma$, the inequality (4.1) follows from (4.31) so the pairs M_0z_i and Nz_i satisfy the hypothesis of Lemma 4.1 for each i. The estimates of this lemma then give

$$d(M_0'z_i, N'z_i) \leq 2\beta + 13200k\gamma(1+\beta). \tag{4.37}$$

Combining all these cases gives the estimate

$$d(M_0', N') \leq 2\beta + 13200k\gamma(1+\beta). \tag{4.38}$$

We then use the estimate $d(M_0', M') \leq 10\gamma$ to obtain

$$d(M', N') \leq 10\gamma + 2\beta + 13200k\gamma(1+\beta), \tag{4.39}$$

exactly as required. □

In order to use Theorem 4.2 to show that the property of having finite length transfers to close subalgebras, we need one final ingredient detailing how the local distance property behaves for close C*-algebras with close commutants.

Lemma 4.3. *Let A and B be C^*-algebras on a Hilbert space $\mathcal{H}$. Suppose that $d(A,B) < \gamma$ and $d(A', B') < \eta$. Suppose that A has propertry LD_k, where $2\eta + 2k\gamma < 1$. Then B has property $LD_{\tilde{k}}$ where*

$$\tilde{k} = \frac{k}{1-2\eta-2k\gamma}. \tag{4.40}$$

Proof. Consider an element $x \in \mathbb{B}(\mathcal{H}) \setminus B'$. By scaling we may assume that $\|\mathrm{ad}(x)|_B\| = 1$. By ultraweak compactness, there exists $b' \in B'$ so that $\|x-b'\| = d(x, B')$. The replacement of x by $x - b'$ allows us to make the further assumption that $\|x\| = d(x, B')$. Our objective now is to estimate $\|x\|$ from above.

Consider $a \in A$, $\|a\| \leq 1$, and choose $b \in B$, $\|b\| \leq 1$, so that $\|a - b\| < \gamma$. Then

$$\|[x,a]\| \leq \|[x,b]\| + \|[x, a-b]\| \leq 1 + 2\gamma\|x\|. \tag{4.41}$$

Thus $\|\mathrm{ad}(x)|_A\| \leq 1 + 2\gamma\|x\|$. Let $T = \{t \in A' \colon \|x - t\| \leq \|x\|\}$, non-empty since $0 \in T$. The triangle inequality shows that each $t \in T$ satisfies $\|t\| \leq 2\|x\|$. For each $t \in T$, choose $s \in B'$ so that $\|t - s\| \leq \eta\|t\| \leq 2\eta\|x\|$. Then

$$\|x - t\| \geq \|x - s\| - \|t - s\| \geq \|x\| - 2\eta\|x\|. \tag{4.42}$$

Letting $t \in T$ vary, this yields

$$d(x, A') \geq (1 - 2\eta)\|x\|. \tag{4.43}$$

Since A has property LD_k, we obtain

$$(1-2\eta)\|x\| \leq d(x, A') \leq k\|\mathrm{ad}(x)|_A\| \leq k + 2k\gamma\|x\|. \tag{4.44}$$

This implies that

$$\|x\| \leq \frac{k}{1-2\eta-2k\gamma}. \tag{4.45}$$

Since we also have $\|x\| = d(x, B')$ and $\|\mathrm{ad}(x)|_B\| = 1$, this last inequality states that B has property $LD_{\tilde{k}}$ for

$$\tilde{k} = \frac{k}{1 - 2\eta - 2k\gamma}, \tag{4.46}$$

completing the proof. □

We are now in a position to establish the main result of this section: that C*-algebras sufficiently close to those of finite length also have finite length.

Theorem 4.4. *Let C be a C^*-algebra and let A and B be two C^*-subalgebras of C. Suppose that $d(A, B) < \gamma$, and that A has finite length at most ℓ with length constant at most K. Write $k = K\ell/2$,*

$$\beta = K\Big((1 + 316800k\gamma + 528\gamma)^\ell - 1\Big), \tag{4.47}$$

and

$$\eta = 10\gamma + 2\beta + 13200k\gamma(1 + \beta). \tag{4.48}$$

If the inequalities

$$24(12\sqrt{2}k + 4k + 1)\gamma < 1/2200, \quad 2\eta + 2k\gamma < 1 \tag{4.49}$$

are satisfied, then B has property $D_{\tilde{k}}$ for

$$\tilde{k} = \frac{k}{1 - 2\eta - k\gamma}. \tag{4.50}$$

In particular B has finite length and the length of B is at most $\lfloor 2\tilde{k} \rfloor \leq \lfloor K\ell \rfloor$.

