Perturbations of nuclear C^*-algebras Version 1.4 — not for distribution

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Abstract

Kadison and Kastler introduced a natural metric on the collection of all C^*-subalgebras of the bounded operators on a separable Hilbert space. They conjectured that sufficiently close algebras are unitarily conjugate. We establish this conjecture when one algebra is separable and nuclear. We also consider one-sided versions of these notions, obtaining embeddings from certain near inclusions involving separable nuclear C^*-algebras. At the end of the paper we demonstrate how our methods lead to improved characterisations of some of the types of algebras that are of current interest in the classification programme.

1 Introduction

Kadison and Kastler initiated the uniform perturbation theory of operator algebras in [31]. They considered the collection of all operator algebras acting on a fixed separable Hilbert space and equipped this set with a metric induced by the Hausdorff distance between the unit balls. In general terms, this means that two algebras A and B are close if each element in the unit ball of one algebra can be closely approximated by an element in the unit ball of the other algebra. They conjectured that suitably close operator algebras must be unitarily conjugate and this has been verified in various situations. For von Neumann algebras, the injective case was settled in [9] and [48]. It is also known to be true for certain special classes of separable nuclear C*-algebras, including the separable AF algebras [11, 33, 43, 44]. In this paper, our primary purpose is to give an affirmative answer to Kadison and Kastler's question when one algebra is a separable nuclear C*-algebra, a hypothesis which automatically implies the same property for nearby algebras. In this introduction we discuss these results in qualitative terms; precise estimates will be given in the main text.

The original question of Kadison and Kastler leads naturally to the more general one-sided situation of near inclusions of two C*-algebras A and B introduced by the first named author in [11]. Heuristically, this means that every element of the unit ball of A can be approximated closely by some element of the unit ball of B, but we do not require a reverse approximation. The reformulation of Kadison and Kastler's question in this context is to ask whether an embedding of A into B can be found whenever A is very nearly included in B. We are also able to resolve this problem positively in three situations:

(i) when A is separable, has finite nuclear dimension and no hypotheses are imposed on B;

(ii) when A is separable, unital and has approximately inner half flip and no hypotheses are imposed on B;

(iii) when A is separable and both algebras are nuclear.

The notion of *nuclear dimension* was recently introduced by the last named author and Zacharias in [56] and extends the *decomposition rank* defined by Kirchberg and the last named author in [36]. Noncommutative topological dimension is particularly relevant in Elliott's programme to classify nuclear C*-algebras by K-theoretic invariants; in fact, all separable simple nuclear C*-algebras presently covered by known classification theorems have finite nuclear dimension.

The first positive answer to Kadison and Kastler's question was given independently by the first named author [8] and Phillips [42] when one of A or B is a type I von Neumann algebra. Combining the results of [9] with those of Raeburn and Taylor [48], the question was subsequently answered positively when one of A or B is an injective von Neumann algebra. This work was formulated variously in terms of injectivity, hyperfiniteness and Property P, since it predates Connes' work on the equivalence of these notions [13]. The most general statement of these results was later given in [11, Corollary 4.4], where it was shown that if A is an injective von Neumann algebra acting non-degenerately on some Hilbert space H and B is a C*-subalgebra of $\mathbb{B}(H)$, which is sufficiently close to A, then there exists a unitary operator $u \in (A \cup B)''$ with $uAu^* = B$. Furthermore, this unitary can be taken to be close to the identity, by which we mean that $\|u - I_H\|$ can be controlled in terms of the distance between A and B. A continuous path of unitaries u_t connecting $u_0 = I_H$ to $u_1 = u$ then leads to a continuous deformation $A_t = u_t A u_t^*$, $0 \leq t \leq 1$, of $A_0 = A$ into $A_1 = B$. In this way we can regard B as a *perturbation* of A. The other situation in which Kadison and Kastler's question has been resolved for von Neumann algebras is for two close von Neumann subalgebras A and B of a common *finite* von Neumann algebra [10], where again there is a unitary $u \in (A \cup B)''$ close to the identity with $uAu^* = B$. Perturbation problems have also been studied in the ultraweakly closed non-self-adjoint setting [16, 38, 46].

We now turn to perturbation results for C*-algebras. A consequence of the results for injective von Neumann algebras described above and Connes' work [13] is that any C*-algebra which is sufficiently close to a nuclear C*-algebra is also nuclear, [11, Theorem 6.5]. Perturbation results of the form that two sufficiently close C*-algebras A and B must be unitarily conjugate by a unitary $u \in (A \cup B)''$ have been established under the following sets of hypotheses:

(i) Either A or B is separable and AF [11] (see also [43]).

(ii) Either A or B is continuous trace and either unital or separable [44].

Perturbation results have also been established for certain extensions of the C*-algebras in the classes above by Khoshkam in [33]. On the near inclusion side, Johnson has obtained embeddings from near inclusions $A \subset_\gamma B$ when B is a separable subhomogeneous C*-algebra, [30]. In contrast to the injective von Neumann algebra case, we cannot always expect to obtain an estimate which controls $\|u - I_H\|$ in terms of the distance between A and B. Indeed, Johnson [28] has constructed two faithful representations of $C[0,1] \otimes \mathbb{K}$ on some separable Hilbert space whose images A and B are unitarily conjugate and can be taken to be arbitrarily close, but for which there is no isomorphism $\theta : A \to B$ with $\|\theta(x) - x\| \leq \|x\|/70$ for $x \in A$. Here, $\mathbb{K}$ denotes the algebra of compact operators.

Kadison and Kastler's original paper [31] shows that sufficiently close von Neumann algebras have the same type decompositions. Inspired by this work, various authors have examined properties of close operator algebras: [41] shows that close C*-algebras have isomorphic lattices of ideals and homeomorphic spectra, while the work of Khoshkam, [32, 33, 34] shows that sufficiently close nuclear C*-algebras have isomorphic K-groups and so are KK-equivalent if both satisfy the UCT. The first four authors consider problems of this nature related to the similarity length in [12].

In full generality, Kadison and Kastler's question has a negative answer for close nuclear

C*-algebras. In [6] Choi and the first named author gave examples of arbitarily close nuclear C*-algebras which are not $*$-isomorphic. These examples are even approximately finite dimensional, but they are not separable. Our first main theorem, stated qualitatively below and quantitatively as Theorem 4.3, shows that separability is the only obstruction to an isomorphism result for nuclear C*-algebras.

Theorem A. *Let H be a Hilbert space and let A and B be* C*-*subalgebras of* $\mathbb{B}(H)$. *If A is separable and nuclear and B is sufficiently close to A, then there is a surjective $*$-isomorphism $\alpha : A \to B$.*

Johnson's examples [28] show that we cannot demand that the isomorphism of Theorem A is uniformly close to the inclusion of A into $\mathbb{B}(H)$. However, our methods do allow us to specify a finite set X of the unit ball of A and construct an isomorphism $\alpha : A \to B$ which almost fixes X. This additional control is crucial in obtaining a unitary in the von Neumann algebra $\mathrm{W}^*(A, B, I_H)$ which conjugates A into B (provided the underlying Hilbert space is separable). Our second main theorem accomplishes this and so gives a complete answer to Kadison and Kastler's question when one algebra is a separable nuclear C*-algebra. The quantitative version of this theorem is Theorem 5.4 in the text.

Theorem B. *Let H be a separable Hilbert space and let A and B be* C*-*subalgebras of* $\mathbb{B}(H)$. *If A is separable and nuclear and B is sufficiently close to A, then there exists a unitary operator $u \in (A \cup B)''$ with $uAu^* = B$.*

To put the techniques involved in Theorem A into context, it is helpful to first discuss the perturbation results for injective von Neumann algebras from [9]. Given two close injective von Neumann algebras $M, N \subseteq \mathbb{B}(H)$, take a conditional expectation $\Phi : \mathbb{B}(H) \to N$. If we restrict Φ to M we obtain a completely positive and contractive map (cpc map) $\Phi|_M$ from M into N which is uniformly close to the inclusion of M into $\mathbb{B}(H)$. Then $\Phi|_M$ is almost a multiplicative map, in that

$$\sup_{u \in \mathcal{U}(M)} \|\Phi(u)\Phi(u^*) - 1\|$$

is small. The main idea behind [9] is to use injectivity of M to show that a completely positive contractive normal map $\Psi : M \to P$ which is almost multiplicative must be uniformly close to a $*$-homomorphism. The $*$-homomorphism is obtained from Ψ by integrating the Stinespring projection for Ψ over the unitary groups of an increasing family of dense finite dimensional subalgebras of M and so this can be regarded as an averaging result. Modulo technicalities regarding normality, this procedure can be applied to $\Phi|_M$ to obtain a $*$-homomorphism from

M onto N which is close to the inclusion of M into $\mathbb{B}(H)$. A second averaging argument is then used to show that such maps are spatially implemented. The general question of when approximately multiplicative maps between Banach algebras are close to multiplicative maps has been studied by Johnson, [29].

Now suppose that we have two close C*-algebras A and B on some Hilbert space H and that one of these algebras is separable and nuclear. Results from [11] show that both algebras must then be separable and nuclear. We do not have a conditional expectation onto B which we can use to obtain a cpc map $A \to B$ uniformly close to the inclusion of A into $\mathbb{B}(H)$, but we can use Arveson's extension theorem, [1], to produce completely positive maps from $A \to B$ which approximate this inclusion on finite subsets of the unit ball of A (see Proposition 2.16). Accordingly we look to develop point-norm versions of the averaging techniques from [9] for nuclear C*-algebras. This is the subject of Section 3 and the critical ingredient is the *amenability* of a nuclear C*-algebra established by Haagerup in [22]. The first of these lemmas (Lemma 3.2) enables us to obtain cpc maps $A \to B$ which are almost multiplicative on a a finite set Y of the unit ball of A up to a specified tolerance ε from cpc maps $A \to B$ which are almost multiplicative on some finite set X of the unit ball of A up to a fixed tolerance of $1/17$. The set X can be thought of as a Følner set for Y and ε. We can apply this lemma to the cpc maps arising from Arveson's extension theorem to produce cpc maps $A \to B$ which are almost multiplicative on arbitrary finite sets of the unit ball of A up to an an arbitrary small tolerance. The next stage is to construct a $*$-homomorphism $A \to B$ as a point norm limit of these maps. Our second lemma (Lemma 3.4) enables us to conjugate these maps by unitaries to ensure this point norm convergence. An intertwining argument (Lemma 4.1) inspired by [11, Theorem 6.1] and those in the classification programme gives a $*$-isomorphism $A \to B$. At this point separability of A is crucial.

It is perhaps worth noting that we use a variety of characterisations of nuclearity in the course of the proof of Theorem A. The equivalence between nuclearity of A and injectivity of A^{**} is used in [11] to show that nuclearity transfers to close C*-algebras. The characterization by the completely positive approximation property, due to Choi and Effros [7], allows us to find cpc maps $A \to B$, and the amenability of nuclear C*-algebras, [22], is an essential ingredient for converting these maps into a $*$-isomorphism $A \to B$.

We now turn to Theorem B. Earlier results from [11] and elementary techniques enable us to reduce to the situation of two close separable nuclear C*-algebras A and B which are non-degenerately represented on some separable Hilbert space and have the same ultraweak closure. By repeatedly applying Theorem A and Lemma 3.4, we can construct a sequence

$\{u_n\}_{n=1}^\infty$ of unitaries so that $\lim \mathrm{Ad}(u_n)$ converges in point norm topology to an isomorphism between A and B. If this sequence was *-strongly convergent to a unitary, then this unitary would implement a spatial isomorphism between A and B. However there is no reason why this should be so; indeed the sequence $\{u_n\}_{n=1}^\infty$ is not even guaranteed to have a non-zero ultraweak accumulation point. Instead we explicitly modify the sequence $\{u_n\}_{n=1}^\infty$ to force the required *-strong convergence while still retaining control over the point norm limit of $\lim \mathrm{Ad}(u_n)$. This adjustment procedure is inspired by Bratteli's classification of representations of AF algebras [3, Section 4], which in turn builds upon the work of Powers for UHF algebras in [47]. A key observation in this work is that if A is a C*-algebra non-degenerately represented on H with ultraweak operator closure $M = A''$ and F is a finite dimensional C*-subalgebra of A, then $(F' \cap A)'' = F' \cap M$. The Kaplansky density theorem then enables unitaries in M commuting with F to be approximated in *-strong topology by unitaries in A commuting with F. In Lemma 3.7 we prove a Kaplansky density theorem for unitaries with a uniform spectral gap which approximately commute with suitable finite sets, again using amenability of nuclear C*-algebras. Using this result, a technical argument enables us to make the suitable adjustments to the sequence of unitaries $\{u_n\}_{n=1}^\infty$ described above to prove Theorem B. This is the subject of Section 5.

The procedure used to obtain Theorem A forms the basis of our near inclusion results. Given a sufficiently small near inclusion of A in B with A separable and nuclear, our intertwining argument gives an embedding $A \hookrightarrow B$ whenever we can produce cpc maps $A \to B$ which almost fix finite sets in the unit ball of A (up to a tolerance depending on the near inclusion constant). In two situations the existence of these maps is immediate: when B is nuclear the maps are given by Arveson's extension theorem, while if A is unital and has approximately inner half flip the maps are given by [11, Proposition 6.7]. The third, and least restrictive, hypothesis under which we can produce these maps is when A has finite nuclear dimension. Here further work is required to construct our cpc maps. Using the completely positive approximation property for nuclear C*-algebras, [7], we can approximately factorise the identity map on a nuclear C*-algebra A through matrix algebras $\mathbb{M}_k$ using cp maps. When A has finite nuclear dimension n, these factorisations have additional structure: the maps $\mathbb{M}_k \to A$ decompose as the sum of $(n+1)$ cpc maps which preserve orthogonality (order zero maps). The main technical lemma of Section 6 is a perturbation result for order zero maps $\mathbb{M}_k \to A$ producing a nearby cpc map $\mathbb{M}_k \to B$ whenever A is nearly contained in B. Combining this with the intertwining argument gives the theorem below, which is stated quantitatively as Theorem 6.10.

Theorem C. *Let H be a Hilbert space and let A and B be* C*-*subalgebras of* $\mathbb{B}(H)$. *If A is separable and has finite nuclear dimension and is nearly contained in B, then there is an embedding $A \hookrightarrow B$.*

Our paper is organised as follows. In the next section we set out the notation used in the paper, establish some basic facts and recall a number of results from the literature for the reader's convenience. In Section 3 we discuss amenability for C*-algebras and give our point norm averaging results and Kaplansky density result. Section 4 contains the intertwining argument (Lemma 4.1) which combines the averaging results of Section 3 to prove Theorem A (Theorem 4.2, Theorem 4.3). We also give two near inclusion results and some other consequences at this stage. Section 5 contains the proof of Theorem B. We begin Section 6 with a review of order zero maps between C*-algebras and prove our perturbation theorem for these maps, using this to establish Theorem C. We also recall the salient facts about the nuclear dimension for the readers convenience. We end the paper in Section 7 with some sample applications of our techniques to other situations. Firstly we give a strengthened local characterisation of inductive limits of finitely presented weakly semiprojective nuclear C*-algebras. Secondly we revisit our perturbation theorem for order zero maps to show that the resulting map can also be taken of order zero, and we close by presenting an improved characterisation of $\mathcal{Z}$-stability for nuclear C*-algebras.

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2 Preliminaries

In this section we establish notation and recall some standard results. We begin with Kadison and Kastler's metric on operator algebras from [31].

Definition 2.1. Let C be a C*-algebra. We equip the collection of C*-subalgebras of C with a metric d by applying the Hausdorff metric to the unit balls of these subalgebras. That is $d(A, B) < \gamma$ if, and only if, for each x in the unit ball of either A or B, there exists y in the unit ball of the other algebra with $\|x - y\| < \gamma$.

In the second half of the paper, we shall also use the notion of near containment introduced in [11].

Definition 2.2. Suppose that A and B are C*-subalgebras of a C*-algebra C and let $\gamma > 0$. Write $A \subseteq_\gamma B$ if given x in the unit ball of A there exists $y \in B$ with $\|x - y\| \leq \gamma$. Note that we do not require that y lie in the unit ball of B. Write $A \subset_\gamma B$ if there exists $\gamma' < \gamma$ with $A \subseteq_{\gamma'} B$. For subsets X and Y of the unit ball of a C*-algebra with X finite, the notation $X \subseteq_\gamma Y$ will mean that each $x \in X$ has a corresponding element $y \in Y$ satisfying $\|x - y\| \leq \gamma$ and $X \subset_\gamma Y$ will mean that each $x \in X$ has a corresponding element $y \in Y$ such that $\|x - y\| < \gamma$.

Remark 2.3. An equivalent notion of distance between C*-algebras was introduced in [11] using near containments. Define $d_0(A, B)$ to be the infimum of all γ for which $A \subseteq_\gamma B$ and $B \subseteq_\gamma A$. The difference between d_0 and d arises from the fact that we do not require the y in Definition 2.2 to lie in the unit ball of B. It is immediate that $d(A, B) \leq d_0(A, B) \leq 2d(A, B)$, but the function d_0 does not appear to satisfy the triangle inequality; for this reason we prefer to work with the metric d.

The following proposition is folklore. We will use it to obtain surjectivity of our isomorphisms.

Proposition 2.4. *Let A and B be* C**-algebras on a Hilbert space with $A \subseteq B$ and $B \subset_1 A$. Then $A = B$.*

The next proposition records some standard estimates. The first statement follows from Lemma 2.7 of [8] and the second can be found as [40, Lemma 6.2.1].

Proposition 2.5. *1. Let x be an operator on a Hilbert space H with $\|x - I_H\| < 1$ and let u be the unitary in the polar decomposition of x. Then $\|u - I_H\| \leq \sqrt{2}\|x - I_H\|$.*
2. Let p and q be projections in a unital C**-algebra A with $\|p - q\| < 1$. Then there is a unitary $u \in A$ with $upu^* = q$ and $\|u - 1\| \leq \sqrt{2}\|p - q\|$.*

On occasion we will need to lift a near inclusion $A \subset_\gamma B$ to a near inclusion of a tensor product $A \otimes D \subset_\mu B \otimes D$. This can be done when D is nuclear and A has Kadison's similarity property [11, Theorem 3.1]. The version of these facts below is taken from 2.10-2.12 of [12] specialised to the case when A is nuclear (and so has length 2 with length constant at most 1).

Proposition 2.6 ([12, Corollary 2.12])**.** *Let $A, B \subseteq \mathbb{B}(H)$ be* C**-algebras with $A \subset_\gamma B$ for some $\gamma > 0$ and A nuclear. Given any nuclear* C**-algebra D, we have $A \otimes D \subseteq_{2\gamma+\gamma^2} B \otimes D$ inside $\mathbb{B}(H) \otimes D$.*

We also need a version of the previous proposition for finite sets which we state in the context of amplification by matrix algebras, see [12, Remark 2.11].

Proposition 2.7. *Let A be a nuclear* C**-algebra on some Hilbert space H. Then for each $n \in \mathbb{N}$ and each finite subset X of the unit ball of $A \otimes \mathbb{M}_n$, there is a finite subset Y of the unit ball of A with the following property. Whenever B is another* C**-algebra on H with $Y \subseteq_\gamma B$ for some $\gamma > 0$, then $X \subseteq_{2\gamma+\gamma^2} B \otimes \mathbb{M}_n$. In particular if $A \subset_\gamma B$, then $A \otimes \mathbb{M}_n \subset_{2\gamma+\gamma^2} B \otimes \mathbb{M}_n$ for all $n \in \mathbb{N}$.*

Remark 2.8. Note that the same result holds for rectangular matrices, i.e. under the same hypotheses as the proposition, given a finite subset X of the unit ball of $\mathbb{M}_{1\times r}(A \otimes \mathbb{M}_n)$ for some $r, n \in \mathbb{N}$, then there is a corresponding finite subset Y of the unit ball of A such that whenever $Y \subseteq_\gamma B$, then $X \subseteq_{2\gamma+\gamma^2} \mathbb{M}_{1\times r}(B \otimes \mathbb{M}_n)$. This follows by working in $\mathbb{M}_r(\mathbb{B}(H) \otimes \mathbb{M}_n) \cong \mathbb{B}(H) \otimes \mathbb{M}_{rn}$ and then cutting down after the approximations have been performed.

In Sections 4 and 5 we need to transfer nuclearity and separabilty to close subalgebras, so that our main results only require hypotheses on one algebra. The next two results enable us to do this. The first is due to the first named author and requires the equivalence between nuclearity of A and injectivity of the bidual A^{**}, see the account in [4]. The second is folklore, appearing in the proof of [11, Theorem 6.1], for example. We give a proof of the latter statement for completeness.

Proposition 2.9 ([11, Theorem 6.5])**.** *Let A and B be* C**-subalgebras of some* C**-algebra C with $d(A, B) < 1/101$. Then A is nuclear if, and only if, B is nuclear.*

Proposition 2.10. *Let A and B be* C**-subalgebras of some* C**-algebra C with $B \subset_{1/2} A$. If A is separable, then B is separable.*

Proof. Suppose that A is separable and suppose that $B \subset_{1/2} A$. Fix $0 < \gamma < 1/2$ with $B \subset_\gamma A$. Take $0 < \varepsilon < 1 - 2\gamma$, and let $\{b_i : i \in I\}$ be a maximal set of elements in the unit ball of B such that $\|b_i - b_j\| \geq 1 - \varepsilon$ when $i \neq j$. For each $i \in I$, find an operator a_i in the unit ball of A with $\|a_i - b_i\| < \gamma$. For $i \neq j$, we have $\|a_i - a_j\| \geq 1 - \varepsilon - 2\gamma > 0$, so separability of A implies that I is countable. Let C be the closed linear span of $\{b_i : i \in I\}$.

If $C \neq B$, take a unit vector $x \in B/C$ and write $x = y + C$ for some y with $1 \leq \|y\| < 1+\varepsilon$. Let $\tilde{y} = y/\|y\|$. If $\|\tilde{y} - b_i\| < 1-\varepsilon$ for some i, then

$$\|x\|_{B/C} \leq \|y - b_i\| \leq \|y - \tilde{y}\| + \|\tilde{y} - b_i\| < \varepsilon + 1 - \varepsilon = 1, \tag{2.1}$$

which is a contradiction. Thus we can adjoin y to $\{b_i : i \in I\}$ contradicting maximality. Hence $B = C$ and so B is separable. □

In Section 5 we shall conjugate C*-algebras by unitaries to reduce to the situation of close separable nuclear C*-algebras which have the same ultraweak closure using the next two propositions. The first is [12, Proposition 3.2], while the second was established in [11]. Note that the distance used in [11] is the quantity d_0 described in Remark 2.3. We restate the result we need in terms of d and add a shared unit assumption which is implicit in the original version.