Proof. Let $\pi : B \to \mathbb{B}(\mathcal{K})$ be a representation of B on a Hilbert space $\mathcal{K}$, and let $\rho : C \to \mathbb{B}(\mathcal{H})$ be a representation of C on a larger Hilbert space $\mathcal{H}$ so that ρ extends π (see [2, Proposition II.6.4.11], for example). Since $\rho(A)$ is a quotient of A, the length of $\rho(A)$ is at most ℓ with length constant at most K. Theorem 4.2 gives

$$d(\rho(A)', \rho(B)') < 10\gamma + 2\beta + 13200k\gamma(1 + \beta) = \eta, \tag{4.51}$$

where the first inequality of (4.49) is the estimate required to apply Theorem 4.2. The second inequality of (4.49) is the hypothesis of Lemma 4.3 and so $\pi(B)$ has property $LD_{\tilde{k}}$, where $\tilde{k}$ is given by (4.50). Since the representation π of B was arbitrary, B has property $D_{\tilde{k}}$, as required. The final statement of the Theorem follows from Proposition 2.8. □

Remark 4.5. While it is obvious that sufficiently small choices of γ will allow us to satisfy (4.49), the dependence of these inequalities on K and ℓ does not make clear the range of admissible values for this constant. We consider here one example. Suppose that $\ell = 3$ and $K = 1$ for A, a situation that occurs when A is a stable but non-nuclear C*-algebra, for instance. Then direct calculation shows that (4.49) is satisfied for $\gamma < 10^{-7}$. □

We now turn to some immediate applications of Theorem 4.4. The first corollary follows from [12, Theorem 3.1].

Corollary 4.6. *Let A, B be C*-algebras of some C*-algebra C and let E be a nuclear C*-algebra. Suppose that A has finite length at most ℓ and length constant at most K and suppose that $d(A,B) < \gamma$. Let k, β, η be as in Theorem 4.4. Provided γ satisfies the inequalities (4.49), then*

$$B \otimes E \subset_\mu A \otimes E, \tag{4.52}$$

where

$$\mu = \frac{6k\gamma}{1 - 2\eta - k\gamma}. \tag{4.53}$$

In particular $d(A \otimes E, B \otimes E) < 2\mu$.

Using the fact that the distance between any two C*-subalgebras of the same C*-algebra is at most 1, we get the following alternative formulation of the previous corollary.

Corollary 4.7. *For each $\ell \geq 1$ and $K \geq 1$ there exists a constant $L_{\ell,K}$ (which can be found explicitly) such that whenever A, B are C*-subalgebras of some C*-algebra C such that A has length at most ℓ and length constant at most K, then*

$$d(A \otimes E, B \otimes E) \leq L_{\ell,K} d(A,B) \tag{4.54}$$

for every nuclear C-algebra E.*

Raeburn and Taylor [39] showed the existence of a constant $\gamma_0 > 0$ with the property that if two von Neumann algebras M and N have $d(M,N) < \gamma_0$, then M is injective if and only if N is injective. As a consequence (using [23, Lemma 5] and that a C*-algebra A is nuclear if and only if A^{**} is injective), it follows that two C*-algebras A and B with $d(A,B) < \gamma_0$ are either both nuclear or both non-nuclear. This argument was given in [12, Theorem 6.5], in which it was also shown that one can take $\gamma_0 = 1/101$. Finite dimensional C*-algebras have length 1 (with length constant 1). In [38], Pisier characterised nuclearity using the similarity length showing that a C*-algebra is nuclear if and only if it has length at most 2 (it then follows that the length constant must be 1). We can use this characterisation and Theorem 4.4 to recapture the stablity of nuclearity under small perturbations: if we take $\ell = 2$ and $K = 1$ in Theorem 4.4, then there is certainly a constant γ_0 for which $\gamma < \gamma_0$ satisfy (4.49) and the $\tilde{k}$ given in (4.50) has $\tilde{k} < 3/2$. It follows that if A is nuclear and $d(A,B) < \gamma_0$, then B has length at most 2 so is nuclear. We obtain a similar statement for algebras of higher lengths, though as our results do not enable us to control the length constant we must restrict to the case of length constant 1, (although no example of a C*-algebra with finite length and length constant strictly larger than 1 is known). In the case $\ell(A) = 3$ and $d(A,B) < \gamma_0$, we obtain the exact value $\ell(B) = 3$, since any smaller value would imply nuclearity of B and hence of A, by the preceding remarks. This would give the contradiction $\ell(A) \leq 2$. We record this discussion in the following result, using the notation above.

Corollary 4.8. *For each $\ell \geq 1$, there exists a constant $\gamma_\ell > 0$ such that if A and B are two C*-subalgebras of a C*-algebra C with $d(A,B) < \gamma_\ell$ and A has length at most ℓ with length constant at most 1, then B has length at most ℓ. In particular, if $\ell(A) = 3$ and γ_3 is chosen to be less than γ_0, then $\ell(B) = 3$.*

5. K-theory and traces

The classification programme for nuclear C*-algebras was introduced by Elliott in [17], in which separable AF C*-algebras were classified by their *local semigroups*, the Murray-von Neumann equivalence classes of projections with addition defined where it makes sense. In [32], J. Phillips and Raeburn showed that sufficiently close C*-algebras have isomorphic local semigroups and deduced that sufficiently close separable AF C*-algebras must be isomorphic. Subsequently, Khoshkam examined the K-theory of close subalgebras in [26], showing that sufficiently close nuclear C*-algebras have isomorphic K-groups and so opened the road to using classification results to resolve perturbation problems. As Khoshkam notes, the argument of [26] only uses nuclearity to lift a near inclusion $A \subset_\gamma B$ with A nuclear to near inclusions $A \otimes \mathbb{M}_n \subset_{6\gamma} B \otimes \mathbb{M}_n$ for all n, via property D_1 and [12, Theorem 3.1].

Let A, B be C*-subalgebras of a C*-algebra C. Write $\tilde{C}$ for the C*-algebra obtained by adding a new unit I to C (even if C already has a unit) and let $\tilde{A} = C^*(A, I)$ and $\tilde{B} = C^*(B, I)$ so that $\tilde{A}$ and $\tilde{B}$ share the same unit. Recall that $K_0(A)$ is the kernel of the natural map $K_0(\tilde{A}) \to K_0(\mathbb{C}I) \cong \mathbb{Z}$ and that $K_1(A)$ is naturally isomorphic to $K_1(\tilde{A})$ (as $K_1(\mathbb{C}I) = \{0\}$).

Theorem 5.1 (Khoshkam — [26, Proposition 2.4, Remark 2.5])**.** *Let A, B be C*-subalgebras of a C*-algebra C. Suppose that there exists $\gamma \leq 1/3$ such that $A \otimes \mathbb{M}_n \subset_\gamma B \otimes \mathbb{M}_n$ for all $n \in \mathbb{N}$. Then there are homomorphisms $\Phi_0 : K_0(A) \to K_0(B)$ and $\Phi_1 : K_1(A) \to K_1(B)$ defined as follows.*

- (i) *Given a projection $p \in \mathbb{M}_n(\tilde{A})$, choose a projection $q \in \mathbb{M}_n(\tilde{B})$ with $\|p - q\| < \sqrt{2}\gamma$. Define $\Phi_0([p]_0) = [q]_0$. This is well defined and extends to a homomorphism $K_0(\tilde{A}) \to K_0(\tilde{B})$ which induces a homorphism $\Phi_0 : K_0(A) \to K_0(B)$.*
- (ii) *Given a unitary $u \in \mathbb{M}_n(\tilde{A})$, choose a unitary $v \in \mathbb{M}_n(\tilde{B})$ with $\|u - v\| < \sqrt{2}\gamma$. Define $\Phi_1([u]_1) = [v]_1$. This is well defined and extends to a homomorphism $K_1(A) \cong K_1(\tilde{A}) \to K_1(\tilde{B}) \cong K_1(B)$.*

If, in addition, $B \otimes \mathbb{M}_n \subset_{\gamma'} A \otimes \mathbb{M}_n$ for some $\gamma' \leq 1/3$ and for all $n \in \mathbb{N}$, then Φ_0 and Φ_1 are isomorphisms.