Proposition 2.11. *Let A and B be C*-subalgebras of a unital C*-algebra C satisfying $d(A,B) < \gamma < 1/4$. Then A is unital if and only if B is unital. In this case there is a unitary $u \in C$ with $\|u - 1_C\| < 2\sqrt{2}\gamma$ and $u1_Au^* = 1_B$.*

Proposition 2.12 ([11, Corollary 4.2 (c)])**.** *Let M and N be injective von Neumann algebras on a Hilbert space H which share the same unit. If $d(M,N) < \gamma < 1/8$, then there exists a unitary $u \in (M \cup N)''$ with $uMu^* = N$ and $\|u - I_H\| \leq 12\gamma$.*

Distances between maps restricted to finite sets will be a recurring theme in the paper, as will be maps which act almost as $*$-homomorphisms, so we formalise these with the following notational definitions.

Definition 2.13. Given two maps $\phi_1, \phi_2 : A \to B$ between normed spaces, a subset $X \subseteq A$ and $\varepsilon > 0$, write $\phi_1 \approx_{X,\varepsilon} \phi_2$ if $\|\phi_1(x) - \phi_2(x)\| \leq \varepsilon$ for $x \in X$. When A and B are both subspaces of $\mathbb{B}(H)$, we use ι to denote the inclusion maps so $\phi_1 \approx_{X,\varepsilon} \iota$ means $\|\phi_1(x) - x\| \leq \varepsilon$ for $x \in X$.

Definition 2.14. Let A and B be C*-algebras, X a subset of A and $\varepsilon > 0$. A bounded linear map $\phi : A \to B$ is an *(X,ε)-approximate $*$-homomorphism* if it is a completely positive contractive map (cpc map) and the inequality

$$\|\phi(x)\phi(x^*) - \phi(xx^*)\| \leq \varepsilon, \quad x \in X \cup X^*, \tag{2.2}$$

is satisfied.

The following well-known consequence of Stinespring's theorem, which can be found as [35, Lemma 7.11], shows why we only consider pairs of the form x, x^* in the previous definition.

Proposition 2.15. *Let $\phi : A \to B$ be a cpc map between* C*-*algebras. For $x, y \in A$, we have*

$$\|\phi(xy) - \phi(x)\phi(y)\| \leq \|\phi(xx^*) - \phi(x)\phi(x^*)\|^{1/2}\|y\|. \tag{2.3}$$

Given a near inclusion $A \subset_\gamma B$, the next two results give conditions under which we can find cpc maps $A \to B$ which approximate on finite sets the inclusion map of A into the bounded operators on the underlying Hilbert space. Our first result of this type is an application of Arveson's extension theorem [1]. It uses the characterisation of nuclearity, due to Choi and Effros, [7], in terms of approximate factorisations of the identity map by completely positive maps through matrix algebras. The second is a version of [11, Proposition 6.7], for unital C*-algebras with approximately inner half flip. We include a proof here to remove the implicit assumption of a shared unit of the original version. A third result of this type will be obtained in Section 6 when A has finite nuclear dimension.

Proposition 2.16. *Let A and B be two* C*-*algebras on a Hilbert space H with B nuclear. Given a finite set X in the unit ball of A with $X \subset_\gamma B$, there exists a cpc map $\phi : A \to B$ such that $\|\phi(x) - x\| \leq 2\gamma$ for $x \in X$.*

Proof. Since $X \subset_\gamma B$, we may choose $\gamma' < \gamma$ so that $X \subset_{\gamma'} B$. Let $\varepsilon = 2(\gamma - \gamma') > 0$. Find a finite subset Y of B such that $X \subseteq_{\gamma'} Y$. Use the nuclearity of B to find a finite dimensional C*-algebra F and cpc maps $\alpha : B \to F$ and $\beta : F \to B$ so that $\beta \circ \alpha \approx_{Y,\varepsilon} \mathrm{id}_B$. Arveson's extension theorem, [1], allows us to extend α to a cpc map $\tilde{\alpha} : \mathbb{B}(H) \to F$. For a given $x \in X$, choose $y \in Y$ so that $\|x - y\| \leq \gamma'$. Then

$$\|\beta(\tilde{\alpha}(x)) - x\| \leq \|\beta(\tilde{\alpha}(x - y))\| + \|\beta(\tilde{\alpha}(y)) - y\| + \|x - y\| \leq 2\gamma' + \varepsilon = 2\gamma, \tag{2.4}$$

so the result holds with $\phi = \beta \circ \tilde{\alpha}$. □

Recall that a unital C*-algebra A has approximately inner half flip if the following condition is satisfied. Given a finite subset $F \subseteq A$ and $\varepsilon > 0$, there exists a unitary u in the spatial C*-tensor product $A \otimes A$ so that

$$\|u(1 \otimes x)u^* - x \otimes 1\|_{A \otimes A} < \varepsilon, \quad x \in F. \tag{2.5}$$

Such C*-algebras are automatically nuclear by the proof of Proposition 2.8 of [19]. The next result is established by reducing to the case where A and B are both unital with the same unit and then following [11, Proposition 6.7].

Proposition 2.17. *Let A and B be a C^*-algebras on a Hilbert space H. Suppose that A is unital with approximately inner half flip and $A \subset_\gamma B$ for $\gamma < 1/2$. Then for all finite sets X in the unit ball of A, there exists a cpc map $\phi : A \to B$ with*

$$\|\phi(x) - x\| \leq 8\alpha + 4\alpha^2 + 4\sqrt{2}\gamma, \quad x \in X, \tag{2.6}$$

where $\alpha = (4\sqrt{2}+1)\gamma + 4\sqrt{2}\gamma^2$.

Proof. Fix $\gamma' < \gamma$ so that $A \subseteq_{\gamma'} B$. By Lemma 2.1 of [8], find a projection $p \in B$ with $\|p - 1_A\| \leq 2\gamma'$. By Part 2 of Lemma 2.5 find a unitary u on H with $upu^* = 1_A$ and $\|u - I_H\| \leq \sqrt{2}\|p - 1_A\|$. Define $B_0 = u(pBp)u^* = 1_A uBu^* 1_A$ so that A and B_0 are both unital and share the same unit. Given x in the unit ball of A, there exists $y \in B$ with $\|x - y\| \leq \gamma'$. Then

$$\begin{aligned}\|x - 1_A uyu^* 1_A\| &\leq \|x - uyu^*\| \leq \|x - y\| + 2\|u - I_H\|\|y\| \\ &\leq \gamma' + 4\sqrt{2}\gamma'(1+\gamma').\end{aligned} \tag{2.7}$$

Then $A \subseteq_{\alpha'} B_0$, where $\alpha' = \gamma' + 4\sqrt{2}\gamma'(1+\gamma')$.

Fix a finite subset X of the unit ball of A and set $\varepsilon = \alpha - \alpha' > 0$. As A has approximately inner half flip, find a unitary $v \in A \otimes A$ with $\|v(1 \otimes x)v^* - x \otimes 1\| < \varepsilon$ for $x \in X$. Since A is nuclear, Proposition 2.6 gives some w in the unit ball of $B_0 \otimes A$ with $\|v - w\| \leq 2(2\alpha' + \alpha'^2)$. Given a state ψ on A, let $R_\psi : \mathrm{C}^*(B_0, A) \to B_0$ be the cpc slice map induced by ψ and note that $R_\psi(B_0 \otimes A) = B_0$. Then $\phi(x) = u^* R_\psi(w(1 \otimes x)w^*)u$ defines a cpc map $A \to B$ since $u^* B_0 u \subseteq B$. We have

$$\begin{aligned}\|\phi(x) - x\| &\leq \|\phi(x) - R_\psi(w(1 \otimes x)w^*)\| + \|R_\psi(w(1 \otimes x)w^*) - R_\psi(v(1 \otimes x)v^*)\| \\ &\quad + \|R_\psi(v(1 \otimes x)v^*) - x\| \\ &\leq 2\|u - I_H\| + 2\|w - v\| + \varepsilon \\ &\leq 4\sqrt{2}\gamma + 8\alpha + 4\alpha^2, \quad x \in X,\end{aligned} \tag{2.8}$$

so (2.6) holds. □

As is often the case in perturbation theory, improved constants can be obtained in Proposition 2.17 under the assumption that A and B share the same unit. This is also true of many of our subsequent results. In order to provide a unified treatment of the unital and non-unital cases, we use two distinct unitizations in this paper, detailed below.

Notation 2.18. Given a C*-algebra A, we write $A^\dagger$ for the following unitisation of A. When A is concretely represented on a Hilbert space H, the algebra $A^\dagger$ is obtained as $C^*(A, e_A)$, where e_A is the support projection of A. Given a cpc map $\phi : A \to B$, with A non-unital, it is well known that we can extend ϕ to a unital completely positive map (ucp map) $\tilde{\phi} : A^\dagger \to B^\dagger$ by defining $\tilde{\phi}(1_{A^\dagger}) = 1_{B^\dagger}$ (see [50, Section 1.2] or [4, Section 2.2]). However in the next section we will consider cpc maps $\phi : A \to B$ where A may be unital and it is convenient to also convert these to ucp maps. To this end we introduce another unitisation of A. Take a faithful representation π of A on a Hilbert space H and form the Hilbert space $\widetilde{H} = H \oplus \mathbb{C}$. The representation π extends to $\widetilde{H}$ by $x \mapsto \begin{pmatrix} \pi(x) & 0 \\ 0 & 0 \end{pmatrix}$. We define $\widetilde{A} = C^*(\pi(A), I_{\widetilde{H}})$. This construction is independent of the faithful representation π, and we note that A is always a proper norm closed ideal in $\widetilde{A}$. It follows that any cpc map $\phi : A \to B$ can be extended to a ucp map $\tilde{\phi} : \widetilde{A} \to B^\dagger$. □

Many of our results have a hypothesis of the form $d(A, B) < \gamma \leq K$ where K is some absolute constant, for example $K = 10^{-11}$ in Theorem 5.4. These constants are certainly not optimal, even for our methods, although we have attempted to make them as tight as possible. In the interests of readability, we have often rounded up multiple square roots to appropriate integer values.

3 Approximate averaging in nuclear C*-algebras

In this section we exploit the amenability of nuclear C*-algebras to establish point norm versions of the averaging results for injective von Neumann algebras in [9]. These techniques inevitably introduce small error terms not found in the von Neumann results since we are unable to take an ultraweak limit without possibly leaving the C*-algebra. We also use these methods to prove a Kaplansky density result for approximate relative commutants for use in Section 5. We begin with a review of the theory of amenability as it applies to C*-algebras.

Johnson defined amenability in the context of Banach algebras as a cohomological property [27] and characterised amenable Banach algebras as those having a virtual diagonal [26, Proposition 1.3]. Given a Banach algebra A, let $A \widehat{\otimes} A$ be the projective tensor product. A *virtual diagonal* for A is an element ω of $(A \widehat{\otimes} A)^{**}$, with $a\omega = \omega a$ and $m^{**}(\omega)a = a$ for all $a \in A$, where $m : A \widehat{\otimes} A \to A$ is the multiplication map $x \otimes y \mapsto xy$. When A is a C*-algebra we can replace the latter condition by $m^{**}(\omega) = 1_{A^{**}}$. As explained in [26], virtual diagonals play the role in the context of Banach algebras that the invariant mean plays for amenable groups.

Connes showed that any amenable C*-algebra is nuclear [14] and Haagerup subsequently established the converse [22] using the non-commutative Grothendieck-Haagerup-Pisier inequality, [23, 45], to build on earlier partial results of Bunce and Paschke [5]. Haagerup's proof not only obtains a virtual diagonal for a unital nuclear C*-algebra but also shows that it can be taken in the ultraweakly closed convex hull of $\{x^* \otimes x : x \in A,\ \|x\| \leq 1\}$, [22, Theorem 3.1]. This additional property will be of crucial importance subsequently. In the unital setting, it is natural to ask whether the virtual diagonal can be chosen from the ultraweakly closed convex hull of $\{u^* \otimes u : u \in \mathcal{U}(A)\}$ as occurs for the group algebras C$^*(G)$ of a discrete amenable group G. This is not always the case; this property is equivalent to the concept of strong amenability for unital C*-algebras [22, Lemma 3.4]. The Cuntz algebras $\mathcal{O}_n$ provide examples of nuclear C*-algebras which are not strongly amenable [49].

An *approximate diagonal* for a Banach algebra A is a bounded net (x_α) in $A \widehat{\otimes} A$ with $\|x_\alpha a - a x_\alpha\|_{A \widehat{\otimes} A} \to 0$ and $\|m(x_\alpha)a - a\|_A \to 0$ for all $a \in A$. In the unital C*-algebra case we can replace the second condition by $\|m(x_\alpha) - 1_A\| \to 0$. Approximate diagonals are closely connected to virtual diagonals since any ultraweak accumulation point of an approximate diagonal gives a virtual diagonal, while in [26, Lemma 1.2] Johnson uses a Hahn-Banach argument to create an approximate diagonal from a virtual diagonal. Just as a virtual diagonal plays the role of an invariant mean, the elements of an approximate diagonal play the role of Følner sets. To support this viewpoint, note that if G is a countable discrete amenable group and $(F_n)_n$ an increasing exhaustive sequence of Følner sets, then $(\frac{1}{|F_n|}\sum_{g \in F_n} g^* \otimes g)_n$ is an approximate diagonal for C$^*(G)$. Combining Haagerup's work [22] on the location of a virtual diagonal in a nuclear C*-algebra with [26, Lemma 1.2] gives the following result.

Lemma 3.1. *Let A be a unital nuclear* C**-algebra. Given a finite set $X \subseteq A$ and $\varepsilon > 0$, there exist contractions $\{a_i\}_{i=1}^n$ in A and non-negative constants $\{\lambda_i\}_{i=1}^n$ summing to* 1 *such that*

$$\left\| \sum_{i=1}^n \lambda_i a_i^* a_i - 1_A \right\| < \varepsilon \tag{3.1}$$

and

$$\left\| x \left(\sum_{i=1}^n \lambda_i a_i^* \otimes a_i \right) - \left(\sum_{i=1}^n \lambda_i a_i^* \otimes a_i \right) x \right\|_{A \widehat{\otimes} A} < \varepsilon, \quad x \in X. \tag{3.2}$$

If in addition A is strongly amenable, then each of the a_i's can be taken to be a unitary in A and so (3.1) can be replaced by $\sum_{i=1}^n \lambda_i a_i^* a_i = 1_A$.

Our next lemma is a point-norm version of Lemma 3.3 of [9] for nuclear C*-algebras.

Lemma 3.2. *Let A and D be* C*-*algebras and suppose that A is nuclear. Given a finite subset X of the unit ball of A, $\varepsilon > 0$, and $0 < \mu < (25\sqrt{2})^{-1}$, there exists a finite subset Y of the unit ball of A (depending only on X, A, ε and μ) with the following property: if $\phi : A \to D$ is a (Y, γ)-approximate $*$-homomorphism for some $\gamma \leq 1/17$, then there exists an (X, ε)-approximate $*$-homomorphism $\psi : A \to D$ with $\|\phi - \psi\| \leq 8\sqrt{2}\gamma^{1/2} + \mu$.*

Proof. Let $\eta > 0$ be such that if $0 \leq p_0 \leq 1$ is an operator with spectrum contained in $\Omega = [0, 0.496] \cup [0.504, 1]$ and q is the spectral projection of p_0 for $[0.504, 1]$, then the inequality $\|xp_0 - p_0x\| < \eta$ for a contractive operator x implies that $\|xq - qx\| < \varepsilon$. The existence of η (depending only on ε) follows from a standard functional calculus argument using uniform approximations of $\chi_{[0.504,1]}$ by polynomials on Ω. See [2, p. 332].

Nuclearity of A implies nuclearity of the unital C*-algebra $\widetilde{A}$ obtained by adjoining a (possibly additional) unit (see Notation 2.18). By Lemma 3.1 we may choose $\{\tilde{a}_i\}_{i=1}^n$ from the unit ball of $\widetilde{A}$ and non-negative constants $\{\lambda_i\}_{i=1}^n$ summing to 1 satisfying

$$\left\| \sum_{i=1}^n \lambda_i \tilde{a}_i^* \tilde{a}_i - 1_{\widetilde{A}} \right\| < (4\sqrt{2})^{-1}\mu \tag{3.3}$$

and

$$\left\| x \left(\sum_{i=1}^n \lambda_i \, \tilde{a}_i^* \otimes \tilde{a}_i \right) - \left(\sum_{i=1}^n \lambda_i \, \tilde{a}_i^* \otimes \tilde{a}_i \right) x \right\|_{\widetilde{A} \,\hat{\otimes}\, \widetilde{A}} < \eta, \qquad x \in X. \tag{3.4}$$

Each $\tilde{a}_i$ has the form $\alpha_i + z_i$ for $\alpha_i \in \mathbb{C}$ and $z_i \in A$. Projection to the quotient $\widetilde{A}/A$ shows that $|\alpha_i| \leq 1$ and so $\|z_i\| \leq 2$. Define $a_i = z_i/2$, $1 \leq i \leq n$, so that each a_i lies in the unit ball of A and $\tilde{a}_i = \alpha_i + 2a_i$. We then take the set Y to be $\{a_i, a_i^* : 1 \leq i \leq n\}$, a finite subset of the unit ball of A. We now verify that Y has the desired properties.

Consider a cpc map $\phi : A \to D$ satisfying the inequalities

$$\|\phi(a_i^* a_i) - \phi(a_i^*)\phi(a_i)\| \leq \gamma, \quad \|\phi(a_i a_i^*) - \phi(a_i)\phi(a_i^*)\| \leq \gamma, \quad i = 1, \ldots, n, \tag{3.5}$$

for some $\gamma \leq 1/17$. Let $\tilde{\phi} : \widetilde{A} \to D^\dagger$ be the canonical extension of ϕ to a ucp map. Assume that $D^\dagger$ is faithfully represented on a Hilbert space H so that I_H is the unit of $D^\dagger$. Stinespring's representation theorem allows us to find a larger Hilbert space K and a unital $*$-representation $\pi : \widetilde{A} \to \mathbb{B}(K)$ such that

$$\tilde{\phi}(x) = p\pi(x)p, \qquad x \in \widetilde{A}, \tag{3.6}$$

where p is the orthogonal projection of K onto H. Define

$$p_0 = \sum_{i=1}^{n} \lambda_i \pi(\tilde{a}_i^*) p \pi(\tilde{a}_i). \tag{3.7}$$

Then $0 \leq p_0 \leq 1$, and this operator lies in $\mathrm{C}^*(\pi(A), p)$ since this is an ideal in $\mathrm{C}^*(\pi(\widetilde{A}), p)$ containing p. Using (3.3), we obtain the estimate

$$\begin{aligned} \|p_0 - p\| &= \left\| \sum_{i=1}^{n} \lambda_i (\pi(\tilde{a}_i)^* p - p\pi(\tilde{a}_i^*))\pi(\tilde{a}_i) + \sum_{i=1}^{n} \lambda_i p\pi(\tilde{a}_i^* \tilde{a}_i) - p \right\| \\ &\leq \sum_{i=1}^{n} \lambda_i \|\pi(\tilde{a}_i)^* p - p\pi(\tilde{a}_i)^*\| + \left\| p\pi\left(\sum_{i=1}^{n} \lambda_i \tilde{a}_i^* \tilde{a}_i - 1_{\tilde{A}} \right) \right\| \\ &\leq 2 \sum_{i=1}^{n} \lambda_i \|\pi(a_i)^* p - p\pi(a_i)^*\| + (4\sqrt{2})^{-1}\mu. \end{aligned} \tag{3.8}$$

We are in position to closely follow [9, Lemma 3.3] for the remainder of the proof. Firstly

$$\|\pi(a_i^*)p - p\pi(a_i^*)\|^2 = \max\{\|p\pi(a_i)(1-p)\pi(a_i^*)p\|, \|p\pi(a_i^*)(1-p)\pi(a_i)p\|\}, \tag{3.9}$$

from the estimate on [9, p. 4]. For $a \in A$, we have

$$\phi(aa^*) - \phi(a)\phi(a^*) = p\pi(a)(1-p)\pi(a)^* p, \tag{3.10}$$

so taking $a = a_i$ and $a = a_i^*$ gives the estimate

$$\|\pi(a_i^*)p - p\pi(a_i^*)\|^2 \leq \gamma, \qquad 1 \leq i \leq n, \tag{3.11}$$

from (3.5). Using (3.11) at the end of (3.8) yields the inequality

$$\|p_0 - p\| \leq 2\gamma^{1/2} + (4\sqrt{2})^{-1}\mu < 0.496, \tag{3.12}$$

where the latter inequality follows from the bounds on γ and μ. Define β to be the constant $2\gamma^{1/2} + (4\sqrt{2})^{-1}\mu$. The spectrum of p_0 is contained in $[0, \beta] \cup [1-\beta, 1]$. Letting q denote the spectral projection of p_0 for $[1-\beta, 1]$, (or equivalently $[0.504, 1]$), the functional calculus gives $\|p_0 - q\| \leq \beta$, and so $\|p - q\| \leq 2\beta < 1$ from (3.12). Both p and q lie in $\mathrm{C}^*(\pi(\widetilde{A}), p)$, and so there is a unitary w in this C^*-algebra so that $wpw^* = q$ and $\|I_K - w\| \leq 2\sqrt{2}\beta$, from Proposition 2.5 part 2. Define a cpc map $\psi : A \to \mathbb{B}(H)$ by

$$\psi(x) = pw^*\pi(x)wp, \qquad x \in A. \tag{3.13}$$

Since $w^*\pi(A)w \subseteq \mathrm{C}^*(\pi(A), p)$, we observe that ψ maps A into D because $p\pi(A)p$ is the range of $\phi : A \to D$. The estimate

$$\|\phi - \psi\| \leq 2\|I_K - w\| \leq 4\sqrt{2}\beta = 8\sqrt{2}\gamma^{1/2} + \mu \tag{3.14}$$

is immediate from (3.13).

It remains to check that ψ is an (X, ε)-approximate $*$-homomorphism. The definition of the projective tensor norm ensures that the map $y \otimes z \mapsto \pi(y)p\pi(z)$ extends to a contraction from $\widetilde{A} \mathbin{\widehat{\otimes}} \widetilde{A}$ into $\mathbb{B}(K)$. Applying this map to (3.4) gives

$$\|\pi(x)p_0 - p_0\pi(x)\| < \eta, \qquad x \in X, \tag{3.15}$$

using the definition of p_0 from (3.7). The choice of η then gives

$$\|\pi(x)q - q\pi(x)\| < \varepsilon, \qquad x \in X. \tag{3.16}$$

Thus, for $x \in X$,

$$\|\psi(xx^*) - \psi(x)\psi(x^*)\| = \|pw^*\pi(xx^*)wp - pw^*\pi(x)q\pi(x^*)wp\| < \varepsilon, \tag{3.17}$$

using (3.16) and the relation $pw^*q = pw^*$. □

Remark 3.3. (i). If we choose to regard γ as fixed in the previous lemma, then we could add the constraint $\mu \leq \nu\gamma^{1/2}$ (where ν is a fixed but arbitrary positive constant) to the hypotheses. This would change the concluding inequality to $\|\phi - \psi\| \leq (8\sqrt{2} + \nu)\gamma^{1/2}$, a form that is convenient for subsequent estimates.
(ii). With the hypothesis of Lemma 3.2, if A is strongly amenable then so too is $\widetilde{A}$. As such we can replace the estimate (3.3) by the identity $\sum_{i=1}^{n} \lambda_i \tilde{a}_i{}^* \tilde{a}_i = 1_{\widetilde{A}}$. It then follows that we can take $\mu = 0$ and obtain an estimate $\|\phi - \psi\| \leq 8\sqrt{2}\gamma^{1/2}$. Without strong amenability, if we want Y to be independent of γ we are forced to introduce the constant μ upon which our estimates do not send $\|\phi - \psi\|$ to zero as $\gamma \to 0$. □

We now turn to our second averaging lemma which is the analogue of Proposition 4.2 of [9]. It is important to ensure that our point-norm version of this result handles not just $*$-homomorphisms but also (Y, δ)-approximate $*$-homomorphisms for sufficiently large Y and small δ as these are the outputs of Lemma 3.2. Taking this into account gives the following lemma.

Lemma 3.4. *Let A be a nuclear* C^**-algebra and let D be a* C^**-algebra. Given a finite set X in the unit ball of A and $\varepsilon > 0$, there exist a finite set Y in the unit ball of A and $\delta > 0$,* *both depending only on X, A and ε, with the following property. If $\phi_1, \phi_2 : A \to D$ are (Y,δ)-approximate $*$-homomorphisms such that $\phi_1 \approx_{Y,\gamma} \phi_2$, for some $\gamma \le 13/150$, then there exists a unitary $u \in D^\dagger$ satisfying $\|u - I_{D^\dagger}\| < 2\sqrt{2}\gamma + 5\sqrt{2}\delta$ and $\phi_1 \approx_{X,\varepsilon} \mathrm{Ad}(u) \circ \phi_2$.*

Note that $\mathrm{Ad}(u) \circ \phi_2$ maps A into D, since D is an ideal in $D^\dagger$.

Proof. Fix a finite set X in the unit ball of A and an $\varepsilon > 0$. By considering polynomial approximations to $t^{1/2}$ on $[0,1]$, we may find $\eta > 0$, depending only on ε, so that the inequality $\|s^*sy - ys^*s\| < \eta$ for contractive operators s and y implies the relation $\|\, |s|y - y|s|\, \| < \varepsilon/4$ (see [2, p. 332]).

Let δ satisfy

$$0 < \delta \le \min\{\eta^2/100, \varepsilon^2/400, 1/200\}. \tag{3.18}$$

By Lemma 3.1, we may find $\{\tilde{a}_i\}_{i=1}^n$ in the unit ball of $\widetilde{A}$ and non-negative constants $\{\lambda_i\}_{i=1}^n$ summing to 1, such that

$$\left\| \sum_{i=1}^n \lambda_i \tilde{a}_i^* \tilde{a}_i - 1_{\widetilde{A}} \right\| < \delta, \tag{3.19}$$

and

$$\left\| \sum_{i=1}^n \lambda_i (x\tilde{a}_i^* \otimes \tilde{a}_i - \tilde{a}_i^* \otimes \tilde{a}_i x) \right\|_{\widetilde{A} \,\widehat{\otimes}\, \widetilde{A}} < \delta^{1/2}, \qquad x \in X \cup X^*. \tag{3.20}$$

As in Lemma 3.2, each $\tilde{a}_i$ can be written as $\alpha_i + 2a_i$, where α_i is a constant and a_i is in the unit ball of A. Let $Y = \{a_1, \ldots, a_n\}$. We now show that Y and δ have the desired property.

Suppose that $\phi_1, \phi_2 : A \to D$ are (Y,δ)-approximate $*$-homormorphisms satisfying $\phi_1 \approx_{Y,\gamma} \phi_2$ for some $\gamma \le 13/150$, and let $\tilde{\phi}_1, \tilde{\phi}_2 : \widetilde{A} \to D^\dagger$ be their unital completely positive extensions. Define

$$s = \sum_{i=1}^n \lambda_i \tilde{\phi}_1(\tilde{a}_i{}^*) \tilde{\phi}_2(\tilde{a}_i). \tag{3.21}$$

Since $x \otimes y \mapsto \tilde{\phi}_1(x)\tilde{\phi}_2(y)$ is contractive on $\widetilde{A} \,\widehat{\otimes}\, \widetilde{A}$, the inequality

$$\left\| \sum_{i=1}^n \lambda_i \tilde{\phi}_1(x\tilde{a}_i^*) \tilde{\phi}_2(\tilde{a}_i) - \sum_{i=1}^n \lambda_i \tilde{\phi}_1(\tilde{a}_i^*) \tilde{\phi}_2(\tilde{a}_i x) \right\| < \delta^{1/2}, \qquad x \in X \cup X^*, \tag{3.22}$$

follows from (3.20). Now $\tilde{a}_i = \alpha_i + 2a_i$ and ϕ_1 and ϕ_2 are (Y,δ)-approximate $*$-homomorphisms so Proposition 2.15 gives the inequalities

$$\|\tilde{\phi}_1(x\tilde{a}_i^*) - \tilde{\phi}_1(x)\tilde{\phi}_1(\tilde{a}_i^*)\| \le 2\|\phi_1(a_i^* a_i) - \phi_1(a_i^*)\phi_1(a_i)\|^{1/2} < 2\delta^{1/2}, \tag{3.23}$$

and

$$\|\tilde{\phi}_2(\tilde{a}_i x) - \tilde{\phi}_2(\tilde{a}_i)\tilde{\phi}_2(x)\| \le 2\|\phi_2(a_i a_i^*) - \phi_2(a_i)\phi_2(a_i^*)\| < 2\delta^{1/2}, \tag{3.24}$$

for x in the unit ball of A, and so in particular for $x \in X \cup X^*$, and $1 \leq i \leq n$. Then (3.22), (3.23) and (3.24) combine to yield

$$\|\phi_1(x)s - s\phi_2(x)\| < 5\delta^{1/2}, \qquad x \in X \cup X^*. \tag{3.25}$$

Taking adjoints gives

$$\|s^*\phi_1(x) - \phi_2(x)s^*\| < 5\delta^{1/2}, \qquad x \in X \cup X^*. \tag{3.26}$$

Since

$$\begin{aligned} s^*s\phi_2(x) - \phi_2(x)s^*s &= s^*\phi_1(x)s + s^*(s\phi_2(x) - \phi_1(x)s) - \phi_2(x)s^*s \\ &= (s^*\phi_1(x) - \phi_2(x)s^*)s + s^*(s\phi_2(x) - \phi_1(x)s), \end{aligned} \tag{3.27}$$

the inequality

$$\|s^*s\phi_2(x) - \phi_2(x)s^*s\| < 10\delta^{1/2}, \qquad x \in X \cup X^*, \tag{3.28}$$

follows from (3.25) and (3.26).

The choice of δ gives $10\delta^{1/2} \leq \eta$ so, defining z to be $|s|$,

$$\|z\phi_2(x) - \phi_2(x)z\| < \varepsilon/4, \qquad x \in X \cup X^*, \tag{3.29}$$

by definition of η. Now

$$\begin{aligned} \|s - 1_{D^\dagger}\| &= \left\| \sum_{i=1}^n \lambda_i \tilde{\phi}_1(\tilde{a}_i^*)\tilde{\phi}_2(\tilde{a}_i) - 1_{D^\dagger} \right\| \\ &\leq \left\| \sum_{i=1}^n \lambda_i \tilde{\phi}_1(\tilde{a}_i^*)\tilde{\phi}_1(\tilde{a}_i) - 1_{D^\dagger} \right\| + 2\gamma \\ &\leq \left\| \tilde{\phi}_1 \left(\sum_{i=1}^n \lambda_i \tilde{a}_i^* \tilde{a}_i - 1_{\tilde{A}} \right) \right\| + \left\| \sum_{i=1}^n \lambda_i (\tilde{\phi}_1(\tilde{a}_i^*)\tilde{\phi}_1(\tilde{a}_i) - \tilde{\phi}_1(\tilde{a}_i^*\tilde{a}_i)) \right\| + 2\gamma \\ &\leq \delta + 4\delta + 2\gamma = 5\delta + 2\gamma, \end{aligned} \tag{3.30}$$

using (3.19), $\phi_1 \approx_{Y,\gamma} \phi_2$, and that ϕ_1 is a (Y, δ)-approximate $*$-homomorphism. Since $5\delta + 2\gamma < 1$, (3.30) gives invertibility of s, and the unitary in the polar decomposition $s = uz$ lies in $D^\dagger$ and satisfies $\|u - 1_{D^\dagger}\| \leq 5\sqrt{2}\delta + 2\sqrt{2}\gamma$ by part 1 of Proposition 2.5. Then

$$\begin{aligned} \|z - 1_{D^\dagger}\| &= \|u^*s - 1_{D^\dagger}\| = \|s - u\| \\ &\leq \|s - 1_{D^\dagger}\| + \|u - 1_{D^\dagger}\| \\ &\leq 5(1 + \sqrt{2})\delta + 2(1 + \sqrt{2})\gamma < 1/2. \end{aligned} \tag{3.31}$$

From this we obtain $\|z^{-1}\| \leq 2$ so, for $x \in X$,

$$\begin{aligned}\|\phi_1(x) - u\phi_2(x)u^*\| &= \|\phi_1(x)u - u\phi_2(x)\| \\ &\leq \|\phi_1(x)uz - u\phi_2(x)z\|\|z^{-1}\| \\ &\leq 2\|\phi_1(x)s - s\phi_2(x)\| + 2\|uz\phi_2(x) - u\phi_2(x)z\| \\ &\leq 10\delta^{1/2} + 2\|z\phi_2(x) - \phi_2(x)z\| < 10\delta^{1/2} + \varepsilon/2 \leq \varepsilon, \end{aligned} \tag{3.32}$$

using the definition of δ and equations (3.25) and (3.29). □

Remark 3.5. (i) Having found one pair (Y, δ) for which Lemma 3.4 holds, it is clear that we may enlarge Y and decrease δ. Thus we may assume that $X \subseteq Y$, and we are at liberty to take δ as small as we wish.
(ii) We have chosen to formulate Lemma 3.4 so that δ does not depend on γ. However, if we demand that γ lies in the interval $[\gamma_0, 13/150]$ for some $\gamma_0 > 0$, then we could add the extra constraint $\delta \leq (5\sqrt{2})^{-1}\nu\gamma_0$ to (3.18), where ν is a fixed but arbitrarily small positive number. This changes the estimate on u from $\|1_{D^\dagger} - u\| < 2\sqrt{2}\gamma + 5\sqrt{2}\delta$ to $\|1_{D^\dagger} - u\| < (2\sqrt{2} + \nu)\gamma$. In particular, we can take ν to be so small that the estimate $\|1_{D^\dagger} - u\| \leq 3\gamma$ holds. We will use this form of the lemma repeatedly in Section 5, for a finite range of γ simultaneously.
(iii) If A, and hence $\widetilde{A}$ is strongly amenable, then we can replace the estimate (3.19) by $\sum_{i=1}^{n} \lambda_i \tilde{\phi}_1(\tilde{a}_i{}^*)\tilde{\phi}_2(\tilde{a}_i{}^*) = 1_{D^\dagger}$. However this does not allow us to obtain a stronger version of the lemma for (Y, δ)-approximate $*$-homomorphisms. If we further insist that both ϕ_1 and ϕ_2 are $*$-homomorphisms (rather than just approximate ones), then we can take $\delta = 0$ in Lemma 3.4 in the strongly amenable case. □

Next we give two Kaplansky density style results for approximate relative commutants. Consider a unital C*-algebra A acting non-degenerately on some Hilbert space H and let $M = A''$. Given a finite dimensional C*- subalgebra F of A, one can average over the compact unitary group of F to see that the relative comutant $F' \cap A$ is *-strongly dense in the the relative commutant $F' \cap M$. The next lemma is a version of this for approximate relative commutants, replacing averaging over a compact unitary group by an argument using approximate diagonals.

Lemma 3.6. *Let A be a unital* C*-*algebra faithfully and non-degenerately represented on a Hilbert space H with strong closure $M = A''$. Let X be a finite subset of the unit ball of A which is contained within a nuclear* C*-*subalgebra A_0 of A with $1_A \in A_0$. Given constants $\varepsilon, \mu > 0$, there exists a finite set Y in the unit ball of A_0 and $\delta > 0$ with the following* *property. Given a finite subset S of the unit ball of H and m in the unit ball of M such that*

$$\|my - ym\| \leq \delta, \quad y \in Y, \tag{3.33}$$

there exists an element $a \in A$ with $\|a\| \leq \|m\|$ such that

$$\|ax - xa\| \leq \varepsilon, \quad x \in X \tag{3.34}$$

and

$$\|a\xi - m\xi\| < \mu, \ \|a^*\xi - m^*\xi\| < \mu \quad \xi \in S. \tag{3.35}$$

If m is self-adjoint, then a can be taken self-adjoint.

Proof. Fix $0 < \delta < \mu/4$. By Lemma 3.1 we can find positive scalars $\{\lambda_i\}_{i=1}^n$ summing to 1 and elements $\{b_i\}_{i=1}^n$ in the unit ball of A_0 satisfying

$$\left\| \sum_{i=1}^n \lambda_i b_i^* b_i - 1_A \right\| \leq \delta \tag{3.36}$$

and

$$\left\| \sum_{i=1}^n \lambda_i (x b_i^* \otimes b_i - b_i^* \otimes b_i x) \right\|_{A_0 \,\widehat{\otimes}\, A_0} < \varepsilon, \quad x \in X. \tag{3.37}$$

Now define Y to be $\{b_1, \ldots, b_n\}$. If $m \in M$ has $\|m\| \leq 1$ and satisfies (3.33), then

$$\begin{aligned} \left\| m - \sum_{i=1}^n \lambda_i b_i^* m b_i \right\| &\leq \left\| m - \sum_{i=1}^n \lambda_i b_i^* b_i m \right\| + \left\| \sum_{i=1}^n \lambda_i b_i^* (b_i m - m b_i) \right\| \\ &\leq \delta + \delta = 2\delta, \end{aligned} \tag{3.38}$$

from (3.33) and (3.36).

Given a finite set S in the unit ball of A, the Kaplansky density theorem gives $a_0 \in A$ with $\|a_0\| \leq \|m\|$ satisfying

$$\|(a_0 - m) b_i \xi\| < \mu/2, \ \|(a_0 - m)^* b_i \xi\| < \mu/2, \quad i = 1, \ldots, n, \quad \xi \in S. \tag{3.39}$$

If $m = m^*$, then we can additionally insist that $a_0 = a_0^*$. Define

$$a = \sum_{i=1}^n \lambda_i b_i^* a_0 b_i \tag{3.40}$$

which is self-adjoint if m, and hence a_0, is self-adjoint. Furthermore, $\|a\| \leq \|a_0\| \leq \|m\|$. The definition of the projective tensor product norm shows that $y \otimes z \mapsto y a_0 z$ gives a contractive map of $A_0 \,\widehat{\otimes}\, A_0$ into A. Using (3.37) and (3.40), we conclude that

$$\|ax - xa\| < \varepsilon, \quad x \in X. \tag{3.41}$$

For $\xi \in S$,

$$\|(a-m)\xi\| \le \left\| \Big(\sum_{i=1}^{n} \lambda_i b_i^*(a_0 - m) b_i \Big) \xi \right\| + \left\| \Big(\sum_{i=1}^{n} \lambda_i b_i m b_i^* - m \Big) \xi \right\|$$
$$< \mu/2 + 2\delta, \tag{3.42}$$

from (3.38) and (3.39). The choice of δ gives $\|(a-m)\xi\| < \mu$. Similarly, $\|(a-m)^*\xi\| < \mu$ for $\xi \in S$. □

In Section 5, we will use the following version of Lemma 3.6 for unitaries. The hypothesis $\|u - I_H\| \le \alpha < 2$ which we impose below is to ensure a gap in the spectrum, permitting us to take a continuous logarithm. This is essential for our methods.

Lemma 3.7. *Let A be a unital C^*-algebra faithfully and non-degenerately represented on a Hilbert space H with strong closure $M = A''$. Let X be a finite subset of the unit ball of A which is contained within a nuclear C^*-algebra A_0 of A with $1_A \in A_0$. Given constants $\varepsilon_0, \mu_0 > 0$ and $0 < \alpha < 2$, there exists a finite set Y in the unit ball of A_0 and $\delta_0 > 0$ with the following property. Given a finite set S in the unit ball of H and a unitary $u \in M$ satisfying $\|u - I_H\| \le \alpha$ and*

$$\|uy - yu\| \le \delta_0, \qquad y \in Y, \tag{3.43}$$

there exists a unitary $v \in A$ such that

$$\|vx - xv\| \le \varepsilon_0, \qquad x \in X, \tag{3.44}$$
$$\|v\xi - u\xi\| < \mu_0, \quad \|v^*\xi - u^*\xi\| < \mu_0, \qquad \xi \in S, \tag{3.45}$$

and $\|v - I_H\| \le \|u - I_H\| \le \alpha$.

Proof. Fix $0 < \alpha < 2$. The result is obtained from Lemma 3.6 using some polynomial approximations and the following two observations.

Firstly, there is an interval $[-c, c]$ with $0 < c < \pi$ so that $|1 - e^{i\theta}| \le \alpha$ if and only if θ lies in this interval modulo 2π. Any unitary u satisfying $\|u - 1\| \le \alpha$ has spectrum contained in the arc $\{e^{i\theta} : \ -c \le \theta \le c\}$ on which there is a continuous logarithm. By approximating $\log z$ on the arc by polynomials in the complex variables z and $\bar{z}$, we obtain the following. Given $\delta > 0$, there exists $\delta_0 > 0$ so that if $u \in M$ is a unitary satisfying $\|u - 1\| \le \alpha$ and

$$\|uy - yu\| \le \delta_0 \tag{3.46}$$

for some $y \in \mathbb{B}(H)$ with $\|y\| \le 1$, then

$$\left\| \frac{\log u}{\pi} y - y \frac{\log u}{\pi} \right\| \le \delta, \tag{3.47}$$

as in [2, p. 332]. Note that this deduction also requires (3.46) to hold with u replaced by u^*, but this is immediate from the algebraic identity $u^*y - yu^* = u^*(yu - uy)u^*$.

Secondly, as we show below, the map $x \mapsto e^{i\pi x}$ is uniformly strong-operator continuous on the unit ball in $M_{\mathrm{s.a.}}$ in the following sense. Given $\mu_0 > 0$, there exists $\mu > 0$ with the following property. For each finite subset S of the unit ball of H and $h \in M_{\mathrm{s.a.}}$ with $\|h\| \le 1$, there exists a finite subset S' of the unit ball of H (depending only on S, μ_0 and h) such that the inequalities

$$\|(e^{i\pi h} - e^{i\pi k})\xi\| < \mu_0, \quad \|(e^{i\pi h} - e^{i\pi k})^*\xi\| < \mu_0, \quad \xi \in S, \tag{3.48}$$

are valid whenever an element k in the unit ball of $M_{\mathrm{s.a.}}$ satisfies

$$\|(h-k)\xi\| < \mu, \quad \xi \in S'. \tag{3.49}$$

This follows by considering polynomial approximations of $e^{i\pi t}$ for $t \in [-1,1]$. Given $\mu_0 > 0$, let $p(t) = \sum_{j=0}^r \lambda_j t^j$ be a polynomial in t with $\sup_{-1\le t\le 1} |p(t) - e^{i\pi t}| < \mu_0/3$ and define

$$\mu = \frac{\mu_0}{3r\sum_{j=0}^r |\lambda_j|}. \tag{3.50}$$

Given h and S, define $S' = \{h^j\xi : \xi \in S,\ 0 \le j < r\}$ and suppose that (3.49) holds. For $\xi \in S$ and $0 \le j \le r$, we compute

$$\begin{aligned}\|(h^j - k^j)\xi\| &\le \|(h^{j-1} - k^{j-1})h\xi\| + \|k^{j-1}(h-k)\xi\| \\ &\le \|(h^{j-2} - k^{j-2})h^2\xi\| + \|k^{j-2}(h-k)h\xi\| + \|k^{j-1}(h-k)\xi\| \\ &\le \dots \\ &\le \sum_{m=0}^{j-1} \|(h-k)h^m\xi\| < r\mu,\end{aligned} \tag{3.51}$$

so that

$$\|(p(h) - p(k))\xi\| \le \sum_{j=0}^r |\lambda_j|\|(h^j - k^j)\xi\| < \sum_{j=0}^r |\lambda_j| r\mu = \mu_0/3, \quad \xi \in S, \tag{3.52}$$

and, similarly,

$$\|(p(h) - p(k))^*\xi\| \le \mu_0/3, \quad \xi \in S. \tag{3.53}$$

The estimates in (3.48) follow.