The choices required to define the maps above can be made. The remarks of [26, 1.4] show that if $A \otimes \mathbb{M}_n \subset_\gamma B \otimes \mathbb{M}_n$, then $\tilde{A} \otimes \mathbb{M}_n \subset_{2\gamma} \tilde{B} \otimes \mathbb{M}_n$. The estimates given in Proposition 3.1 can then be used to make the necessary choices.

Remark 5.2. The map Φ_0 also preserves the order structure of K_0. Write $K_0(A)^+$ for the positive cone in $K_0(A)$ which consists of the classes $[p]_0$ in $K_0(A)$ corresponding to projections p in $\mathbb{M}_n(A)$ for some n and write $\Sigma(A)$ for the *scale* in $K_0(A)$ which consists of the classes $[p]_0$ in $K_0(A)$ corresponding to projections in A. Then, provided the γ of the previous theorem satisfies $(2+\sqrt{2})\gamma < 1$, it follows that Φ_0 has $\Phi_0(K_0(A)^+) \subseteq K_0(B)^+$ and $\Phi_0(\Sigma(A)) \subseteq \Sigma(B)$. This condition on γ is needed so that, for example given a projection p in some $\mathbb{M}_n(A)$, there is a projection $q_0 \in \mathbb{M}_n(B)$ with $\|p - q_0\| \leq 2\gamma$ by Proposition 3.1 (i). By definition $\Phi_0([p]_0) = [q]_0$, where q is a projection in $\mathbb{M}_n(\tilde{B})$ with $\|p - q_0\| < \sqrt{2}\gamma$. Thus $\|q - q_0\| < 1$ so $[q]_0 = [q_0]_0 \in K_0(B)^+$. □

Combining Khoshkam's work with our analysis in section 4 gives the following general result, showing that sufficiently close algebras have isomorphic (ordered) K-theories provided one algebra has finite length.

Corollary 5.3. *Let A and B be C*-subalgebras of a C*-algebra C. Suppose that A has length at most ℓ and length constant at most K. Suppose further that $d(A,B)=\gamma$ for some γ satisfying (4.49) and such that the μ of Corollary 4.6 satisfies $\mu < 1/(2+\sqrt{2})$. Then $\Phi_* : K_*(A) \to K_*(B)$ is an isomorphism preserving the order structure and scale on K_0.*

In the finite case K-theory alone is not sufficient to classify a large class of simple separable nuclear C*-algebras and so the Elliott invariant has been expanded to include tracial information. In the (finite) non-unital case, the Elliott invariant consists of the data

$$((K_0(A), K_0(A)^+, \Sigma(A)), K_1(A), T(A), \rho_A), \tag{5.1}$$

where $T(A)$ is the cone of positive tracial functionals on A and ρ_A the natural pairing $K_0(A) \times T(A) \to \mathbb{R}$ given by extending $([p]_0, \tau) \mapsto (\tau \otimes \mathrm{tr}_n)(p)$, when p is a projection in $A \otimes \mathbb{M}_n$ and tr_n is the unique trace on $\mathbb{M}_n$ with $\mathrm{tr}_n(I_{\mathbb{M}_n}) = n$. We refer to [40] for a discussion of these invariants and an account of the classification programme.

In the rest of this section our objective is to examine traces on close C*-algebras. Suppose we are given a near inclusion $A \subset_\gamma B$ of unital C*-algebras which share the same unit and a tracial state τ on B. This induces a state $K_0(\tau)$ on $K_0(B)$. If A has finite length and γ is sufficiently small, then we can obtain a state $K_0(\tau) \circ \Phi_0$ on $K_0(A)$ by composing with Khoshkam's map $\Phi_0 : K_0(A) \to K_0(B)$ of Theorem 5.1. By Theorem 3.3 of [3], $K_0(\tau) \circ \Phi_0$ arises from a quasitrace A. If A is additionally assumed to be exact, then Haagerup's result [20] shows that this quasitrace is actually a trace on A. In this way we obtain a map from the tracial states on B into those on A. We prefer a more direct approach passing through the bidual, which gives an isomorphism between the trace states of close C*-algebras without assuming exactness.