We can now deduce the lemma from Lemma 3.6. Assume then that X, ε_0, and μ_0 are given and let A_0 be a nuclear C*-subalgebra of A containing X and 1_A. By means of another polynomial approximation argument choose $\varepsilon > 0$ so that if $k \in M_{\mathrm{s.a.}}$, $\|k\| \le 1$, and

$$\|xk - kx\| \le \varepsilon, \qquad x \in X, \tag{3.54}$$

then

$$\|xe^{i\pi k} - e^{i\pi k}x\| \leq \varepsilon_0, \qquad x \in X, \tag{3.55}$$

as in [2, p. 332]. Let $\mu > 0$ be the constant corresponding via the second observation above to μ_0 and apply Lemma 3.6 to (X, ε, μ), producing a finite set Y in the unit ball of A and a constant $\delta > 0$. Let δ_0 be the constant corresponding to δ given by our first observation so that (3.46) implies (3.47).

Suppose that we have a unitary $u \in M$ satisfying $\|u - I_H\| \leq \alpha$ and

$$\|uy - yu\| \leq \delta_0, \qquad y \in Y. \tag{3.56}$$

Define $h = -i \log u/\pi$ in $M_{\text{s.a.}}$. The definition of δ_0 gives

$$\|hy - yh\| \leq \delta, \qquad y \in Y. \tag{3.57}$$

Given a finite set S in the unit ball of H, let S' be the finite set corresponding to h, S and μ_0 from the uniform strong-operator continuity of $x \mapsto e^{i\pi x}$. Putting this set into Lemma 3.6, we can find a self-adjoint operator k in the unit ball of A with $\|k\| \leq \|h\|$ which satisfies (3.49) and (3.54). Let $v = e^{i\pi k}$, which has $\|v - I_H\| \leq \|u - I_H\| \leq \alpha$. By definition of ε, (3.55) holds and this gives (3.44). Similarly the definition of μ ensures that (3.49) implies (3.48) so that (3.45) also holds. □

Remark 3.8. In [37, Section 4] it is shown that Haagerup's proof that nuclear C*-algebras are amenable also implies that every C*-algebra A has the following property: given an irreducible representation π of A on H, a finite subset X of A, a finite subset S of H and $\varepsilon > 0$, there exist $\{a_i\}_{i=1}^n$ in A such that

1. $\|\sum_{i=1}^n a_i a_i^*\| \leq 1$;
2. $\sum_{i=1}^n \pi(a_i a_i^*)\xi = \xi$ for $\xi \in S$;
3. $\|[\sum_{i=1}^n a_i y a_i^*, x]\| \leq \varepsilon$ for all $x \in X$ and all y in the unit ball of A.

Although we do not need it in this paper, using the property above in place of nuclearity in the proof of Lemma 3.6, gives the result below. Comparing this with Lemma 3.6, the key difference is that the set S below must be specified with X and ε, whereas in Lemma 3.6 the set S can be chosen after m is specified. When we use these results in Section 5, being able to choose S at this late stage is important to our argument.

Proposition 3.9. *Let A be* C*-*algebra represented irreducibly on a Hilbert space H. Let X be a finite subset of the unit ball of A, S be a finite subset of the unit ball of H and $\varepsilon > 0$. Then there exists a finite subset Y of the unit ball of A and $\delta > 0$ with the following property. Given any $m \in \mathbb{B}(H)$ with $\|my - ym\| \leq \delta$ for $y \in Y$, there exists $a \in A$ with $\|a\| \leq \|m\|$, $\|ax - xa\| \leq \varepsilon$ for $x \in X$ and $\|(m-a)\xi\| \leq \varepsilon$ for $\xi \in S$.*

4 Approximation on finite sets and isomorphisms

In this section we establish the qualitative version of Theorem A: that C*-algebras sufficiently close to a separable nuclear C*-algebra A must be isomorphic to A. To do this, we use an approximation approach inspired by the intertwining arguments of [11, Theorem 6.1] and those in the classification programme (see, for example, [21]). This is presented in Lemma 4.1, where we have given a general formulation in terms of the existence of certain completely positive contractions. This is designed for application in several contexts where this hypothesis can be verified, and so it forms the basis for all our subsequent near inclusion results as well as Theorem A.

Lemma 4.1. *Suppose that A and B are* C*-*algebras on some Hilbert space H and that A is separable and nuclear. Suppose that there exists a constant $\eta > 0$ satisfying $\eta < 1/210000$ such that for each finite subset Z of the unit ball of A, there is a completely positive contraction $\phi : A \to B$ satisfying $\phi \approx_{Z,\eta} \iota$. Then, given any finite subset X_A of the unit ball of A and $0 < \mu < 1/2000$, there exists an injective $*$-homomorphism $\alpha : A \to B$ with $\alpha \approx_{X_A, 8\sqrt{6}\eta^{1/2}+\eta+\mu} \iota$. If, in addition, $B \subset_{1/5} A$, then we can take α to be surjective.*

Proof. Let $\{a_n\}_{n=0}^{\infty}$ be a dense sequence in the unit ball of A, where $a_0 = 0$. Fix a finite subset X_A of the unit ball of A. Given $\mu < 1/2000$, define $\nu = 2\mu < 1/4000$. We will construct inductively sequences $\{X_n\}_{n=0}^{\infty}$, $\{Y_n\}_{n=0}^{\infty}$ of finite subsets of the unit ball of A, a sequence $\{\delta_n\}_{n=0}^{\infty}$ of positive constants, a sequence $\{\theta_n : A \to B\}_{n=0}^{\infty}$ of completely positive contractions, and a sequence of unitaries $\{u_n\}_{n=1}^{\infty}$ in $B^\dagger$ which satisfy the following conditions.

(a) The sets $\{X_n\}_{n=0}^{\infty}$ are increasing, $a_n \in X_n$ for $n \geq 0$, and $X_A \subseteq X_1$.

(b) $\delta_n \leq 2^{-n}$ for $n \geq 0$, and given any two (Y_n, δ_n)-approximate $*$-homomorphisms $\phi_1, \phi_2 : A \to B$ satisfying $\phi_1 \approx_{Y_n, 2(8\sqrt{6}\eta^{1/2}+\eta+\nu)} \phi_2$, there exists a unitary $u \in B^\dagger$ with $\mathrm{Ad}(u) \circ \phi_1 \approx_{X_n, 2^{-n}\nu} \phi_2$. This unitary can be chosen to satisfy $\|u - 1_{B^\dagger}\| \leq 4\sqrt{2}(8\sqrt{6}\eta^{1/2} + \eta + \nu) + \nu$.

(c) $X_n \subseteq Y_n$ and θ_n is a (Y_n, δ_n)-approximate $*$-homomorphism for $n \geq 0$.

(d) $\theta_n \approx_{Y_n, 8\sqrt{6}\eta^{1/2}+\eta+\nu} \iota$ for $n \geq 0$.

(e) $\mathrm{Ad}(u_n) \circ \theta_n \approx_{X_n, 2^{-(n-1)}\nu} \theta_{n-1}$ and $\|u_n - 1_{B^\dagger}\| \leq 4\sqrt{2}(8\sqrt{6}\eta^{1/2} + \eta + \nu) + \nu$ for $n \geq 1$. When $n = 1$, we take $u_1 = 1$.

If $B \subset_{1/5} A$, then the separability of A ensures the separability of B by Proposition 2.10. In this case fix a dense sequence $\{b_n\}_{n=0}^{\infty}$ in the unit ball of B with $b_0 = 0$, and in this case we shall require the following extra condition for our induction:

(f) $d(u_{n-1}^* \dots u_1^* b_i u_1 \dots u_{n-1}, X_n) \leq 2/5$ for $0 \leq i \leq n$.

Assuming for the moment that the induction has been accomplished, we first show that conditions (a)-(e) allow us to construct the embedding $\alpha : A \hookrightarrow B$. Define $\alpha_n = \mathrm{Ad}(u_1 \dots u_n) \circ \theta_n$ for $n \geq 1$ so $\alpha_1 = \theta_1$ since $u_1 = 1$. For a fixed integer k and an element $x \in X_k$, we have

$$\begin{aligned}\|\alpha_{n+1}(x) - \alpha_n(x)\| &= \|\mathrm{Ad}(u_1 \dots u_n) \circ \mathrm{Ad}(u_{n+1}) \circ \theta_{n+1}(x) - \mathrm{Ad}(u_1 \dots u_n) \circ \theta_n(x)\| \\ &= \|\mathrm{Ad}(u_{n+1}) \circ \theta_{n+1}(x) - \theta_n(x)\| \\ &\leq 2^{-n}\nu, \quad n \geq k, \end{aligned} \tag{4.1}$$

using (e). Density of the X_k's in the unit ball of A then shows that the sequence $\{\alpha_n\}_{n=1}^{\infty}$ converges in the point norm topology to a completely positive contraction $\alpha : A \to B$. Condition (c) implies that α is a $*$-homomorphism since each α_n is a (Y_n, δ_n)-approximate $*$-homomorphism, $\lim_{n\to\infty} \delta_n = 0$, and $\cup_{n=0}^{\infty} Y_n$ is dense in the unit ball of A. For each $x \in X_n$, it follows from (4.1) that

$$\begin{aligned}\|\alpha_n(x) - \alpha(x)\| &\leq \sum_{m=n}^{\infty} \|\alpha_{m+1}(x) - \alpha_m(x)\| \\ &\leq \sum_{m=n}^{\infty} 2^{-m}\nu = 2^{-(n-1)}\nu, \end{aligned} \tag{4.2}$$

and, in particular, that

$$\|\alpha(x) - \theta_1(x)\| = \|\alpha(x) - \alpha_1(x)\| \leq \nu, \quad x \in X_A \subseteq X_1. \tag{4.3}$$

Thus, from (4.2) and (d),

$$\begin{aligned}\|\alpha(x)\| &\geq \|\alpha_n(x)\| - 2^{-n}\nu \\ &= \|\theta_n(x)\| - 2^{-n}\nu \\ &\geq \|x\| - (8\sqrt{6}\eta^{1/2} + \eta + \nu) - 2^{-n}\nu, \quad x \in X_n. \end{aligned} \tag{4.4}$$

Letting $n \to \infty$ in (4.4), it follows that

$$\|\alpha(x)\| \geq \|x\| - (8\sqrt{6}\eta^{1/2} + \eta + \nu), \tag{4.5}$$

for any x in the unit ball of A, using the collective density of the X_n's. This shows that α is injective since $\|\alpha(x)\| \geq 1 - (8\sqrt{6}\eta^{1/2} + \eta + \nu) > 0$ for x in the unit sphere of A. Thus α is a $*$-isomorphism of A into B. Then, for $x \in X_A \subseteq X_1$, the estimate

$$\begin{aligned}\|\alpha(x) - x\| &\leq \|\alpha(x) - \theta_1(x)\| + \|\theta_1(x) - x\| \\ &\leq \nu + 8\sqrt{6}\eta^{1/2} + \eta + \nu = 8\sqrt{6}\eta^{1/2} + \eta + \mu \end{aligned} \tag{4.6}$$

follows from (4.3) and hypothesis (d).

In the case that $B \subset_{1/5} A$, we shall show that the additional assumption (f) gives $B \subset_1 \alpha(A)$ and so α is surjective by Proposition 2.4. Indeed, given $0 \leq i \leq n$, find $x \in X_n$ with $\|x - u_{n-1}^* \dots u_1^* b_i u_1 \dots u_{n-1}\| \leq 2/5$. Then

$$\begin{aligned}\|\alpha_n(x) - b_i\| &= \|u_1 \dots u_n \theta_n(x) u_n^* \dots u_1^* - b_i\| \\ &\leq \|u_n \theta_n(x) u_n^* - x\| + \|x - u_{n-1}^* \dots u_1^* b_i u_1 \dots u_{n-1}\| \\ &\leq 2\|u_n - 1_{B^\dagger}\| + \|\theta_n(x) - x\| + \|x - u_{n-1}^* \dots u_1^* b_i u_1 \dots u_{n-1}\| \\ &\leq 8\sqrt{2}(8\sqrt{6}\eta^{1/2} + \eta + \nu) + 2\nu + (8\sqrt{6}\eta^{1/2} + \eta + \nu) + 2/5 \\ &\leq 0.94 < 1, \end{aligned} \tag{4.7}$$

using (d), (e), and the upper bounds $\eta < 1/210000$ and $\nu < 1/4000$. The claim follows from the density of $\{b_i\}_{i=0}^{\infty}$ in the unit ball of B. It remains to complete the inductive construction.

We begin the induction trivially, setting $X_0 = Y_0 = \{a_0\} = \{0\}$, $u_0 = 1$, $\delta_0 = 1$ and taking any completely positive contraction $\theta_0 : A \to B$. Suppose that the construction is complete up to the n-th stage. Define $X_{n+1} = X_A \cup X_n \cup \{a_{n+1}\} \cup Y_n$ so that condition (a) holds. When $B \subset_{1/5} A$, we will have the same near containment for their unit balls with 2/5 replacing 1/5 (see Definition 2.2 and Remark 2.3). Thus we can extend X_{n+1} so that condition (f) holds. Now use Lemma 3.4 and Remark 3.5 (i) to find a finite set Y_{n+1} in the unit ball of A, containing X_{n+1}, and $0 < \delta_{n+1} < \min\{\delta_n, 2^{-(n+1)}, (5\sqrt{2})^{-1}\nu\}$ so that condition (b) holds. This is possible because direct calculation shows that $2(8\sqrt{6}\eta^{1/2} + \eta + \nu) < 13/150$ from the upper bounds on η and ν, and the estimate $\|u - 1_{B^\dagger}\| \leq 4\sqrt{2}(8\sqrt{6}\eta^{1/2} + \eta + \nu) + \nu$ follows from $\delta_n \leq (5\sqrt{2})^{-1}\nu$. Now use Lemma 3.2 to find a finite set $Z \supset Y_{n+1}$ in the unit ball of A so that given a $(Z, 3\eta)$-approximate $*$-homomorphism $\phi : A \to B$, we can find a (Y_{n+1}, δ_{n+1})-approximate $*$-homomorphism $\psi : A \to B$ with

$$\|\phi - \psi\| \leq 8\sqrt{2}(3\eta)^{1/2} + \nu = 8\sqrt{6}\eta^{1/2} + \nu. \tag{4.8}$$

This is possible because $3\eta < 1/17$. Let

$$Z' = Z \cup Z^* \cup \{zz^* : z \in Z \cup Z^*\}. \tag{4.9}$$

From the hypothesis, there is a completely positive contraction $\phi : A \to B$ with $\phi \approx_{Z',\eta} \iota$, whereupon ϕ is a $(Z, 3\eta)$-approximate $*$-homomorphism. The definition of Z then gives us a (Y_{n+1}, δ_{n+1})-approximate $*$-homomorphism $\theta_{n+1} : A \to B$ with $\|\phi - \theta_{n+1}\| \leq 8\sqrt{6}\eta^{1/2} + \nu$ verifying (c). It follows that $\theta_{n+1} \approx_{Y_{n+1}, 8\sqrt{6}\eta^{1/2}+\eta+\nu} \iota$ and so condition (d) holds. Since θ_n and θ_{n+1} are (Y_n, δ_n)-approximate $*$-homomorphisms which satisfy $\theta_n \approx_{Y_n, 2(8\sqrt{6}\eta^{1/2}+\eta+\nu)} \theta_{n+1}$, there exists a unitary u_{n+1} in $B^\dagger$ with $\mathrm{Ad}(u_{n+1}) \circ \theta_{n+1} \approx_{X_n, 2^{-n}\nu} \theta_n$ and $\|u_{n+1} - 1_{B^\dagger}\| \leq 4\sqrt{2}(8\sqrt{6}\eta^{1/2} + \eta + \nu) + \nu$ from the inductive version of condition (b). This gives condition (e). This last step is not required when $n = 0$, as $X_0 = \{0\}$ so we can take $u_1 = 1$ in this case. □

Using Proposition 2.16, it follows that sufficiently close separable and nuclear C*-algebras are isomorphic. In fact we only need one near inclusion to be relatively small.

Theorem 4.2. *Suppose that A and B are separable nuclear* C*-*algebras on some Hilbert space H with $A \subset_\gamma B$ and $B \subset_\delta A$ for*

$$\gamma \leq 1/420000, \quad \delta \leq 1/5. \tag{4.10}$$

Then A and B are isomorphic.

Proof. Take $0 < \gamma' < \gamma$ so that $A \subset_{\gamma'} B$. Then Proposition 2.16 provides the cpc maps $A \to B$ required to use Lemma 4.1 when $\eta = 2\gamma'$ and so the result follows. □

The qualitative version of Theorem A also follows as algebras close to a separable and nuclear C*-algebra must again be separable and nuclear. While the examples of [28] show that it is not possible in general to obtain isomorphisms which are uniformly close to the identity between close separable nuclear C*-algebras, Lemma 4.1 does enable us to control the behaviour of our isomorphisms on finite subsets of the unit ball.

Theorem 4.3. *Let A and B be* C*-*algebras on some Hilbert space with*

$$d(A, B) < \gamma < 1/420000. \tag{4.11}$$

If A is nuclear and separable, then A and B are isomorphic. Furthermore, given finite sets X and Y in the unit balls of A and B respectively, there exists a surjective $*$-isomorphism $\theta : A \to B$ with

$$\|\theta(x) - x\|, \ \|\theta^{-1}(y) - y\| \leq 28\gamma^{1/2}, \quad x \in X, \ y \in Y. \tag{4.12}$$

Proof. As $d(A, B) < 1/101$, Proposition 2.9 ensures that B is nuclear. Choose γ' so that $d(A, B) < \gamma' < \gamma$ and enlarge the set X if necessary so that $Y \subset_{\gamma'} X$. Proposition 2.16 shows that the hypotheses of Lemma 4.1 hold for $\eta = 2\gamma'$ so A and B are isomorphic. Furthermore, for a constant $0 < \mu < 1/2000$, Lemma 4.1 provides a surjective $*$-isomorphism $\theta : A \to B$ with

$$\|\theta(x) - x\| < 8\sqrt{6}(2\gamma')^{1/2} + 2\gamma' + \mu, \quad x \in X. \tag{4.13}$$

For each $y \in Y$, there exists $x \in X$ with $\|x - y\| \leq \gamma'$. As $\|\theta^{-1}(y) - x\| = \|y - \theta(x)\|$ it follows that

$$\|\theta^{-1}(y) - y\| \leq 2\|x - y\| + \|\theta(x) - y\| < 8\sqrt{6}(2\gamma')^{1/2} + 4\gamma' + \mu. \tag{4.14}$$

In these inequalities, a suitably small choice of μ and the inequality $\gamma' < \gamma < 1/420000$ lead to upper estimates of $28\gamma^{1/2}$ in both cases. □

Lemma 4.1 is also the technical tool behind our three near inclusion results. We can deduce two of these results now, using Propositions 2.16 and 2.17 to obtain the cpc maps required to use Lemma 4.1. Our third, and most general result, which only assumes that A has finite nuclear dimension is postponed until Section 6.

Corollary 4.4. *Let γ be a constant satisfying $0 < \gamma < 1/420000$, and consider a near inclusion $A \subset_\gamma B$ of C^*-algebras on a Hilbert space H, where A and B are nuclear and A is separable. Then A embeds into B and, moreover, for each finite subset X of the unit ball of A there exists an injective $*$-homomorphism $\theta : A \to B$ with $\theta \approx_{X,28\gamma^{1/2}} \iota$.*

Proof. Proposition 2.16 shows that A and B satisfy the hypotheses of Lemma 4.1 when $\eta = 2\gamma$. The number 28 appears from the inequality

$$8\sqrt{6}(2\gamma)^{1/2} + 2\gamma + \mu < 28\gamma^{1/2}, \tag{4.15}$$

which is valid for a sufficiently small choice of μ. □

Corollary 4.5. *Let γ be a constant satisfying $0 < \gamma < 1/12600000 \approx 7.9 \times 10^{-8}$. Consider a near inclusion $A \subset_\gamma B$ where A is unital, separable and has approximately inner half flip. Then A embeds into B and, moreover, for each finite subset X of the unit ball of A there exists an injective $*$-homomorphism $\theta : A \to B$ with $\theta \approx_{X,152\gamma^{1/2}} \iota$.*

Proof. Recall that the proof of Proposition 2.8 of [19] shows that A is nuclear. Write $\alpha = (4\sqrt{2} + 1)\gamma + 4\sqrt{2}\gamma^2$. Proposition 2.17 shows that the hypotheses of Lemma 4.1 hold for

$\eta = 8\alpha + 4\alpha^2 + 4\sqrt{2}\gamma$ and the bound on γ is chosen so that $\eta < 60\gamma < 1/210000$. The number 152 appears from the inequality

$$8\sqrt{6}\eta^{1/2} + \eta + \mu < 152\gamma^{1/2}, \tag{4.16}$$

which is valid for a sufficiently small choice of μ. □

We briefly examine stability under tensoring by the strongly self-absorbing algebras introduced in [52]. Recall that a separable unital C*-algebra D is *strongly self-absorbing* if there is an isomorphism between D and $D \otimes D$ which is approximately unitarily equivalent to the embedding $D \hookrightarrow D \otimes D$, $x \mapsto x \otimes 1_D$. Such algebras automatically have approximately inner half flip [52, Proposition 1.5] and so are simple and nuclear. A separable C*-algebra A is *D-stable* if $A \otimes D \cong A$. When A is unital, an equivalent formulation of D-stability for A is the following condition: given finite sets X in A and Y in D and $\varepsilon > 0$, there exists an embedding $\phi : D \hookrightarrow A$ such that $\|\phi(y)x - x\phi(y)\| < \varepsilon$ for all $x \in X$ and $y \in Y$ ([52, Theorem 2.2]).

Corollary 4.6. *Let C and D be* C*-*algebras with D separable and strongly self-absorbing in the sense of [52]. Then, within the space of separable* C*-*subalgebras of C containing 1_C, the D-stable* C*-*algebras form a closed subset.*

Proof. Suppose that $A \subseteq C$ is a unital C*-subalgebra such that, for any $\gamma > 0$, there is a D-stable C*-algebra $B \subseteq C$ with $d(A, B) < \gamma$. We have to show that A is D-stable. To this end, note that any finite subset X of A may be approximated arbitrarily well by a finite subset $\bar{X}$ in a nearby unital D-stable B. Now for a finite subset Y of D, there is an embedding of D into B such that Y almost commutes with $\bar{X}$. From Corollary 4.5 we obtain an embedding of D into A which sends Y to a close subset $\bar{Y} \subseteq A$. Now if D and B were sufficiently close, $\bar{Y}$ and X will almost commute. By [52, Theorem 2.2], this is enough to ensure that A is D-stable. □

We have not been able to decide whether the D-stable subalgebras also form an open subset. However, Corollary 4.5 immediately gives results for embeddings of strongly self-absorbing C*-algebras, since these have approximately inner half flip.

Corollary 4.7. *Let $A \subset_\gamma B$ be a near inclusion of* C*-*algebras on a Hilbert space H, where $0 < \gamma < 1/12600000 \approx 7.9 \times 10^{-8}$, and let D be a strongly self-absorbing C*-algebra. If A admits an embedding of D, then so does B. Moreover, on finite subsets of the unit ball of D one may choose the embedding into B to be within $152\gamma^{1/2}$ of the embedding into A.*

Recent progress in the structure theory of nuclear C*-algebras suggests that the preceding corollaries are particularly interesting in the case where $D = \mathcal{Z}$, the Jiang–Su algebra. Moreover, $\mathcal{Z}$-stability is relevant also for non-nuclear C*-algebras; for example it will be shown in [25] that it implies finite length (hence Kadison's similarity property). We have not been able to establish whether $\mathcal{Z}$-stability is preserved under closeness, i.e., to answer the following question: If A and B are sufficiently close, and A is $\mathcal{Z}$-stable, is B $\mathcal{Z}$-stable as well? The previous corollary at least shows that the existence of embeddings of $\mathcal{Z}$ is preserved under close containment. The existence of such embeddings is highly nontrivial (even for otherwise well-behaved C*-algebras, such as simple, unital AH algebras, cf. [15]).

5 Unitary Equivalence

In the previous section we have shown that two close separable nuclear C*- subalgebras A and B of $\mathbb{B}(H)$ are $*$-isomorphic. We now show in Theorem 5.4 that there is a unitary u such that $uAu^* = B$ when H is separable. This establishes Theorem B of the introduction and gives a complete answer to Kadison and Kastler's question from [31] in this context.

The technicalities of the upcoming proofs warrant some additional overall explanation of the methods employed. Before describing our approach, we recall our unitisation conventions from Notation 2.18, so that if A and B have the same ultraweak closure, then $A^\dagger$ and $B^\dagger$ have the same unit, e_A, where e_A is the support projection of A (and of B).

In Theorem 5.3 below, we obtain the unitary that implements a $*$-isomorphism under the additional assumption that A and B have the same ultraweak closure. This restriction is removed in Theorem 5.4 by using known perturbation results for injective von Neumann algebras to reduce to the situation of Theorem 5.3. The assumption that A and B have the same ultraweak closure enables us to approximate *-strongly unitaries in $C^*(A, B, 1)$ with a uniform spectral gap by unitaries in A or B using the Kaplansky density result of Lemma 3.7. The basic idea of our proof is to construct a sequence of unitaries $\{u_n\}_{n=1}^\infty$ in $C^*(A, B, 1)$ using Lemma 3.4 and Theorem 4.3 so that $\lim_{n\to\infty} \mathrm{Ad}(u_n)$ exists in the point-norm topology and defines a surjective $*$-isomorphism from A onto B. If this sequence converged *-strongly to a unitary u, then this would implement unitary equivalence. However, there is no reason to believe that this happens. Thus we produce a modified sequence $\{v_n\}_{n=1}^\infty$ of unitaries which explicitly converges *-strongly, while maintaining the requirement that $\lim_{n\to\infty} \mathrm{Ad}(v_n)$ gives a surjective $*$-isomorphism from A onto B. Lemma 3.7 enables us to approximate *-strongly the unitaries obtained in $C^*(A, B, 1)$ by unitaries in $B^\dagger$. In principle, the idea is to multiply by unitaries produced by Lemma 3.7, to ensure *-strong convergence. In order for $\lim_{n\to\infty} \mathrm{Ad}(v_n)$ (in the point-norm topology) to exist and define a $*$-isomorphism, it is essential that these unitaries can be taken to approximately commute with suitable finite sets. In practice, we have further technical hurdles to overcome. One instance of this is the need to ensure that the unitaries to which we apply Lemma 3.7 have the required uniform spectral gap. This leads us to split off two technical lemmas (Lemmas 5.1 and 5.2) which combine the results of Lemmas 3.4, 3.7 and Theorem 4.3 in exactly the correct order for use in the inductive step of Theorem 5.3. We advise the reader to begin this sequence of results with the latter one, referring back to the preceding two as needed.

The bounds on the constant γ in the next three results are chosen so that whenever we wish to use Lemma 3.4 or Theorem 4.3, it is legitimate to do so. In particular, the choice ensures that $392\gamma^{1/2} \leq 13/150$, so that Lemma 3.4 and Remark 3.5 (ii) can be applied with γ replaced by $\alpha\gamma^{1/2}$ for $\alpha \leq 392$. It also guarantees the validity of (5.46), an inequality that governs the overall bound on γ. In part (VIII) of the following lemma, the inequality $1848\gamma^{1/2} \leq 1848 \times 10^{-4} < 1$ will allow us to use Lemma 3.7.

Lemma 5.1. *Suppose that A and B are separable and nuclear* C**-algebras acting non-degerately on a Hilbert space H. Suppose that $A'' = B'' = M$ and $d(A, B) < \gamma \leq 10^{-8}$. Given finite subsets $X_0, Z_{0,B}$ of the unit ball of B, a finite subset $Z_{0,A}$ of the unit ball of A, and constants $\varepsilon_0 > 0$, $\mu_0 > 0$, there exist finite subsets Y_0, Z_0 of the unit ball of B, $\delta_0 > 0$, a unitary $u_0 \in M$ and a surjective $*$-isomorphism $\sigma : B \to A$ with the following properties:*

(I) $X_0 \subseteq Y_0$.

(II) $\delta_0 < \varepsilon_0/2$.

(III) $Z_{0,B} \subseteq Z_0$.

(IV) $\sigma \approx_{Z_0,28\gamma^{1/2}} \iota$ *and* $\sigma^{-1} \approx_{Z_{0,A},28\gamma^{1/2}} \iota$.

(V) $\sigma \approx_{Y_0,\delta_0/2} \mathrm{Ad}(u_0)$.

(VI) $\|u_0 - 1_M\| \leq 84\gamma^{1/2}$.

(VII) *Given any $*$-homomorphism $\psi : B \to D$ for some unital* C**-subalgebra D of M with $\psi \approx_{Z_0,364\gamma^{1/2}} \iota$, there exists a unitary $w_0 \in$* C$^*(A, D, 1_M)$ *with $\|1_M - w_0\| \leq 1176\gamma^{1/2}$ such that*

$$\mathrm{Ad}(w_0) \circ \psi \approx_{Y_0,\delta_0/2} \sigma. \tag{5.1}$$

(VIII) *Given a unitary* $v_0 \in M$ *with* $\|v_0 - u_0\| \leq 1848\gamma^{1/2}$ *and* $\mathrm{Ad}(v_0) \approx_{Y_0,\delta_0} \sigma$*, and given a finite subset* S *of the unit ball of* H*, there exists a unitary* $v_0' \in B^\dagger$ *satisfying* $\|v_0' - 1_{B^\dagger}\| \leq 1848\gamma^{1/2}$*,* $\mathrm{Ad}(v_0 v_0') \approx_{X_0,\varepsilon_0} \sigma$*, and*

$$\|(v_0 v_0' - u_0)\xi\|, \quad \|(v_0 v_0' - u_0)^*\xi\| < \mu_0, \qquad \xi \in S. \tag{5.2}$$

Proof. By Lemma 3.7 applied to the unitisation $B^\dagger$, there is a finite subset $\widetilde{Y}_0 \subseteq B^\dagger$ and $\delta_0 > 0$ such that if $u \in M$ is a unitary satisfying $\|u - 1_M\| \leq 1848\gamma^{1/2}$ and

$$\|\tilde{y}u - u\tilde{y}\| \leq 3\delta_0, \qquad \tilde{y} \in \widetilde{Y}_0, \tag{5.3}$$

and a finite subset S_0 of H is given, then there exists a unitary $v \in B^\dagger$ satisfying $\|v - 1_{B^\dagger}\| \leq 1848\gamma^{1/2}$,

$$\|xv - vx\| \leq \varepsilon_0/2, \qquad x \in X_0, \tag{5.4}$$

and

$$\|(v-u)\xi\|,\ \|(v-u)^*\xi\| < \mu_0, \qquad \xi \in S_0. \tag{5.5}$$

In applying this lemma, we have replaced ε_0 by $\varepsilon_0/2$ and δ_0 by $3\delta_0$. We may replace δ_0 by any smaller number, ensuring that condition (II) holds.

Each $\tilde{y}_i \in \widetilde{Y}_0$ can be written $\tilde{y}_i = \alpha_i + 2y_i$ for scalars α_i and elements y_i in the unit ball of B, and let Y_1 denote the collection of these y_i's. Define $Y_0 = Y_1 \cup X_0$, so that condition (I) is satisfied. If the inequality

$$\|yu - uy\| < 3\delta_0/2, \qquad y \in Y_0, \tag{5.6}$$

holds for a particular unitary $u \in M$, then it also holds for $y \in Y_1 \subseteq Y_0$, implying (5.3) and hence (5.4) and (5.5).

Since $*$-homomorphisms are (X,δ)-approximate $*$-homomorphisms for any set X and any $\delta > 0$, we may apply Lemma 3.4 and Remark 3.5 (ii) for any unital C*-subalgebra D of M to conclude that there exists a finite set Z_0 in the unit ball of B with the following property. If $\phi_1, \phi_2 : B \to D$ are $*$-homomorphisms with $\phi_1 \approx_{Z_0,\alpha\gamma^{1/2}} \phi_2$, where $\alpha \in \{28,\ 392\}$, then there exists a unitary $w_0 \in D$ with $\|1_D - w_0\| \leq 3\alpha\gamma^{1/2}$ and $\mathrm{Ad}(w_0) \circ \phi_1 \approx_{Y_0,\delta_0/2} \phi_2$. By increasing Z_0 to contain $Z_{0,B}$, condition (III) is satisfied.

By Theorem 4.3 we may now choose a surjective $*$-isomorphism $\sigma : B \to A$ so that condition (IV) holds. Since σ and ι are $*$-homomorphisms of B into $\mathrm{C}^*(A, B, 1_M) \subseteq M$, there then exists a unitary $u_0 \in M$ with $\|u_0 - 1_M\| \leq 84\gamma^{1/2}$ so that $\sigma \approx_{Y_0,\delta_0/2} \mathrm{Ad}(u_0)$, using $\alpha = 28$ above. This establishes conditions (V) and (VI).

From condition (IV), we have $\sigma \approx_{Z_0,28\gamma^{1/2}} \iota$. If $\psi : B \to D \subseteq M$ is another $*$-homomorphism satisfying $\psi \approx_{Z_0,364\gamma^{1/2}} \iota$, then $\psi \approx_{Z_0,392\gamma^{1/2}} \sigma$. Taking $\alpha = 392$ above, the choice of Z_0 allows us to find a unitary $w_0 \in M$ with $\|1_M - w_0\| \leq 1176\gamma^{1/2}$ so that condition (VII) is satisfied.

It only remains to establish (VIII). Suppose that $v_0 \in M$ satisfies $\|v_0 - u_0\| \leq 1848\gamma^{1/2}$ and $\mathrm{Ad}(v_0) \approx_{Y_0,\delta_0} \sigma$, and that a finite subset S of the unit ball of H is given. Since $\sigma \approx_{Y_0,\delta_0/2} \mathrm{Ad}(u_0)$ from condition (V), we obtain

$$\mathrm{Ad}(v_0) \approx_{Y_0,3\delta_0/2} \mathrm{Ad}(u_0), \tag{5.7}$$

implying that

$$\|v_0^* u_0 y - y v_0^* u_0\| \leq 3\delta_0/2, \qquad y \in Y_0. \tag{5.8}$$

Since $\|1 - v_0^* u_0\| \leq 1848\gamma^{1/2}$, we can take $S_0 = S \cup v_0^* S$ so the choice of $\widetilde{Y}_0$ at the start of the proof allows us to find a unitary $v_0' \in B^\dagger$ with the following properties:

1. $\|1 - v_0'\| \leq 1848\gamma^{1/2}$,
2. $\|v_0' x - x v_0'\| < \varepsilon_0/2, \quad x \in X_0$,
3. $\|(v_0' - v_0^* u_0)\xi\| < \mu$, $\|(v_0' - v_0^* u_0)^* v_0^* \xi\| < \mu, \quad \xi \in S$.

The third condition above gives (5.2), while the second shows that $\mathrm{Ad}(v_0') \approx_{X_0,\varepsilon_0/2} \iota$. Since $X_0 \subseteq Y_0$ and $\delta_0 < \varepsilon_0/2$, we have $\mathrm{Ad}(v_0) \approx_{X_0,\varepsilon_0/2} \sigma$. It follows that $\mathrm{Ad}(v_0 v_0') \approx_{X_0,\varepsilon_0} \sigma$, and condition (VIII) is proved. □

Lemma 5.2. *Suppose that A and B are separable and nuclear C^*-algebras acting non-degerately on a Hilbert space H. Suppose that $A'' = B'' = M$ and $d(A, B) < \gamma \leq 10^{-8}$. Given finite subsets X in the unit ball of A and Z_B in the unit ball of B and constants $\varepsilon, \mu > 0$, there exist finite subsets Y of the unit ball of A and Z of the unit ball of B, a constant $\delta > 0$, a unitary $u \in M$, and a surjective $*$-isomorphism $\theta : A \to B$ with the following properties.*

(i) $\delta < \varepsilon$.

(ii) $X \subseteq_\varepsilon Y$.

(iii) $\|u - 1_M\| \leq 252\gamma^{1/2}$.

(iv) $\theta \approx_{Y,\delta} \mathrm{Ad}(u)$.

(v) $\theta \approx_{X,364\gamma^{1/2}} \iota$, $\theta^{-1} \approx_{Z_B,364\gamma^{1/2}} \iota$.

(vi) *Given a surjective $*$-isomorphism $\phi : A \to B$ with $\phi^{-1} \approx_{Z,364\gamma^{1/2}} \iota$, there exists a unitary $w \in B^\dagger$ with $\|w - u\| \leq 1596\gamma^{1/2}$ and $\mathrm{Ad}(w) \circ \phi \approx_{Y,\delta/2} \theta$.*

(vii) *Given a unitary $v \in M$ with $\|v - u\| \leq 1848\gamma^{1/2}$ and $\mathrm{Ad}(v) \approx_{Y,\delta} \theta$, and given any finite subset S of the unit ball of H, there exists a unitary $v' \in B^\dagger$ with $\|1_{B^\dagger} - v'\| \leq 1848\gamma^{1/2}$, $\mathrm{Ad}(v'v) \approx_{X,\varepsilon} \theta$, and*

$$\|(v'v - u)\xi\|,\ \|(v'v - u)^*\xi\| < \mu, \qquad \xi \in S. \tag{5.9}$$

Proof. By Lemma 3.4 and Remark 3.5 (ii), there exists a finite subset Z_1 of the unit ball of A with the following property. Given a unital C*-subalgebra D of M, if $\phi_1, \phi_2 : A \to D$ are $*$-homomorphisms such that $\phi_1 \approx_{Z_1,56\gamma^{1/2}} \phi_2$ then there is a unitary $w_1 \in D$ satisfying $\|w_1 - 1_D\| \leq 168\gamma^{1/2}$ and $\phi_1 \approx_{X,\varepsilon/3} \mathrm{Ad}(w_1) \circ \phi_2$.

By Theorem 4.3, we may choose a surjective $*$-isomorphism $\beta : A \to B$ with the property that $\beta \approx_{Z_1,28\gamma^{1/2}} \iota$. Then define a finite subset X_0 of the unit ball of B by $X_0 = \beta(X)$. Taking $\varepsilon_0 = \varepsilon/3$, $\mu_0 = \mu$, $Z_{0,A} = X$ and $Z_{0,B} = \beta(Z_1) \cup Z_B$, we may apply Lemma 5.1 to obtain $Y_0, Z_0, \delta_0, \sigma$ and u_0 satisfying conditions (I)–(VIII) of this lemma. We then define $Z = Z_0 \subseteq B$ and $\delta = \delta_0$. By Lemma 5.1 (II), $\delta < \varepsilon_0/2 < \varepsilon$ so condition (i) holds. By Lemma 5.1 (IV), $\sigma \approx_{Z_0,28\gamma^{1/2}} \iota$, so $\sigma \circ \beta \approx_{Z_1,56\gamma^{1/2}} \mathrm{id}_A$, since $\beta(Z_1) \subseteq Z_{0,B} \subseteq Z_0$ from Lemma 5.1 (III). By the choice of Z_1 above, there exists a unitary $w_1 \in A^\dagger \subseteq M$ with $\|1_M - w_1\| \leq 168\gamma^{1/2}$ so that

$$\mathrm{Ad}(w_1) \approx_{X,\varepsilon/3} \sigma \circ \beta. \tag{5.10}$$

Now define a surjective $*$-isomorphism $\theta : A \to B$ to be $\sigma^{-1} \circ \mathrm{Ad}(w_1)$. Then $\theta^{-1} = \mathrm{Ad}(w_1^*) \circ \sigma$, and (5.10) can be rewritten as

$$\theta \approx_{X,\varepsilon/3} \beta, \tag{5.11}$$

which implies that

$$\theta^{-1} \approx_{X_0,\varepsilon/3} \beta^{-1}. \tag{5.12}$$

If $z \in Z_0$ then, from Lemma 5.1 (IV),

$$\|\sigma(z) - z\| \leq 28\gamma^{1/2}. \tag{5.13}$$

Thus

$$\begin{aligned}\|\theta^{-1}(z)-z\| &= \|\mathrm{Ad}(w_1^*)(\sigma(z)-z)+\mathrm{Ad}(w_1^*)(z)-z\|\\ &\leq 28\gamma^{1/2}+2\|w_1^*-1_M\|\\ &\leq 364\gamma^{1/2}, \qquad z\in Z_0. \end{aligned}\tag{5.14}$$

Consequently $\theta^{-1}\approx_{Z_0,364\gamma^{1/2}}\iota$, so the second statement of condition (v) holds since $Z_B\subseteq Z_{0,B}\subseteq Z_0$. From Lemma 5.1 (IV), $\sigma^{-1}\approx_{Z_{0,A},28\gamma^{1/2}}\iota$, so a similar calculation leads to $\theta\approx_{Z_{0,A},364\gamma^{1/2}}\iota$, establishing the first statement of condition (v), since $X=Z_{0,A}$.

We now define $u=u_0^*w_1$. By Lemma 5.1 (VI), $\|u_0-1_M\|\leq 84\gamma^{1/2}$ and so

$$\begin{aligned}\|u-1_M\| &= \|w_1-u_0\|\leq\|w_1-1_M\|+\|1_M-u_0\|\\ &\leq 168\gamma^{1/2}+84\gamma^{1/2}=252\gamma^{1/2},\end{aligned}\tag{5.15}$$

giving condition (iii). Now let

$$Y=w_1^*\sigma(Y_0)w_1=\theta^{-1}(Y_0),\tag{5.16}$$

where the latter equality comes from the definition of θ. From Lemma 5.1 (V), $\sigma\approx_{Y_0,\delta/2}\mathrm{Ad}(u_0)$, and so $\sigma^{-1}\approx_{\sigma(Y_0),\delta/2}\mathrm{Ad}(u_0^*)$. The relation $\theta=\sigma^{-1}\circ\mathrm{Ad}(w_1)$ then gives

$$\theta\approx_{w_1^*\sigma(Y_0)w_1,\delta/2}\mathrm{Ad}(u),\tag{5.17}$$

since $u=u_0^*w_1$ and so condition (iv) holds. By definition, $Y=\theta^{-1}(Y_0)$ and $\beta(X)=X_0$. Moreover, $X_0\subseteq Y_0$ from Lemma 5.1 (I). Using (5.12), we see that

$$X=\beta^{-1}(X_0)\subseteq_{\varepsilon/3}\theta^{-1}(X_0)\subseteq\theta^{-1}(Y_0)=Y,\tag{5.18}$$

verifying condition (ii).

Now consider a surjective $*$-isomorphism $\phi:A\to B$ with $\phi^{-1}\approx_{Z,364\gamma^{1/2}}\iota$, which we extend canonically to a unital $*$-isomorphism of $A^\dagger$ onto $B^\dagger$, also denoted ϕ. By Lemma 5.1 (VII), there is a unitary $w_0\in C^*(A,1_M)=A^\dagger\subseteq M$ with $\|w_0-1_M\|\leq 1176\gamma^{1/2}$ such that

$$\mathrm{Ad}(w_0)\circ\phi^{-1}\approx_{Y_0,\delta/2}\sigma.\tag{5.19}$$

Each $y\in Y$ can be written as $\mathrm{Ad}(w_1^*)\circ\sigma(y_0)=\theta^{-1}(y_0)$ for some $y_0\in Y_0$, so

$$\begin{aligned}\|\theta(y)-\phi(w_0^*w_1yw_1^*w_0)\| &= \|y_0-(\phi\circ\mathrm{Ad}(w_0^*)\circ\sigma)(y_0)\|\\ &= \|(\mathrm{Ad}(w_0)\circ\phi^{-1})(y_0)-\sigma(y_0)\|\\ &\leq\delta/2,\end{aligned}\tag{5.20}$$

from (5.19). Thus

$$\theta \approx_{Y,\delta/2} \phi \circ \mathrm{Ad}(w_0^* w_1). \tag{5.21}$$

Then

$$\phi \circ \mathrm{Ad}(w_0^* w_1) = \mathrm{Ad}(\phi(w_0^* w_1)) \circ \phi, \tag{5.22}$$

since $w_0^* w_1 \in A^\dagger$. If we define $w = \phi(w_0^* w_1) \in B^\dagger$, then $\theta \approx_{Y,\delta/2} \mathrm{Ad}(w) \circ \phi$ from (5.21) and (5.22). Moreover, the earlier estimates $\|w_0 - 1_M\| \leq 1176\gamma^{1/2}$ and $\|w_1 - 1_M\| \leq 168\gamma^{1/2}$ give $\|w_0^* w_1 - 1_M\| \leq 1344\gamma^{1/2}$ and so $\|w - 1_M\| \leq 1344\gamma^{1/2}$. Recalling the estimate of (5.15),

$$\begin{aligned} \|w - u\| &\leq \|w - 1_M\| + \|u - 1_M\| \\ &\leq 1344\gamma^{1/2} + 252\gamma^{1/2} = 1596\gamma^{1/2}, \end{aligned} \tag{5.23}$$

and condition (vi) is verified.