Lemma 5.4. *Suppose that A and B are C*-subalgebras of some C*-algebra C such that $d(A,B) = \gamma$ for some $\gamma < 1/2200$. Then there exists an affine isomorphism $\Psi : T(B) \to T(A)$. Furthermore, given $n \in \mathbb{N}$ and projections $p \in A \otimes \mathbb{M}_n$ and $q \in B \otimes \mathbb{M}_n$ with $\|p - q\| < 1/2 - 10\gamma$, then*

$$(\Psi(\tau) \otimes \mathrm{tr}_n)(p) = (\tau \otimes \mathrm{tr}_n)(q), \quad \tau \in T(B). \tag{5.2}$$

Proof. Working in the universal representation of C, the weak closures M and N of A and B are isometrically isomorphic to A^{**} and B^{**} respectively. Lemma 5 of [23] gives $d(M,N) \le d(A,B)$. By Lemma 3.4, there exists a unitary $u \in (Z(M) \cup Z(N))''$ such that $Z(uMu^*) = Z(N)$ and $\|u - I\| \le 5\gamma$. Write $A_1 = uAu^*$ and $M_1 = uMu^*$ so that $d(M_1, N) \le 11\gamma$. Since $11\gamma < 1/10$, Lemma 3.5 applies. In particular, there is a projection z_{fin} in $Z(M_1) = Z(N)$ such that Mz_{fin} and Nz_{fin} are both finite while $M(I - z_{\mathrm{fin}})$ and $N(I - z_{\mathrm{fin}})$ are both purely infinite.

Now take a positive linear tracial functional τ on B. There is a unique extension τ^{**} to a normal positive linear tracial functional on N. This must factor through the finite part of N and the centre valued trace on this algebra (see [24, 8.2]). It follows that there is a unique positive normal functional ϕ_τ on $Z(N)z_{\mathrm{fin}}$ such that

$$\tau^{**}(x) = \phi_\tau(\mathbb{T}_{Nz_{\mathrm{fin}}}(x)), \quad x \in N, \tag{5.3}$$

where $\mathbb{T}_{Nz_{\rm fin}}$ is the centre valued trace on $Nz_{\rm fin}$. Then $\phi_\tau\circ\mathbb{T}_{M_1z_{\rm fin}}$ defines a normal positive tracial functional on M_1. Define a positive linear functional τ_1 on A_1 by restricting this functional to A_1, i.e.

$$\tau_1=(\phi_\tau\circ\mathbb{T}_{M_1z_{\rm fin}})|_{A_1}. \tag{5.4}$$

Let $\Psi(\tau):A\to\mathbb{C}$ be given by $\Psi(\tau)(x)=\tau_1(uxu^*)$ so $\Psi(\tau)$ is a positive tracial functional on A. The map Ψ is evidently affine. Since every positive tracial functional on A_1 extends uniquely to M_1, where it factors through $M_1z_{\rm fin}$ and $\mathbb{T}_{M_1z_{\rm fin}}$ the map Ψ is onto and so an affine isomorphism between the positive tracial functionals on B and those on A.

We now establish (5.2). Fix $n\in\mathbb{N}$ and projections $p\in A\otimes\mathbb{M}_n$ and $q\in B\otimes\mathbb{M}_n$ with $\|p-q\|<1/2-10\gamma$. Note that $\tau\otimes{\rm tr}_n$ is the restriction of $(\phi_\tau\otimes{\rm tr}_n)\circ\mathbb{T}_{Nz_{\rm fin}\otimes\mathbb{M}_n}$ to $B\otimes\mathbb{M}_n$, while τ_1 gives rise to $\tau_1\otimes{\rm tr}_n$ on $A_1\otimes\mathbb{M}_n$ which is given by restricting $((\phi_\tau\otimes{\rm tr}_n)\circ\mathbb{T}_{M_1z_{\rm fin}\otimes\mathbb{M}_n})$ to $A_1\otimes\mathbb{M}_n$. Then

$$\|(u\otimes I_{\mathbb{M}_n})p(u\otimes I_{\mathbb{M}_n})^*-q\|\le 2\|u-I\|+\|p-q\|<1/2. \tag{5.5}$$

As $d(M_1,N)\le 11\gamma<1/200$, Lemma 3.6 applies and so

$$\mathbb{T}_{M_1z_{\rm fin}\otimes\mathbb{M}_n}(upu^*)=\mathbb{T}_{Nz_{\rm fin}\otimes\mathbb{M}_n}(q). \tag{5.6}$$

Thus $(\tau\otimes{\rm tr}_n)(q)=(\tau_1\otimes{\rm tr}_n)(upu^*)=(\Psi(\tau)\otimes{\rm tr}_n)(p)$. □

Combining the previous lemma with the results of Section 4, it follows that sufficiently close C*-algebras have the same Elliott invariant when one has finite length.