It only remains to verify condition (vii). Consider a unitary $v \in M$ with $\|v - u\| \leq 1848\gamma^{1/2}$, and $\mathrm{Ad}(v) \approx_{Y,\delta} \theta$, and fix a finite subset S of the unit ball of H. Then

$$\mathrm{Ad}(v w_1^*) \approx_{w_1 Y w_1^*,\delta} \theta \circ \mathrm{Ad}(w_1^*) = \sigma^{-1} \tag{5.24}$$

so

$$\mathrm{Ad}(v w_1^*) \approx_{\sigma(Y_0),\delta} \sigma^{-1} \tag{5.25}$$

and

$$\mathrm{Ad}(w_1 v^*) \approx_{Y_0,\delta} \sigma, \tag{5.26}$$

using $\theta = \sigma^{-1} \circ\ \mathrm{Ad}(w_1)$ and $Y = \theta^{-1}(Y_0)$. Since $u = u_0^* w_1$, we have the estimate

$$\|w_1 v^* - u_0\| = \|v^* - u^*\| \leq 1848\gamma^{1/2}. \tag{5.27}$$

Let $S' = S \cup \{w_1\xi\colon\ \xi \in S\}$, a finite subset of the unit ball of H. Then Lemma 5.1 (VIII), with $v_0 = w_1 v^*$, gives a unitary $v_0' \in B^\dagger$ satisfying $\|v_0' - 1_M\| \leq 1848\gamma^{1/2}$, $\mathrm{Ad}(w_1 v^* v_0') \approx_{X_0,\varepsilon/3} \sigma$, and

$$\|(w_1 v^* v_0' - u_0)\xi\|,\ \|(w_1 v^* v_0' - u_0)^* w_1 \xi\| < \mu, \qquad \xi \in S. \tag{5.28}$$

Set $v' = v_0'^*$. Then $\|1_M - v'\| \leq 1848\gamma^{1/2}$, and

$$\|(v^* v'^* - u^*)\xi\|,\ \|(v' v - u)\xi\| < \mu, \qquad \xi \in S. \tag{5.29}$$

Moreover, since $w_1 v^* v'^* = w_1 v^* v_0'$, we obtain

$$\mathrm{Ad}(v^* v'^*) \approx_{X_0,\varepsilon/3} \mathrm{Ad}(w_1^*) \circ \sigma = \theta^{-1}, \tag{5.30}$$

and

$$\mathrm{Ad}(v^*v'^*) \approx_{X_0,2\varepsilon/3} \beta^{-1} \tag{5.31}$$

from (5.30) and (5.12). Thus

$$\mathrm{Ad}(v'v) \approx_{X,2\varepsilon/3} \beta, \tag{5.32}$$

because $X = \beta^{-1}(X_0)$. Since $\theta \approx_{X,\varepsilon/3} \beta$ from (5.11), we obtain $\mathrm{Ad}(v'v) \approx_{X,\varepsilon} \theta$. Thus condition (vii) holds, completing the proof. □

We are now in a position to prove one of the main results of the paper, the unitary implementation of isomorphisms between separable nuclear close C*-algebras. We first prove this under the additional hypothesis that the two algebras have the same ultraweak closure, from which we will deduce the general case subsequently.

Theorem 5.3. *Suppose that A and B are* C*-*algebras acting non-degenerately on a separable Hilbert space H, and that A is separable and nuclear. Suppose that $A'' = B'' = M$ and $d(A,B) < \gamma \leq 10^{-8}$. Then there exists a unitary $u \in M$ such that $uAu^* = B$.*

Proof. Since $\gamma < 1/101$, Propositions 2.9 and 2.10 show that B is also separable and nuclear. Fix dense sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ in the unit balls of A and B respectively, and a dense sequence $\{\xi_n\}_{n=1}^\infty$ in the unit ball of H. We will construct inductively finite subsets $\{X_n\}_{n=0}^\infty$ and $\{Y_n\}_{n=0}^\infty$ of the unit ball of A, finite subsets $\{Z_n\}_{n=0}^\infty$ of the unit ball of B, positive constants $\{\delta_n\}_{n=0}^\infty$, surjective $*$-isomorphisms $\{\theta_n : A \to B\}_{n=0}^\infty$, and unitaries $\{u_n\}_{n=0}^\infty$ in M to satisfy the following conditions.

(1) $a_1, \ldots, a_n \in X_n,\ n \geq 1$.

(2) $X_n \subseteq_{2^{-n}/3} Y_n$ and $\delta_n < 2^{-n},\ n \geq 0$.

(3) $\theta_n \approx_{X_{n-1},2^{-(n-1)}} \theta_{n-1},\ n \geq 1$.

(4) $\theta_n \approx_{Y_n,\delta_n} \mathrm{Ad}(u_n),\ n \geq 0$.

(5) $\|(u_n - u_{n-1})\xi_i\|,\ \|(u_n - u_{n-1})^*\xi_i\| < 2^{-n}$ for $1 \leq i \leq n$.

(6) For each $1 \leq i \leq n$, there exists $x \in X_n$ with $\|\theta_n(x) - b_i\| \leq 9/10$.

(7) Given a surjective $*$-isomorphism $\phi : A \to B$ with $\phi^{-1} \approx_{Z_n,364\gamma^{1/2}} \iota$, there exists a unitary $w \in B^\dagger$ with $\|w - u_n\| \leq 1596\gamma^{1/2}$ and $\mathrm{Ad}\, w \circ \phi \approx_{Y_n,\delta_n/2} \theta_n$.

(8) Given a unitary $v \in M$ with $\|v - u_n\| \leq 1848\gamma^{1/2}$ and $\mathrm{Ad}(v) \approx_{Y_n,\delta_n} \theta_n$, and given any finite subset S of the unit ball of H, there exists a unitary $v' \in B^\dagger$ with $\|v' - 1_M\| \leq 1848\gamma^{1/2}$, $\mathrm{Ad}(v'v) \approx_{X_n,2^{-(n+1)}} \theta_n$, and

$$\|(v'v - u_n)\xi\|,\ \|(v'v - u_n)^*\xi\| < 2^{-(n+1)}, \qquad \xi \in S. \tag{5.33}$$

(9) There exists a unitary $z \in B^\dagger$ with $\|z - u_n\| \leq 252\gamma^{1/2}$.

Conditions (7)–(9) are not needed to derive unitary equivalence but are used in the inductive step. Assuming that the induction has been accomplished, we first show how conditions (1)–(6) establish unitary implementation.

Conditions (1) and (3) imply that the sequence $\{\theta_n\}_{n=1}^\infty$ converges in the point norm topology to a $*$-isomorphism θ of A into B. Fix $i \geq 1$. For a given integer $n \geq i$, condition (6) allows us to choose $x \in X_n$ so that $\|\theta_n(x) - b_i\| \leq 9/10$. By condition (3), $\|\theta_{m+1}(x) - \theta_m(x)\| \leq 2^{-m}$ for $m \geq n$. Thus

$$\begin{aligned}\|\theta(x) - b_i\| &\leq \|\theta_n(x) - b_i\| + \sum_{m=n}^\infty 2^{-m} \\ &\leq 9/10 + 2^{-(n-1)}.\end{aligned} \tag{5.34}$$

Since $n \geq i$ was arbitrary, (5.34) and the density of $\{b_i\}_{i=1}^\infty$ in the unit ball of B show that $d(B, \theta(A)) \leq 9/10$, and so $\theta(A) = B$ by Proposition 2.4. Thus $\theta : A \to B$ is a surjective $*$-isomorphism. Since the unitary group of M is closed in the *-strong topology, condition (5) ensures that the sequence $\{u_n\}_{n=1}^\infty$ converges *-strongly to a unitary $u \in M$. Conditions (1), (2) and (4) then show that $\theta = \mathrm{Ad}(u)$, and so $B = uAu^*$, proving the result.

We start the induction by taking $X_0 = Y_0 = \emptyset$, $Z_0 = \emptyset$, $\delta_0 = 1/2$, $u_0 = 1$, and θ_0 any $*$-isomorphism of A onto B, possible by Theorem 4.3. At this initial level, conditions (1), (3), (5) and (6) do not have meaning, but these will not be used in the inductive step. Conditions (2) and (4) are trivial (as $X_n = Y_n = \emptyset$), while conditions (7) and (9) are satisfied by taking $w = 1_M$ and $z = 1_M$ respectively. In condition (8) given a unitary $v \in M$ with $\|v - u_0\| \leq 1848\gamma^{1/2}$ and any finite subset S of the unit ball of H, take $v' = v^*$ so that $\|v' - 1_M\| = \|v - u_0\|$ and the left hand side of (5.33) vanishes. It remains to carry out the inductive step. In order to distinguish the conditions that we are assuming at level n from those we are proving at the next level, we employ the notations $(\cdot)_n$ and $(\cdot)_{n+1}$ as appropriate.

Now suppose that the various objects have been constructed to satisfy $(1)_n$– $(9)_n$. Let $z \in B^\dagger$ be the unitary of condition $(9)_n$ satisfying $\|z - u_n\| \leq 252\gamma^{1/2}$. Then z^*b_iz lies in the unit ball of B for $1 \le i \le n+1$, so we may choose elements x_i in the unit ball of A such that $\|x_i - z^* b_i z\| < \gamma$, for $1 \le i \le n+1$. Now define

$$X_{n+1} = X_n \cup Y_n \cup \{a_1, \dots, a_{n+1}\} \cup \{x_1, \dots, x_{n+1}\} \tag{5.35}$$

so that condition $(1)_{n+1}$ is satisfied.

In Lemma 5.2, let $X = X_{n+1}$, $\varepsilon = \delta_n/6$, $\mu = 2^{-(n+2)}$ and $Z_B = Z_n$, and let $Y_{n+1} \subseteq A$, $\delta_{n+1} > 0$, $Z_{n+1} \subseteq B$, $u \in M$ and $\theta : A \to B$ be the resulting objects which satisfy conditions (i)–(vii) of that lemma. By $(2)_n$ and Lemma 5.2 (i) $\delta_{n+1} < \varepsilon = \delta_n/6 < 2^{-(n+1)}/3$ and so the inequality $\delta_{n+1} < 2^{-(n+1)}$ holds. Also $X_{n+1} \subseteq_{2^{-(n+1)}/3} Y_{n+1}$ since $(2)_n$ ensures that $\varepsilon \le 2^{-(n+1)}/3$. Thus condition $(2)_{n+1}$ is satisfied. By Lemma 5.2 (v), $\theta^{-1} \approx_{Z_n, 364\gamma^{1/2}} \iota$, so we may apply condition $(7)_n$ to find a unitary $w \in B^\dagger$ with $\|w - u_n\| \le 1596\gamma^{1/2}$ such that $\mathrm{Ad}(w) \circ \theta \approx_{Y_n, \delta_n/2} \theta_n$. From Lemma 5.2 (iv), $\mathrm{Ad}(u) \approx_{Y_{n+1}, \delta_{n+1}} \theta$. Since $Y_n \subseteq X_{n+1} \subset_\varepsilon Y_{n+1}$ and $\delta_{n+1} \le \varepsilon = \delta_n/6$, a simple triangle inequality argument gives $\mathrm{Ad}(u) \approx_{Y_n, \delta_n/2} \theta$. It follows that

$$\mathrm{Ad}(wu) \approx_{Y_n, \delta_n} \theta_n. \tag{5.36}$$

By Lemma 5.2 (iii), $\|u - 1_M\| \le 252\gamma^{1/2}$, so

$$\|wu - u_n\| \le \|w - u_n\| + \|w(u - 1_M)\| = \|w - u_n\| + \|u - 1_M\| \le 1848\gamma^{1/2}. \tag{5.37}$$

Thus the unitary wu satisfies the initial hypotheses of condition $(8)_n$ which we can now apply to the set $S = \{\xi_1, \dots, \xi_{n+1}\}$. Consequently there exists a unitary $v' \in B^\dagger$ with $\|v' - 1_M\| \le 1848\gamma^{1/2}$,

$$\mathrm{Ad}(v'wu) \approx_{X_n, 2^{-(n+1)}} \theta_n, \tag{5.38}$$

and

$$\|(v'wu - u_n)\xi_i\|,\ \|(v'wu - u_n)^*\xi_i\| < 2^{-(n+1)}, \qquad 1 \le i \le n+1. \tag{5.39}$$

Defining $u_{n+1} = v'wu$, we see that condition $(5)_{n+1}$ follows from (5.39). Now $v'w \in B^\dagger$, so we may take z in condition $(9)_{n+1}$ to be this unitary, since

$$\|v'w - u_{n+1}\| = \|v'w - v'wu\| = \|1 - u\| \le 252\gamma^{1/2}. \tag{5.40}$$

Next, define $\theta_{n+1} = \mathrm{Ad}(v'w) \circ \theta$, which maps A onto B because B is an ideal in $B^\dagger$. Lemma 5.2 (iv) gives $\mathrm{Ad}(u) \approx_{Y_{n+1}, \delta_{n+1}} \theta$, so applying $\mathrm{Ad}(v'w)$ results in $\mathrm{Ad}(u_{n+1}) \approx_{Y_{n+1}, \delta_{n+1}} \theta_{n+1}$, proving condition $(4)_{n+1}$.

We turn now to condition $(3)_{n+1}$. The choices of X_{n+1}, Y_{n+1} and δ_{n+1} give the inclusions $X_{n+1} \subseteq_{2^{-(n+1)}/3} Y_{n+1}$, and $X_n \subseteq X_{n+1}$, and also the inequality $\delta_{n+1} < 2^{-(n+1)}/3$. Thus

$$\theta_{n+1} = \mathrm{Ad}(v'w) \circ \theta \approx_{X_n, 2^{-(n+1)}} \mathrm{Ad}(u_{n+1}). \tag{5.41}$$

Combined with (5.38), we obtain

$$\theta_{n+1} \approx_{X_n, 2^{-n}} \theta_n, \tag{5.42}$$

and condition $(3)_{n+1}$ is proved.

The choice of θ from Lemma 5.2 (v) entailed $\theta \approx_{X_{n+1}, 364\gamma^{1/2}} \iota$. Applying $\mathrm{Ad}(v'w)$ to this gives

$$\theta_{n+1} \approx_{X_{n+1}, 364\gamma^{1/2}} \mathrm{Ad}(v'w). \tag{5.43}$$

Recall that, by construction, there are elements $x_i \in X_{n+1}$ such that

$$\|z^* b_i z - x_i\| \leq \gamma, \qquad 1 \leq i \leq n+1, \tag{5.44}$$

and also that $\|z - u_n\| \leq 252\gamma^{1/2}$, $\|v' - 1_M\| \leq 1848\gamma^{1/2}$, and $\|w - u_n\| \leq 1596\gamma^{1/2}$. From these inequalities, it follows that

$$\begin{aligned} \|z x_i z^* - v' w x_i (v'w)^*\| &\leq \|u_n x_i u_n^* - (v'w) x_i (v'w)^*\| + 504\gamma^{1/2} \\ &\leq \|w x_i w^* - (v'w) x_i (v'w)^*\| + 3696\gamma^{1/2} \\ &\leq 2\|v' - 1_M\| + 3696\gamma^{1/2} \leq 7392\gamma^{1/2}. \end{aligned} \tag{5.45}$$

Thus, for $1 \leq i \leq n+1$, it follows from (5.43), (5.44) and (5.45) that

$$\begin{aligned} \|\theta_{n+1}(x_i) - b_i\| &\leq \|\theta_{n+1}(x_i) - (v'w)x_i(v'w)^*\| + \|(v'w)x_i(v'w)^* - z x_i z^*\| + \|z x_i z^* - b_i\| \\ &\leq 364\gamma^{1/2} + 7392\gamma^{1/2} + \gamma \leq 7757\gamma^{1/2} \leq 9/10, \end{aligned} \tag{5.46}$$

since $\gamma \leq 10^{-8}$. This proves condition $(6)_{n+1}$.

We now prove condition $(7)_{n+1}$, so take a surjective $*$-isomorphism $\phi : A \to B$ with $\phi^{-1} \approx_{Z_{n+1}, 364\gamma^{1/2}} \iota$. By Lemma 5.2 (vi), there exists a unitary $w' \in B^\dagger$ with $\|w' - u\| \leq 1596\gamma^{1/2}$ and $\mathrm{Ad}(w') \circ \phi \approx_{Y_{n+1}, \delta_{n+1}/2} \theta$. Apply $\mathrm{Ad}(v'w)$ to this to obtain $\mathrm{Ad}(v'ww') \circ \phi \approx_{Y_{n+1}, \delta_{n+1}/2} \mathrm{Ad}(v'w) \circ \theta$. Since $u_{n+1} = v'wu$, we have $\|v'ww' - u_{n+1}\| \leq 1596\gamma^{1/2}$ and also $\mathrm{Ad}(v'ww') \circ \phi \approx_{Y_{n+1}, \delta_{n+1}/2} \theta_{n+1}$ since $\theta_{n+1} = \mathrm{Ad}(v'w) \circ \theta$. Then $v'ww'$ is the required unitary in condition $(7)_{n+1}$ which now holds.

The last remaining condition is $(8)_{n+1}$. Now θ and u were chosen to satisfy Lemma 5.2 (vii) for Y_{n+1}, δ_{n+1}, $\varepsilon = \delta_n/6 \leq 2^{-(n+1)}/3$, and $\mu = 2^{-(n+2)}$, and we now show that the same is true for $\theta_{n+1} = \mathrm{Ad}(v'w) \circ \theta$ and $u_{n+1} = v'wu$. Given a unitary $v \in M$ with $\|v - v'wu\| \leq 1848\gamma^{1/2}$ and $\mathrm{Ad}(v) \approx_{Y_{n+1}, \delta_{n+1}} \mathrm{Ad}(v'w) \circ \theta$, and given a finite subset S of the unit ball of H, the unitary $w^* v'^* v$ satisfies

$$\|w^* v'^* v - u\| \leq 1848\gamma^{1/2} \quad \text{and} \quad \mathrm{Ad}(w^* v'^* v) \approx_{Y_{n+1}, \delta_{n+1}} \theta. \tag{5.47}$$

Let

$$S' = S \cup \{w^*v'^*\xi\colon\ \xi \in S\}. \tag{5.48}$$

Applying Lemma 5.2 (vii) to S', there is a unitary $\tilde{v} \in B^\dagger$ satisfying $\|\tilde{v} - 1_M\| \leq 1848\gamma^{1/2}$,

$$\mathrm{Ad}(\tilde{v}w^*v'^*v) \approx_{X_{n+1},\varepsilon} \theta, \tag{5.49}$$

and

$$\|(\tilde{v}w^*v'^*v - u)\eta\|,\ \|(\tilde{v}w^*v'^*v - u)^*\eta\| < 2^{-(n+2)}, \qquad \eta \in S'. \tag{5.50}$$

Applying $\mathrm{Ad}(v'w)$ to (5.49) gives

$$\mathrm{Ad}(v'w\tilde{v}w^*v'^*v) \approx_{X_{n+1},\varepsilon} \mathrm{Ad}(v'w) \circ \theta = \theta_{n+1}. \tag{5.51}$$

From the first inequality of (5.50) we obtain

$$\|((v'w\tilde{v}w^*v'^*)v - v'wu)\xi\| < 2^{-(n+2)}, \qquad \xi \in S. \tag{5.52}$$

For $\xi \in S$, put $\eta = w^*v'^*\eta \in S'$ into the second inequality of (5.50), to yield

$$\|((v'w\tilde{v}w^*v'^*)v - v'wu)^*\xi\| < 2^{-(n+2)}, \qquad \xi \in S. \tag{5.53}$$

The unitary $v'w\tilde{v}w^*v'^* \in B^\dagger$ then satisfies the requirements of condition $(8)_{n+1}$ since

$$\|v'w\tilde{v}w^*v'^* - 1_M\| = \|\tilde{v} - 1_M\| \leq 1848\gamma^{1/2}, \tag{5.54}$$

and $\varepsilon \leq 2^{-(n+2)}$. Thus condition $(8)_{n+1}$ holds and the proof is complete. □

In the next result we remove the hypothesis that A and B have the same ultraweak closure from Theorem 5.3 and so establish Theorem B. In the theorem below, the algebras A and B do not necessarily act non-degenerately so we use $\overline{A}^w$ and $\overline{B}^w$ for the ultraweak closures of A and B rather than A'' and B''.

Theorem 5.4. *Let A and B be* C*-*algebras acting on a separable Hilbert space H. Suppose that A is separable and nuclear, and that $d(A,B) < 10^{-11}$. Then there exists a unitary $u \in (A \cup B)''$ such that $uAu^* = B$.*

Proof. Since $d(A,B) < 1/101$, Propositions 2.9 and 2.10 show that B is also separable and nuclear. Choose η so that $d(A,B) < \eta < 10^{-11}$, and denote the support projections of A and B by e_A and e_B respectively. These are the respective units of $\overline{A}^w$ and $\overline{B}^w$. By [31, Lemma 5] we have $d(\overline{A}^w, \overline{B}^w) \le d(A, B)$, so from Proposition 2.11 there is a unitary $u_0 \in \mathrm{W}^*(A, B, I_H)$ such that $u_0 e_A u_0^* = e_B$ and $\|1 - u_0\| \le 2\sqrt{2}\eta$. Let $A_0 = u_0 A u_0^*$. Then

$$d(A_0, A) \le 2\|1 - u_0\| \le 4\sqrt{2}\eta, \tag{5.55}$$

so

$$d(A_0, B) \le (4\sqrt{2} + 1)\eta < 1/8. \tag{5.56}$$

Another use of [31, Lemma 5] shows that the same estimate holds for $d(\overline{A}_0^w, \overline{B}^w)$. These injective von Neumann algebras have the same unit e_B, so we can use Proposition 2.12 to obtain a unitary $v \in \mathrm{W}^*(A, B, I_H)$ so that $v\overline{A}_0^w v^* = \overline{B}^w$ and

$$\|I_H - v\| \le 12 d(\overline{A}_0^w, \overline{B}^w) \le (48\sqrt{2} + 12)\eta. \tag{5.57}$$

If we define $w = vu_0$, then wAw^* and B have identical ultraweak closures, and

$$\|I_H - w\| = \|v^* - u_0\| \le \|I_H - v^*\| + \|I_H - u_0\| \le (50\sqrt{2} + 12)\eta. \tag{5.58}$$

Now define $A_1 = wAw^*$. Then

$$d(A_1, B) \le d(A_1, A) + d(A, B) < 2\|I_H - w\| + \eta < (100\sqrt{2} + 25)\eta < 10^{-8}, \tag{5.59}$$

since $\eta < 10^{-11}$.

Now let K be the range of e_B and restrict A_1 and B to this Hilbert space. The hypotheses of Theorem 5.3 are now met, so there exists a unitary $u_1 \in \mathrm{W}^*(A_1 \cup B) = \mathrm{W}^*(B) \subseteq \mathbb{B}(K)$ so that $u_1 A_1 u_1^* = B$. We extend u_1 to a unitary $u_2 \in B''$ by $u_2 = u_1 + (1 - e_B)$, so that $u_2 A_1 u_2^* = B$. Then $u_2 w A w^* u_2^* = B$, and the proof is completed by defining $u = u_2 w \in \mathrm{W}^*(A, B, I_H)$. □

Corollary 5.5. *Let A be a separable nuclear* C*-*algebra on a separable Hilbert space H. Then the connected component of A for the metric $d(\cdot,\cdot)$ is*

$$V = \{B \colon B = uAu^*,\ u \in \mathcal{U}(\mathbb{B}(H))\}.$$

Proof. Each unitary u may be written $u = e^{ih}$ for a self-adjoint operator h, so A is connected to uAu^* by the path $t \mapsto e^{ith} A e^{-ith}$ for $0 \le t \le 1$. Thus V is contained in the connected component.

If $D \in V^c$, then the open ball of radius 10^{-11} centred at D must lie in V^c, otherwise there exists $B \in V$ with $d(B, D) < 10^{-11}$. If this were the case then, by Theorem 5.4, D would be unitarily equivalent to B and thus to A, placing $D \in V$ and giving a contradiction. Thus V is closed, and it is also open by another application of Theorem 5.4. The result follows. □

6 Near inclusions and nuclear dimension

In this section we return to near inclusions $A \subseteq_\gamma B$, where A is nuclear and separable, and we study the problem of whether A embeds into B for sufficiently small values of γ. For this we will use Lemma 4.1, so the question reduces to finding cpc maps from A to B which closely approximate the inclusion map of A into the underlying $\mathbb{B}(H)$ on finite subsets of the unit ball of A. Making use of the nuclearity of A to approximately factorise id_A through matrix algebras, we see that the core question is this: do cpc maps $\theta : \mathbb{M}_n \to A$ perturb to nearby cpc maps $\tilde{\theta} : \mathbb{M}_n \to B$? The obvious approach is to use the well known identification of θ with a positive element of the ball of radius n of $\mathbb{M}_n(A)$, approximate this by a positive element of $\mathbb{M}_n(B)$ and take the associated cp map $\tilde{\theta} : \mathbb{M}_n \to B$. However, we will lose control of $\|\theta - \tilde{\theta}\|$ which will depend on n, forcing us to employ other methods. We do not know the answer in full generality, but will be able to give a positive solution for the class of order zero maps, defined in Definition 6.1. This will enable us to pass from a near inclusion $A \subseteq_\gamma B$ to an embedding of A into B whenever A has finite nuclear dimension (see Theorem 6.10 which is the quantitative version of Theorem C in the introduction). The nuclear dimension (see Definition 6.6 below) of a C*-algebra, like its forerunner the decomposition rank [36], is defined by requiring the existence of suitable cpc approximate point-norm factorisations $\phi : A \to F$ and $\psi : F \to A$ for id_A through finite dimensional C*-algebras F with the map ψ splitting as a finite sum of order zero maps. We note that in all classes of nuclear separable C*-algebras which have so far been classified, the constituent algebras have finite nuclear dimension. We begin by defining order zero maps; all such maps in this paper will be contractions so we absorb this into the definition.

Definition 6.1. Let F and A be C*-algebras. An *order zero map* is a cpc map $\phi : F \to A$ which preserves orthogonality in the following sense: if $e, f \in F^+$ and $ef = 0$, then $\phi(e)\phi(f) = 0$.

Order zero maps are more general than $*$-homomorphisms, but nevertheless have many pleasant structural properties. In particular, we will use the following two facts from [53] and [54] (see also [55], where these assertions are established when F is not finite dimensional).

Proposition 6.2. *Let A and F be* C**-algebras with F finite dimensional and let $\phi : F \to A$ be an order zero map. Suppose that A is faithfully represented on H. Then there exists a unique $*$-homomorphism $\pi : F \to \overline{\phi(F)}^w \subseteq \overline{A}^w$ such that*

$$\phi(x) = \pi(x)\phi(1_F) = \phi(1_F)\pi(x), \quad x \in F. \tag{6.1}$$

Proposition 6.3. *Let A and F be* C**-algebras with F finite dimensional. Given an order zero map $\phi : F \to A$, the map $id_{(0,1]} \otimes x \mapsto \phi(x)$ induces a $*$-homomorphism $\rho_\phi : C_0(0,1] \otimes F \to A$. Conversely, given a $*$-homomorphism $\rho : C_0(0,1] \otimes F \to A$, there is an order zero map $\rho_\phi : F \to A$ defined by $x \mapsto \rho(id_{(0,1]} \otimes x)$.*

We can now perturb order zero maps. In the theorem below we do not obtain that ψ is automatically order zero, but will address this point in Theorem 7.8.

Theorem 6.4. *Let A be a* C**-algebra on a Hilbert space H. Given a finite dimensional* C**-algebra F and an order zero map $\phi : F \to A$, there exists a finite set Y in the unit ball of A with the following property. If B is another* C**-algebra on H with $Y \subseteq_\gamma B$ for some $\gamma > 0$, then there exists a cp map $\psi : F \to B$ with*

$$\|\phi - \psi\|_{\mathrm{cb}} \leq (2\gamma + \gamma^2)(2 + 2\gamma + \gamma^2). \tag{6.2}$$

Proof. Let $\phi : F \to A$ be cpc and order zero. By replacing A by the image of the $*$-homomorphism $\rho_\phi : C_0(0,1] \otimes F \to A$ from Proposition 6.3 we may assume that A is nuclear. Let $\pi : F \to \overline{\phi(F)}^w \subseteq \overline{A}^w$ be the unique $*$-homomorphism with

$$\phi(x) = \pi(x)\phi(1_F) = \phi(1_F)\pi(x), \quad x \in F, \tag{6.3}$$

given by Proposition 6.2. Write $F = \mathbb{M}_{n_1} \oplus \cdots \oplus \mathbb{M}_{n_r}$. Without loss of generality, we may assume that π is injective, as if $M_{n_k} \subseteq \ker(\pi)$, then (6.3) ensures that $\phi|_{\mathbb{M}_{n_k}} = 0$ allowing us to remove the $\mathbb{M}_{n_k}$ summand from F. For each $1 \leq k \leq r$, let $\{e_{i,j}^{(k)}\}_{i,j=1}^{n_k}$ be a system of matrix units for $\mathbb{M}_{n_k}$. Let $m = \max\{n_1, \ldots, n_r\}$ and let $\{f_{i,j}\}_{i,j=1}^m$ be a system of matrix units for $\mathbb{M}_m$. For each $1 \leq k \leq r$, define a non-unital $*$-homomorphism $\theta_k : \mathbb{M}_{n_k} \to \mathbb{M}_m$ by $\theta_k\left(e_{i,j}^{(k)}\right) = f_{i,j}$. Define a non-unital $*$-homomorphism $\theta : F \to \mathbb{M}_r(\mathbb{B}(H) \otimes \mathbb{M}_m)$ by

$$\theta(x_1 \oplus \cdots \oplus x_r) = \begin{pmatrix} I_H \otimes \theta_1(x_1) & 0 & \cdots & 0 \\ 0 & I_H \otimes \theta_2(x_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I_H \otimes \theta_r(x_r) \end{pmatrix}. \tag{6.4}$$

For each $1 \leq k \leq r$, define self-adjoint partial isometries

$$s_k = \sum_{i,j=1}^{n_k} \pi\left(e_{i,j}^{(k)}\right) \otimes f_{j,i} \in \pi(F) \otimes \mathbb{M}_m. \tag{6.5}$$

These satisfy

$$s_k(I_H \otimes \theta_k(x_k))s_k^* = \pi(x_k) \otimes \sum_{i=1}^{n_k} f_{i,i}, \quad x_k \in \mathbb{M}_{n_k}. \tag{6.6}$$

By (6.3)

$$(\phi(1_F)\otimes 1_{\mathbb{M}_m})s_k=\sum_{i,j=1}^{n_k}\phi\big(e_{i,j}^{(k)}\big)\otimes f_{j,i}=(\phi(1_{\mathbb{M}_{n_k}})\otimes 1_{\mathbb{M}_m})s_k\in A\otimes\mathbb{M}_m. \tag{6.7}$$

The continuous functional calculus then shows that $f(\phi(1_{M_{n_k}})\otimes 1_{\mathbb{M}_m})s_k\in A\otimes\mathbb{M}_m$, whenever f is a continuous function on $[0,1]$ with $f(0)=0$. In particular, $t_k=(\phi(1_{\mathbb{M}_{n_k}})^{1/2}\otimes 1_{\mathbb{M}_m})s_k\in A\otimes\mathbb{M}_m$ for all k. Let $t\in\mathbb{M}_{1\times r}(A\otimes\mathbb{M}_m)$ be the row matrix $(t_1,\dots,t_r)$. We can then define a completely positive map $\phi_0:F\to A\otimes\mathbb{M}_m$ by

$$\phi_0(x)=t\theta(x)t^*,\quad x\in F. \tag{6.8}$$

For $x=x_1\oplus\cdots\oplus x_r\in F$, use (6.6) to compute

$$\begin{aligned}\phi_0(x)=\sum_{k=1}^r t_k(I_H\otimes\theta_k(x_k))t_k^*&=\sum_{k=1}^r(\phi(1_{\mathbb{M}_{n_k}})^{1/2}\otimes 1)\big(\pi(x_k)\otimes\sum_{i=1}^{n_k}f_{i,i}\big)(\phi(1_{\mathbb{M}_{n_k}})^{1/2}\otimes 1)\\&=\sum_{k=1}^r\big(\phi(x_k)\otimes\sum_{i=1}^{n_k}f_{i,i}\big).\end{aligned} \tag{6.9}$$

In particular, under the identification $\mathbb{B}(H)\cong f_{1,1}(\mathbb{B}(H)\otimes\mathbb{M}_m)f_{1,1}$, we can recover ϕ by

$$\phi(x)=(I_H\otimes f_{1,1})\phi_0(x)(I_H\otimes f_{1,1}),\quad x\in F. \tag{6.10}$$

Now

$$\begin{aligned}\|t\|^2=&\|tt^*\|=\|\sum_{k=1}^r(\phi(1_{\mathbb{M}_{n_k}})^{1/2}\otimes 1_{\mathbb{M}_m})s_ks_k^*(\phi(1_{\mathbb{M}_{n_k}})^{1/2}\otimes 1_{\mathbb{M}_m})\|\\ \le&\|\sum_{k=1}^n\phi(1_{\mathbb{M}_{n_k}})\|=\|\phi(1_F)\|=\|\phi\|\le 1\end{aligned} \tag{6.11}$$

so $\{t\}$ is a finite subset of the unit ball of $\mathbb{M}_{1\times r}(A\otimes\mathbb{M}_m)$. Accordingly Proposition 2.7 gives a finite subset Y of the unit ball of A with the property that whenever B is another C*-algebra on H and $Y\subseteq_\gamma B$, then $\{t\}\subseteq_\mu\mathbb{M}_{1\times r}(B\otimes\mathbb{M}_m)$, where $\mu=2\gamma+\gamma^2$. Assume we are given such a C*-algebra B so that we can find some $u=(u_1,\dots,u_r)\in\mathbb{M}_{1\times r}(B\otimes\mathbb{M}_m)$ with $\|t-u\|\le\mu$ in $\mathbb{M}_{1\times r}(\mathbb{B}(H)\otimes\mathbb{M}_m)$.

Define a cp map $\psi_0:F\to B\otimes\mathbb{M}_m$ by $\psi_0(x)=u\theta(x)u^*$. Use the identification $\mathbb{B}(H)\cong f_{1,1}(\mathbb{B}(H)\otimes\mathbb{M}_m)f_{1,1}$ to define a cp map $\psi:F\to B$ by

$$\psi(x)=f_{1,1}\psi_0(x)f_{1,1},\quad x\in F. \tag{6.12}$$

Finally

$$\|\phi-\psi\|_{\mathrm{cb}}\le\|\phi_0-\psi_0\|_{\mathrm{cb}}\le\|t-u\|\|t\|+\|t-u\|\|u\|\le\mu+\mu(1+\mu), \tag{6.13}$$

exactly as claimed. □

Corollary 6.5. *Let A and B be* C*-*algebras on a Hilbert space H with $A \subseteq_\gamma B$ for some $\gamma > 0$. Then for each finite dimensional C*-algebra F and order zero map $\phi : F \to A$, there exists a cp map $\psi : F \to B$ satisfying*

$$\|\phi - \psi\|_{\mathrm{cb}} \leq (2\gamma + \gamma^2)(2 + 2\gamma + \gamma^2). \tag{6.14}$$

We now work towards our embedding result for a near containment of a separable C*-algebra of finite nuclear dimension. First let us recall the definition of nuclear dimension from [56].

Definition 6.6. Let A be a C*-algebra and $n \geq 0$. Say that A has *nuclear dimension* at most n, written $\dim_{\mathrm{nuc}}(A) \leq n$, if for each finite subset X of A and $\varepsilon > 0$, there exists a finite dimensional C*-algebra F which decomposes as a direct sum $F = F_0 \oplus \cdots \oplus F_n$ and maps $\phi : A \to F$ and $\psi : F \to A$ such that $\psi \circ \phi \approx_{X,\varepsilon} \mathrm{id}_A$, ϕ is cpc and ψ decomposes as $\psi = \sum_{i=0}^n \psi_i$, where each $\psi_i : F_i \to A$ is order zero.

The definition of nuclear dimension is a modification of the *decomposition rank* from [36]. The decomposition rank $\mathrm{dr}(A)$ of A is defined in the same way as the nuclear dimension, but with the additional requirement that the map ψ in the definition above is also cpc. Suprisingly the small change in the definition from decomposition rank to nuclear dimension considerably enlarges the class of C*-algebras with finite dimension (while retaining the permanence properties). Indeed in [36] it is shown that a separable C*-algebra with finite decomposition rank is necessarily quasidiagonal and so stably finite, while in [56] it is shown that the Cuntz algebras $\mathcal{O}_n$ (and all classifiable Kirchberg algebras) have finite nuclear dimension. We need one final structural property of the cp approximations defining nuclear dimension (see [56, Remark 2.2 (iv)]); this is immediate in the unital case.

Proposition 6.7. *Suppose that A is a* C*-*algebra with* $\dim_{\mathrm{nuc}}(A) \leq n$. *Given a finite set $X \subseteq A$ and $\varepsilon > 0$, there exist F and maps ϕ and ψ as in Definition 6.6 with the additional property that $\psi \circ \phi$ is cpc.*

Given a near inclusion $A \subseteq_\gamma B$, where A has finite nuclear dimension, we can now approximate in the point-norm topology the inclusion of A into the underlying $\mathbb{B}(H)$ by cpc maps $A \to B$.

Lemma 6.8. *Let $n \geq 0$. Let D be a* C*-*algebra with* $\dim_{\mathrm{nuc}}(D) \leq n$, *let A be a* C*-*algebra represented on the Hilbert space H and let $\theta : D \to A$ be an order zero map. Given a finite subset X of the unit ball of D and $\varepsilon > 0$, there exists another finite subset Y of the unit ball* *of A with the following property. If B is another* C*-*algebra on H with $Y \subseteq_\gamma B$ for some $\gamma > 0$, then there exists a cpc map $\phi : D \to B$ with*

$$\|\phi(x) - \theta(x)\| \leq 2(n+1)(2\gamma+\gamma^2)(2+2\gamma+\gamma^2) + \varepsilon, \quad x \in X. \tag{6.15}$$

Proof. Given a finite subset X of the unit ball of D and $\varepsilon > 0$, we first use the definition of nuclear dimension in conjuction with Proposition 6.7 to find a finite dimensional C*-algebra F and cp maps $\psi_1 : D \to F$ and $\psi_2 : F \to D$ such that

(i) ψ_1 is cpc;

(ii) F decomposes as $F_0 \oplus \cdots \oplus F_n$ and ψ_2 decomposes as $\psi_2 = \sum_{i=0}^n \psi_{2,i}$, where each $\psi_{2,i} : F_i \to D$ is cpc and order zero;

(iii) $\psi_2 \circ \psi_1$ is contractive and $\psi_2 \circ \psi_1 \approx_{X,\varepsilon} \mathrm{id}_D$.

Theorem 6.4 enables us to find a finite set Y in the unit ball of A such that whenever B is another C*-algebra on H with $Y \subseteq_\gamma B$, we can find cp maps $\tilde{\psi}_{2,i} : F_i \to B$ with

$$\|\theta \circ \psi_{2,i} - \tilde{\psi}_{2,i}\|_{\mathrm{cb}} \leq (2\gamma+\gamma^2)(2+2\gamma+\gamma^2), \tag{6.16}$$

for $i = 0, \ldots, n$. This is the set Y required by the lemma as given such maps, we can define a cp map $\tilde{\psi}_2 : F \to B$ by $\sum_{i=0}^n \tilde{\psi}_{2,i}$ and this satisfies

$$\|\theta \circ \psi_2 - \tilde{\psi}_2\|_{\mathrm{cb}} \leq (n+1)(2\gamma+\gamma^2)(2+2\gamma+\gamma^2). \tag{6.17}$$

We define a cp map $\tilde{\phi} : D \to B$ by $\tilde{\phi} = \tilde{\psi}_2 \circ \psi_1$. Now

$$\|\tilde{\phi} - \theta \circ \psi_2 \circ \psi_1\|_{\mathrm{cb}} \leq (n+1)(2\gamma+\gamma^2)(2+2\gamma+\gamma^2) \tag{6.18}$$

so

$$\|\tilde{\phi}\|_{\mathrm{cb}} \leq 1 + (n+1)(2\gamma+\gamma^2)(2+2\gamma+\gamma^2). \tag{6.19}$$

If $\tilde{\phi}$ is already cpc, we define $\phi = \tilde{\phi}$. Otherwise define $\phi = \tilde{\phi}/\|\tilde{\phi}\|_{\mathrm{cb}}$. Thus

$$\|\phi - \tilde{\phi}\|_{\mathrm{cb}} \leq \|\tilde{\phi}\|_{\mathrm{cb}} - 1 \leq (n+1)(2\gamma+\gamma^2)(2+2\gamma+\gamma^2). \tag{6.20}$$

For $x \in X$,

$$\begin{aligned}\|\phi(x) - \theta(x)\| &\leq \|\phi(x) - \tilde{\phi}(x)\| + \|\tilde{\phi}(x) - (\theta \circ \psi_2 \circ \psi_1)(x)\| + \|(\theta \circ \psi_2 \circ \psi_1)(x) - \theta(x)\| \\ &\leq 2(n+1)(2\gamma+\gamma^2)(2+2\gamma+\gamma^2) + \varepsilon, \end{aligned} \tag{6.21}$$

establishing the result. □

Taking θ to be the identity map on A, the next corollary is immediate, since if $A \subset_\gamma B$, then we can find $\gamma' < \gamma$ with $A \subseteq_{\gamma'} B$ and take

$$\varepsilon = 2(n+1)[(2\gamma+\gamma^2)(2+2\gamma+\gamma^2) - (2\gamma'+\gamma'^2)(2+2\gamma'+\gamma'^2)] \tag{6.22}$$

in the previous lemma.

Corollary 6.9. *Let A and B be* C**-algebras on a Hilbert space H with $A \subset_\gamma B$ for some $\gamma > 0$ and suppose that $\dim_{\mathrm{nuc}}(A) \leq n$ for some $n \geq 0$. For any finite subset X of the unit ball of A, there exists a cpc map $\phi : A \to B$ with*

$$\|\phi(x) - x\| \leq 2(n+1)(2\gamma+\gamma^2)(2+2\gamma+\gamma^2), \quad x \in X. \tag{6.23}$$

The intertwining argument of Lemma 4.1 combines with the previous corollary to give immediately the following quantitative version of Theorem C. The number 20 appearing in (6.24) is an integer estimate for $8\sqrt{6}$ from Lemma 4.1.

Theorem 6.10. *Let $n \geq 0$. Let $A \subseteq_\gamma B$ be a near inclusion of* C**-algebras on a Hilbert space H, let $\eta = 2(n+1)(2\gamma+\gamma^2)(2+2\gamma+\gamma^2)$, and suppose that A is separable with nuclear dimension at most n. If $\eta < 1/210000$, then A embeds into B. Moreover, for each finite subset X of the unit ball of A, there exists an embedding $\theta : A \to B$ with*

$$\|\theta(x) - x\| \leq 20\eta^{1/2}. \tag{6.24}$$

Remark 6.11. The hypotheses on A in the previous theorem are, in particular, satisfied for all separable simple nuclear C*-algebras presently covered by known classification theorems. This includes Kirchberg algebras satisfying the UCT, simple unital C*-algebras with finite decomposition rank for which projections separate traces (and also satisfying the UCT), and transformation group C*-algebras associated to compact minimal uniquely ergodic dynamical systems (see [56] and [51]).

7 Applications

In [3] Bratteli initiated the study of separable approximately finite dimensional (AF) C*-algebras, namely those separable C*-algebras arising as direct limits of finite dimensional C*-algebras. He gave a local characterisation of these algebras, showing that a separable C*-algebra is AF if, and only if, for each $\varepsilon > 0$ and each finite set $X \subseteq A$, there exists a finite dimensional C*-subalgebra A_0 of A such that $X \subset_\varepsilon A_0$. By changing ε if necessary, we can scale the set X above so that it lies in the unit ball of A. This characterisation can be weakened: it is not necessary to be able to approximate a finite set of the unit ball of A *arbitrarily* closely by a finite dimensional C*-algebra; an approximation up to a fixed small tolerance is sufficient to imply that A is AF. The proposition below states this precisely and is implicit in the proof of [11, Theorem 6.1].