Theorem 5.5. *Suppose that A and B are C*-subalgebras of some C*-algebra C. Write $d(A,B)=\gamma$ and suppose that A has length at most ℓ with length constant at most K. Suppose γ satisfies the inequalities (4.49) and the μ of Corollary 4.6 satisfies $\mu<1/(2+\sqrt{2})$. Then there exist isomorphisms $\Phi_*:K_*(A)\to K_*(B)$ between the ordered K-theories of A and B, which preserve the scale and an affine isomorphism $\Psi:T(B)\to T(A)$ such that*

$$\rho_A(x,\Psi(\tau))=\rho_B(\Phi_0(x),\tau)),\quad x\in K_0(A),\ \tau\in T(B). \tag{5.7}$$

6. Kirchberg algebras and real rank zero

A Kirchberg C*-algebra is defined by the properties of being nuclear, purely infinite, simple, and separable. One of the crowning achievements of Elliott's classification programme is the theorem of Kirchberg and C. Phillips which shows that such algebras which also satisfy the UCT are classifiable by their K-theory [29, 27]. In this section we make use of this result to examine perturbation theory for such C*-algebras. Our objective is to show that any C*-algebra sufficiently close to a Kirchberg algebra is again a Kirchberg algebra. By earlier results of Christensen and J. Phillips this amounts to showing that a C*-algebra sufficiently close to a simple separable purely infinite algebra is itself purely infinite. This result can be established directly, but we prefer to use a characterisation due to Zhang [41]. He shows that a simple C*-algebra is purely infinite if and only if it is real rank zero and every non-zero projection is infinite. We will show that these two properties transfer to sufficiently close algebras. This has the advantage of additionally establishing a perturbation result for the property of being real rank zero, which is also of importance in the classification programme for finite C*-algebras. We begin with the second of the two properties in Zhang's characterisation above.

Lemma 6.1. *Let A and B be C*-subalgebras of a C*-algebra C with $d(A,B) < 1/14$. If every non-zero projection in A is infinite, then every non-zero projection in B is infinite.*

Proof. Take $\gamma > 0$ with $d(A,B) < \gamma < 1/14$. Given a non-zero projection p in B, use Proposition 3.1 (i) to find a projection $q \in A$ with $\|p-q\| < 2\gamma$ so that q is non-zero. By hypothesis q is infinite so there exists a partial isometry $v \in A$ with $vv^* < q$ and $v^*v = q$. Take an operator b_0 in the unit ball of B with $\|b_0 - v\| < \gamma$ and define $b = pb_0p$. Then

$$\begin{aligned}\|v-b\| &\le \|qvq - pvq\| + \|pvq - pb_0q\| + \|pb_0q - pb_0p\| \\ &\le \|p-q\| + \|v-b_0\| + \|p-q\| \le 5\gamma. \qquad (6.1)\end{aligned}$$

Now represent C on a Hilbert space $\mathcal{H}$. Let $y = b + (I-p)$ and $x = v + (I-q)$ so that $\|y-x\| \le \|v-b\| + \|p-q\| < 7\gamma$. Since x is an isometry, we have

$$(1-7\gamma)\|\xi\| \le \|y\xi\| = \||y|\xi\|, \quad \xi \in \mathcal{H} \qquad (6.2)$$

and so $|y| \ge (1-7\gamma)I$. As $\gamma < 1/14$, the operator $|y|$ is invertible in $C^*(B,I)$. Thus, in the polar decomposition $y = w_0|y|$, the partial isometry w_0 lies in $C^*(B,I)$. If y were invertible then we would have $\|y^{-1}\| \le (1-7\gamma)^{-1}$ from (6.2). Then

$$\|I - y^{-1}x\| = \|y^{-1}(y-x)\| < 7\gamma/(1-7\gamma) < 1, \qquad (6.3)$$

since $\gamma < 1/14$. Thus $y^{-1}x$ is invertible so x is invertible. This contradiction shows that y is not invertible, and consequently w_0 is not a unitary. Then $w = pw_0p$ is a partial isometry in B and $ww^* \ne p$ while $w^*w = p$. Thus p is an infinite projection in B. □

Recall that a C*-algebra A has *real rank zero* if the self-adjoint operators in A of finite spectrum are dense in the self-adjoint operators of A. Equivalently, a C*-algebra A has real rank zero if and only if it has the *hereditary property* that every hereditary C*-subalgebra of A has an approximate unit of projections, see [2, V.3.29]. This latter condition is amenable to perturbation arguments.

Lemma 6.2. *Suppose that A and B are C*-subalgebras of a unital C*-algebra C, and let γ satisfy $d(A,B) < \gamma < 1/8$. If A has real rank zero, then for all $k \ge 0$ in B, there exists a projection $p \in \overline{kBk}$ such that*

$$\|k - kp\| \le 7\gamma\|k\| \le (7/8)\|k\|. \qquad (6.4)$$

Proof. Suppose that C is faithfully represented on $\mathcal{H}$. Let $k \ge 0$ lie in B, and assume without loss of generality that $\|k\| = 1$. Then choose $h \in A_{\text{s.a.}}$ such that $\|h\| \le 1$, $\|h-k\| < \gamma$, and the spectrum of h is finite (and is contained in $[-\gamma, 1]$). Let $q \in A$ be the spectral projection of h for the interval $[4\gamma, 1]$ and choose, by Proposition 3.1 (i), a projection $r \in B$ with $\|r-q\| < 2\gamma$. Then

$$\begin{aligned}\|k(I-r)\| &\le \|(k-h)(I-r)\| + \|h(q-r)\| + \|h(I-q)\| \\ &< \gamma + 2\gamma + 4\gamma = 7\gamma < 7/8. \qquad (6.5)\end{aligned}$$

For a unit vector $\xi \in r\mathcal{H}$, we have the inequality

$$\|k\xi\| \ge \|hq\xi\| - \|h(r-q)\xi\| - \|(h-k)\xi\|$$

$$
\begin{aligned}
&\geq \|q\xi\|/2 - 2\gamma - \gamma \\
&\geq \|r\xi\|/2 - 4\gamma = 1/2 - 4\gamma > 0. \qquad (6.6)
\end{aligned}
$$

Thus kr is bounded below on $r\mathcal{H}$, so the operator $t = |kr| = (rk^2r)^{1/2}$ is invertible on $r\mathcal{H}$. Let $kr = vt$ be the polar decomposition, where $v \in \mathbb{B}(\mathcal{H})$ is a partial isometry. Using the invertibility of t on $r\mathcal{H}$, it is easy to check that v is the norm limit of the sequence $\{v_n\}_{n=1}^{\infty}$ whose elements are defined by $v_n = kr(t + n^{-1}I)^{-1}$ for $n \geq 1$. This shows that $v \in B$, so the range projection $p = vv^*$ of kr also lies in B. Moreover, $p \in \overline{kBk}$ since this algebra contains each element $v_nv_n^*$, $n \geq 1$. By construction, $(I-p)kr = 0$, so

$$\|k(I-p)\| = \|(I-p)k\| = \|(I-p)k(I-r)\| < 7\gamma < 7/8, \tag{6.7}$$

from (6.5). □

Theorem 6.3. *Let A and B be C^*-subalgebras of a common C^*-algebra C with $d(A,B) < 1/8$. Then A has real rank zero if, and only if, B has real rank zero.*

Proof. Fix a hereditary C^*-subalgebra E of B. Given $x_1, \ldots, x_n \in E$ and $\varepsilon > 0$, we must find a projection $p \in E$ with $\|x_i - x_ip\| < \varepsilon$ for all i. As in the proof of [15, Theorem V.7.3], by taking $x = \sum_i x_i^*x_i$, we have

$$\|x_i - x_ip\|^2 = \|(I-p)x_i^*x_i(I-p)\| \leq \|(I-p)x(I-p)\|. \tag{6.8}$$

Therefore it suffices to consider a single positive element $x \in E$ and find a projection $p \in E$ with $\|(I-p)x(I-p)\| < \varepsilon$.

Let $E_0 = \overline{xBx}$, the hereditary subalgebra of B generated by x so that $E_0 \subseteq E$. Use Lemma 6.2 to find a projection $p_1 \in E_0$ with $\|x - xp_1\| \leq (7/8)\|x\|$ and so

$$\|(I-p_1)x(I-p_1)\| \leq \|x - xp_1\| \leq (7/8)\|x\|. \tag{6.9}$$

The element $(I-p_1)x(I-p_1)$ is a positive element of E_0 and so generates a hereditary subalgebra $E_1 = \overline{(I-p_1)x(I-p_1)B(I-p_1)x(I-p_1)}$ of E_0. We can then use Lemma 6.2 again to find a projection $p_2 \in E_1$ with

$$\|(I-p_1)x(I-p_1)(I-p_2)\| < (7/8)\,\|(I-p_1)x(I-p_1)\| < (7/8)^2\|x\|. \tag{6.10}$$

Since $E_1 \subset \overline{(I-p_1)B(I-p_1)}$, it follows that $p_2 \leq 1-p_1$. Thus $(I-p_1)(I-p_2) = I-(p_1+p_2)$ and we have

$$\|(I-(p_1+p_2))x(I-(p_1+p_2))\| < (7/8)^2\|x\|. \tag{6.11}$$

If we continue in this fashion, we will eventually find orthogonal projections $p_1, \ldots, p_n \in E$ such that

$$\|(I-(p_1+\cdots+p_n))x(I-(p_1+\cdots+p_n))\| < (7/8)^n < \varepsilon, \tag{6.12}$$

exactly as required. □

We are now in a position to prove the main result of this section using the previous results, work of J. Phillips and work of the first named author.

Theorem 6.4. *Let A and B be C^*-subalgebras of a C^*-algebra C with $d(A,B) < 1/101$. If A is a Kirchberg algebra, then B is also a Kirchberg algebra.*

Proof. Since $d(A,B) < 1/80$ and A is simple, Lemma 1.2 of [31] shows that B is simple. Since $d(A,B) < 1/101$ and A is nuclear, Theorem 6.5 of [12] shows that B is also nuclear. Since $d(A,B) < 1/2$, B is separable (this is folklore, see the comments in the proof of [12, Theorem 6.1] for example, or see [14] for a proof). Zhang's characterisation of purely infinite C*-algebras shows that A is real rank zero and every non-zero projection of A is infinite so Theorem 6.3 and Lemma 6.1 show that B has the same properties so is purely infinite. □

The following corollary is immediate from the Kirchberg-Phillips classification theorem [29] and Khoshkam's result [26] that sufficiently close nuclear C*-algebras have isomorphic K-theory.

Corollary 6.5. *Let A and B be C*-algebras of a C*-algebra C with $d(A,B) < 1/101$ and suppose that A and B satisfy the UCT. If A is a Kirchberg algebra, then A is isomorphic to B.*

7. Questions

The most important question in the perturbation theory of operator algebras is undoubtably Kadison and Kastler's original conjecture [23] specialised to the cases of von Neumann algebras or separable C*-algebras (thus excluding the examples from [6]). It would be very interesting to find any class $\mathcal{A}$ of non-injective von Neumann algebras or separable but non-nuclear C*-algebras for which algebras sufficiently close to an algebra A in $\mathcal{A}$ are isomorphic to A. We end the paper with three other natural questions which arose during our investigations.

Question 7.1. Does there exists a constant $\gamma_0 > 0$ such that if A and B are C*-subalgebras of some C*-algebra C with $d(A,B) < \gamma_0$, then A is exact if and only if B is exact?

Question 7.2. Suppose that $\ell \geq 1$ and $K \geq 1$ are given. Does there exist a constant $\gamma_{\ell,K} > 0$ such that if A and B are C*-subalgebras of some C*-algebra C with $d(A,B) < \gamma_{\ell,K}$ and A has length at most ℓ with length constant at most K, then there is a natural isomorphism $\mathrm{Ext}(A) \to \mathrm{Ext}(B)$?

More generally one could also ask how KK-theory behaves in the context of close C*-algebras with finite length.

Question 7.3. Are higher values of the real rank stable under small perturabtions? What happens to the stable rank under small perturbations?

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