Proposition 7.1. *There exists a constant $\gamma_0 > 0$ with the following property. Let A be a separable* C*-*algebra and suppose that for all finite sets X in the unit ball of A, there exists a finite dimensional* C*-*subalgebra A_0 of A such that $X \subset_{\gamma_0} A_0$. Then A is AF.*

Our first objective in this section is to generalise this last result to other inductive limits, which admit a local characterisation. In [21], Elliott gave a local characterisation of the separable $A\mathbb{T}$-algebras (those C*-algebras arising as direct limits of algebras of the form $C(\mathbb{T}) \otimes F$, where F is finite dimensional). Loring developed a theory of finitely presented C*-algebras and showed that local characterisations are possible for inductive limits of finitely presented weakly semiprojective C*-algebras. There are many examples of such algebras, including the dimension drop intervals used in [24]. We refer to Loring's monograph [39] for more examples and background information on these concepts (see also [18]). The proposition below is Lemma 15.2.2 of [39], scaling the finite sets involved into the unit ball.

Proposition 7.2 (Loring)**.** *Suppose that A is a* C*-*algebra containing a (not necessarily nested) sequence of* C*-*subalgebras A_n with the property that for each finite set X of the unit ball of A and each $\varepsilon > 0$, there exists $n \in \mathbb{N}$ with $X \subseteq_\varepsilon A_n$. If each A_n is weakly semiprojective and finitely presented, then A is isomorphic to a direct limit $\varinjlim(A_{k_n}, \phi_n)$ for some subsequence $\{k_n\}_{n=1}^\infty$ and some connecting $*$-homomorphisms $\phi_n : A_{k_n} \to A_{k_{n+1}}$.*

Provided that the building blocks A_n are all nuclear, the arbitrary tolerance $\varepsilon > 0$ appearing above can be replaced by the fixed quantity $1/120000$. The only fact we require about finitely presented weakly semiprojective C*-algebras is the following easy proposition, which is immediate from the definition of weak stability of a finite presentation of a C*-algebra.

Proposition 7.3. *Let A be a finitely presented, weakly semiprojective* C*-*algebra. Then, for each finite subset X of the unit ball of A and each $\varepsilon > 0$, there exists a finite subset Y of the unit ball of A and $\delta > 0$ with the following property. If $\phi : A \to B$ is a (Y, δ)-approximate $*$-homomorphism, then there is a $*$-homomorphism $\psi : A \to B$ with $\psi \approx_{X,\varepsilon} \phi$.*

In the proposition above, the choice of δ depends on both X and ε. Indeed, one proves the proposition by replacing X by a weakly stable generating set for A, reducing ε if necessary, and taking $Y = X$. The result follows since the image of Y under a (Y, δ)-approximate $*$-homomorphism is an η-representation for the presentation Y (where $\eta \to 0$ as $\delta \to 0$). When A is nuclear, Lemma 3.2 can be used to show that the δ appearing in Proposition 7.3 only depends on ε and not on X or A. This approach results in enlarging the set Y.

Lemma 7.4. *Fix $\varepsilon > 0$ and let $\delta < \min(1/17, \varepsilon^2/128)$. Suppose that A is a finitely presented, weakly semiprojective nuclear* C*-*algebra. Then for each finite subset X of the unit ball of A, there exists a finite subset Y of the unit ball of A with the following property. If $\phi : A \to B$ is a (Y, δ)-approximate $*$-homomorphism, then there is a $*$-homomorphism $\psi : A \to B$ with $\psi \approx_{X,\varepsilon} \phi$.*

Proof. Fix $\varepsilon > 0$ and take $0 < \delta < 1/17$ such that $8\sqrt{2}\delta^{1/2} < \varepsilon$. Fix $\mu > 0$ so that $8\sqrt{2}\delta^{1/2} + \mu < \varepsilon$ and write $\varepsilon' = \varepsilon - 8\sqrt{2}\delta^{1/2} - \mu > 0$. Given a finitely presented, weakly semiprojective nuclear C*-algebra A and a finite set X in the unit ball of A, use Proposition 7.3 to find $\delta' > 0$ and a finite set Z in the unit ball of A such that if $\phi_1 : A \to B$ is a (Z, δ')-approximate $*$-homomorphism, then there exists a $*$-homomorphism $\psi : A \to B$ with $\psi \approx_{X,\varepsilon'} \phi_1$. By Lemma 3.2, there is a finite set Y in the unit ball of A such that given any (Y, δ)-approximate $*$-homomorphism $\phi : A \to B$, there is a (Z, δ')-approximate $*$-homomorphism $\phi_1 : A \to B$ with $\|\phi - \phi_1\| \leq 8\sqrt{2}\delta^{1/2} + \mu$. It follows that δ has the property claimed in the lemma. □

We now show that we do not need arbitrarily close approximations in order to detect direct limits of finitely presented weakly semiprojective nuclear C*-algebras.

Theorem 7.5. *Let A be a separable* C*-*algebra and suppose that there is a (not necessarily) nested sequence $\{A_k\}_{k=1}^{\infty}$ of finitely presented, weakly semiprojective nuclear* C*-*subalgebras of A with the following property. For each finite subset X of the unit ball of A, there exists $k \in \mathbb{N}$ such that $X \subset_\eta A_k$ for some η satisfying $\eta < 1/120000$. Then A is isomorphic to a direct limit $\varinjlim(A_{k_n}, \phi_{n+1})$ for some subsequence $\{k_n\}$ and some connecting $*$-homomorphisms $\phi_{n+1} : A_{k_n} \to A_{k_{n+1}}$.*

Proof. Fix a dense sequence $\{a_i\}_{i=1}^{\infty}$ of the unit ball of A. For each $n \geq 1$ we will construct $k_n \in \mathbb{N}$, a dense sequence $\{a_i^{(n)}\}_{i=1}^{\infty}$ in the unit ball of A_{k_n} and a unitary u_n in the unitisation $A^\dagger$. For $n > 1$, we will define connecting $*$-homorphisms $\phi_n : A_{k_{n-1}} \to A_{k_n}$. These objects will satisfy the following properties:

1. $\|a_i - u_1 \dots u_{n-1} a_i^{(n)} u_{n-1}^* \dots u_1^*\| < 1/10$ for all $n \geq 1$ and $i = 1, \dots, n$.

2. For $n > 1$, $\|u_n(\phi_n(x))u_n^* - x\| < 2^{-n}$ whenever $x \in A_{k_{n-1}}$ is of the form

$$(\phi_{n-1} \circ \dots \circ \phi_j)(a_i^{(j-1)}), \quad 1 \leq i \leq n, \quad 2 \leq j \leq n-1. \tag{7.1}$$

3. $\|u_n - 1_{A^\dagger}\| \leq 2/5$.

Once the induction is complete, the second condition above ensures that the following diagram gives an approximate intertwining in the sense of [21, 2.3], where $\iota_n : A_n \hookrightarrow A$ is the inclusion map.

$$\begin{array}{ccccccc}
A_{k_1} & \xrightarrow{\phi_2} & A_{k_2} & \xrightarrow{\phi_3} & A_{k_3} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \mathrm{Ad}(u_1)\circ\iota_1 = \iota_{k_1}} & & \downarrow{\scriptstyle \mathrm{Ad}(u_1u_2)\circ\iota_{k_2}} & & \downarrow{\scriptstyle \mathrm{Ad}(u_1u_2u_3)\circ\iota_{k_3}} & & \\
A & \xrightarrow{\mathrm{id}_A} & A & \xrightarrow{\mathrm{id}_A} & A & \longrightarrow & \cdots
\end{array}$$

In particular [21, 2.3] produces a $*$-homomorphism $\theta : \varinjlim(A_{k_n}, \phi_{n+1}) \to A$ as a point-norm limit. This map is injective since each of the vertical maps is injective. For surjectivity, fix $n \in \mathbb{N}$. Let b_n denote the image of $a_n^{(n)}$ in $\varinjlim(A_{k_n}, \phi_{n+1})$ so that

$$\theta(b_n) = \lim_{m \to \infty} u_1 \dots u_m(\phi_m \circ \dots \circ \phi_{n+1})(a_n^{(n)})u_m^* \dots u_1^* \tag{7.2}$$

in norm. Repeatedly using the second condition, we have

$$\|\theta(b_n) - u_1 \dots u_n a_n^{(n)} u_n^* \dots u_1^*\| \leq \sum_{m>n} 2^{-m}, \tag{7.3}$$

so that

$$\begin{aligned}
\|\theta(b_n) - a_n\| &\leq \sum_{m>n} 2^{-m} + \|u_1 \dots u_n a_n^{(n)} u_n^* \dots u_1^* - a_n\| \\
&\leq \sum_{m>n} 2^{-m} + 2\|u_n - 1_{A^\dagger}\| + \|u_1 \dots u_{n-1} a_n^{(n)} u_{n-1}^* \dots u_1^* - a_n\| \\
&\leq \sum_{m>n} 2^{-m} + 4/5 + 1/10
\end{aligned} \tag{7.4}$$

from conditions 1 and 3. Hence $d(A, \theta(\varinjlim(A_{k_n}, \phi_{n+1}))) \leq 9/10 < 1$ so that θ is surjective by Proposition 2.4.

We start the construction by using the hypothesis to find k_1 and a dense sequence $\{a_i^{(1)}\}_{i=1}^\infty$ so that condition 1 holds. We take $u_1 = 1_{A^\dagger}$ so condition 3 holds and at this first stage condition 2 is empty. Suppose that all objects have been constructed up to and including stage $n-1$ for some $n>1$. Let X be the finite subset of the unit ball of $A_{k_{n-1}}$ consisting of the elements $(\phi_{n-1}\circ\cdots\circ\phi_j)\big(a_i^{(j-1)}\big)$ for $i\leq n$ and $j=2,\dots,n-1$. By Lemma 3.4 (with $\gamma=13/150$, $\delta=1/50$ and B any C*-algebra), there exists a finite subset Y of the unit ball of $A_{k_{n-1}}$ such that, given any two $*$-homomorphisms $\psi_1,\psi_2:A_{k_{n-1}}\to B$ with $\psi_1\approx_{Y,13/150}\psi_2$, there is a unitary $u\in B^\dagger$ with $\|u-1_{B^\dagger}\|\leq 2/5$ and $\mathrm{Ad}(u)\circ\psi_1\approx_{X,2^{-n}}\psi_2$. By enlarging Y, we may assume that $Y\supseteq X$. By Proposition 7.4, there is a finite subset Z of the unit ball of $A_{k_{n-1}}$ such that if $\phi:A_{k_{n-1}}\to B$ is a $(Z,6\eta)$-approximate $*$-homomorphism for some η satisfying $\eta<1/120000=\frac{1}{6}\left(\frac{2}{25}\right)^2\frac{1}{128}$, then there is a $*$-homomorphism $\psi:A_{k_{n-1}}\to B$ with $\psi\approx_{Y,2/25}\phi$. Now use the hypothesis to find $k_n\in\mathbb{N}$ such that

$$Z\cup\{u_{n-1}^*\dots u_1^*a_ju_1\dots u_{n-1}\colon 1\leq j\leq n\}\subset_\eta A_{k_n}. \tag{7.5}$$

Since A_{k_n} is nuclear, Proposition 2.16 gives a cpc map $\phi:A_{k_{n-1}}\to A_{k_n}$ with $\|\phi(z)-z\|\leq 2\eta$ for $z\in Y\cup Z\cup Z^*\cup\{zz^*:z\in Z\cup Z^*\}$. In particular such a map is a $(Z,6\eta)$- approximate $*$-homomorphism and hence there is a $*$-homomorphism $\phi_n:A_{k_{n-1}}\to A_{k_n}$ with $\phi_n\approx_{Y,2/25}\phi$. In particular

$$\|\phi_n(y)-y\|\leq\frac{2}{25}+2\eta\leq\frac{13}{150},\quad y\in Y, \tag{7.6}$$

so our choice of Y gives us a unitary $u_n\in A^\dagger$ with $\|u_n-1_{A^\dagger}\|\leq 2/5$ and

$$\|(\mathrm{Ad}(u_n)\circ\phi_n)(x)-x\|\leq 2^{-n},\quad x\in X, \tag{7.7}$$

and conditions 2 and 3 hold. Since $\{u_{n-1}^*\dots u_1^*a_ju_1\dots u_{n-1}\colon 1\leq j\leq n\}\subset_\eta A_{k_n}$ for some $\eta<1/120000$, we may choose a dense sequence $\big\{a_i^{(n)}\big\}_{i=1}^\infty$ to fulfill condition 1. This completes the induction. □

As an example, Elliott's local characterisation of $A\mathbb{T}$-algebras from [21] can be weakened to give the following statement. Note that the algebras A_0 below are all semiprojective by combining [39, 14.1.7, 14.1.8, 14.2.1, 14.2.2].

Corollary 7.6. *Let A be a separable* C*-*algebra. Suppose that, for any finite set X in the unit ball of A, there exists a* C*-*subalgebra A_0 of A with $X\subset_{1/120000}A_0$, where A_0 has the form $C(\mathbb{T})\otimes F_1\oplus C[0,1]\otimes F_2\oplus F_3$ for finite dimensional* C*-*algebras F_1,F_2 and F_3. Then A is $A\mathbb{T}$.*

In a similar fashion, we obtain a generalised characterisation of the Jiang–Su algebra $\mathcal{Z}$, (see [24, Theorems 2.9 and 6.2]). The algebras $Z_{p,q}$ in the statement are semiprojective from [20].

Corollary 7.7. *Let A be a unital, simple, separable* C*-*algebra with a unique tracial state. Suppose that for each finite subset X of the unit ball of A, there exists a prime dimension drop C*-algebra $Z_{p,q}$ with $X \subset_{1/120000} Z_{p,q}$. Then $A \cong \mathcal{Z}$.*

We now return to our perturbation result for order zero maps (Theorem 6.4) and show that the resulting maps can also be taken of order zero. For simplicity, we establish this result for near inclusions rather than for the context of finite sets used in Theorem 6.4. In the next theorem we will make use of the fact that a map ϕ whose domain is a finite dimensional operator space E is completely bounded with $\|\phi\|_{\rm cb} \leq (\dim E)\,\|\phi\|$, (see [17]).

Theorem 7.8. *Let $A \subset_\gamma B$ be a near inclusion of* C*-*algebras, where γ satisfies*

$$0 < \gamma < 10^{-7}. \tag{7.8}$$

Given a finite dimensional C*-*algebra F and an order zero map $\phi : F \to A$, there exists an order zero map $\psi : F \to B$ satisfying*

$$\|\phi - \psi\|_{\rm cb} < 493\gamma^{1/2}. \tag{7.9}$$

Proof. Let $\gamma' < \gamma$ be such that $A \subseteq_{\gamma'} B$. Denote the linear dimension of F by m and choose $\beta > 0$ so that $3m\beta < \gamma^{1/2}$. Let X_0 be a β-net for the unit ball of F, and let

$$X = \{{\rm id}_{(0,1]} \otimes x \colon x \in X_0\} \subseteq C_0(0,1] \otimes F. \tag{7.10}$$

Now apply Lemma 3.4 and Remark 3.5 (ii) to the nuclear C*-algebra $C_0(0,1] \otimes F$ with γ replaced by $82\gamma^{1/2} < 13/150$, $\varepsilon > 0$ replaced by $\beta > 0$, and D replaced by $C^*(A,B)$. Then there is a finite subset Z of the unit ball of $C_0(0,1] \otimes F$ such that if $\phi_1, \phi_2 : C_0(0,1] \otimes F \to C^*(A,B)$ are $*$-homomorphisms satisfying $\phi_1 \approx_{Z,82\gamma^{1/2}} \phi_2$, then there exists a unitary $u \in C^*(A,B)^\dagger$ with $\|u - 1\| < 246\gamma^{1/2}$ and $\phi \approx_{X,\beta} {\rm Ad}(u) \circ \phi_2$. Choose $\varepsilon > 0$ so that

$$(17\gamma' + \varepsilon)^{1/2} + 2\varepsilon < \sqrt{17}\gamma^{1/2}, \tag{7.11}$$

and then define

$$\eta = 4(2\gamma' + \gamma'^2)(2 + 2\gamma' + \gamma'^2) + \varepsilon < 17\gamma' + \varepsilon, \tag{7.12}$$

since $\gamma' < 10^{-7}$. The C*-algebra $C_0(0,1] \otimes F$ is finitely presented and weakly semiprojective, [39, Chapter 14], and has nuclear dimension 1, [56]. The choice of η gives

$$3\eta < \min\{1/17, (81\gamma^{1/2})^2/128\}, \tag{7.13}$$

since $(81\gamma^{1/2})^2/128 > 51\gamma > 3\eta$. We may now apply Lemma 7.4 with X replaced by Z to conclude that there is a finite subset X_1 of the unit ball of $C_0(0,1]\otimes F$ with the following property. If $\theta : C_0(0,1]\otimes F \to B$ is an $(X_1, 3\eta)$-approximate $*$-homomorphism, then there is a $*$-homomorphism $\pi : C_0(0,1]\otimes F \to B$ with $\pi \approx_{Z,81\gamma^{1/2}} \theta$. We may enlarge X_1 if necessary so that $Z \subseteq X_1$.

Given an order zero map $\phi : F \to A$, Proposition 6.3 gives a $*$-homomorphism $\rho_\phi : C_0(0,1]\otimes F \to A$ defined by $\rho_\phi(\mathrm{id}_{(0,1]}\otimes x) = \phi(x)$ for $x \in F$. The nuclear dimension of $C_0(0,1]\otimes F$ is 1, so Lemma 6.8 gives a cpc map $\theta : C_0(0,1]\otimes F \to B$ with

$$\|\rho_\phi(x) - \theta(x)\| \leq 4(2\gamma' + \gamma'^2)(2 + 2\gamma' + \gamma'^2) + \varepsilon = \eta \tag{7.14}$$

for $x \in X_1 \cup X_1^* \cup \{xx^* \colon x \in X_1 \cup X_1^*\}$. Since ρ_ϕ is a $*$-homomorphism, θ is an $(X_1, 3\eta)$-approximate $*$-homomorphism. By choice of X_1, there is a $*$-homomorphism $\pi : C_0(0,1]\otimes F \to B$ with $\pi \approx_{Z,81\gamma^{1/2}} \theta$, and the inequality $\eta < 17\gamma$ leads to $\pi \approx_{Z,82\gamma^{1/2}} \rho_\phi$. Thus the choice of Z ensures the existence of a unitary u with $\|u-1\| \leq 246\gamma^{1/2}$ such that $\rho_\phi \approx_{X,\beta} \mathrm{Ad}(u)\circ\pi$. Define an order zero map $\psi : F \to B$ by $\psi(x) = \pi(\mathrm{id}_{(0,1]}\otimes x)$ for $x \in F$. Then $\phi \approx_{X_0,\beta} \mathrm{Ad}(u)\circ\psi$. Since X_0 is a β-net for the unit ball of A, a simple approximation argument gives

$$\|\phi(x) - (\mathrm{Ad}(u)\circ\psi)(x)\| \leq 3\beta\|x\|, \qquad x \in F. \tag{7.15}$$

Recalling that F has dimension m, we find that

$$\|\phi - \mathrm{Ad}(u)\circ\psi\|_{\mathrm{cb}} \leq m\|\phi - \mathrm{Ad}(u)\circ\psi\| \leq 3m\beta < \gamma^{1/2}. \tag{7.16}$$

We also have the estimate

$$\|\psi - \mathrm{Ad}(u)\circ\psi\|_{\mathrm{cb}} \leq 2\|u-1\| < 492\gamma^{1/2}, \tag{7.17}$$

and the desired conclusion $\|\phi-\psi\|_{\mathrm{cb}} \leq 493\gamma^{1/2}$ follows from the previous two inequalities. $\square$

We end by using our methods to give a new characterisation of when a separable nuclear C*-algebra is D-stable, where D is any separable strongly self-absorbing C*-algebra. For simplicity we only state and prove a unital version, but it seems clear that, with some extra effort, one can use [52, Theorem 2.3] to give a non-unital version as well. We first establish some notation. For a C*-algebra A, $\prod_{n=1}^\infty A$ will denote the space of bounded sequences with entries from A while $\sum_{n=1}^\infty A$ is the ideal of sequences $\{a_n\}_{n=1}^\infty$ for which $\lim_{n\to\infty}\|a_n\| = 0$. We write A_∞ for the quotient space and π for the quotient map of $\prod_{n=1}^\infty A$ onto A_∞. We identify A with a subalgebra of A_∞ by first regarding A as the algebra of constant sequences in $\prod_{n=1}^\infty A$ and then applying π. The relative commutant $A' \cap A_\infty$ is the algebra of central sequences.

Theorem 7.9. *Let D be a separable unital strongly self-absorbing* C^**-algebra and let γ satisfy $0 \le \gamma \le 1/169$. Suppose that A is a separable unital nuclear* C^**-algebra and that, for any finite subsets X and Y of the unit balls of A and D respectively, there exists a ucp (Y,γ)-approximate $*$-homomorphism $\theta : D \to A$ with $\|\theta(y)x - x\theta(y)\| \le \gamma$ for $x \in X$ and $y \in Y$. Then A is D-stable.*

Proof. Suppose that we are given a finite set Z (containing 1_D) in the unit ball of D and some $\varepsilon > 0$. Since $4\gamma \le 1/17$ and D is nuclear, Lemma 3.2 shows that there exists a finite set Y (also containing 1_D) in the unit ball of D so that, if $\phi_Y : D \to A_\infty \cap A'$ is a $(Y, 4\gamma)$-approximate $*$-homomorphism, then there is a (Z,ε)-approximate $*$-homomorphism $\psi_{Z,\varepsilon} : D \to A_\infty \cap A'$ near to ϕ_Y.

Let $\{a_n\}_{n=1}^\infty$ be dense in the unit ball of A. Fix n and use Lemma 3.1 to find positive real numbers $(\lambda_i)_{i=1}^m$ summing to 1 and contractions $\{b_i\}_{i=1}^m$ in A such that $\|\sum_{i=1}^m \lambda_i b_i^* b_i - 1_A\| < 1/n$ and

$$\left\| \sum_{i=1}^m \lambda_i (a_j b_i^* \otimes b_i - b_i^* \otimes b_i a_j) \right\|_{A \widehat{\otimes} A} < 1/n, \qquad 1 \le j \le n. \tag{7.18}$$

By our hypotheses, there is a ucp (Y,γ)-approximate $*$-homomorphism $\theta : D \to A$ with $\|\theta(y)b_i - b_i\theta(y)\| \le \gamma$ for $y \in Y \cup Y^* \cup \{yy^* : y \in Y \cup Y^*\}$ and $i = 1, \dots, m$. Define a cpc map $\phi_n : D \to A$ by $\phi_n(x) = \sum_{i=1}^m \lambda_i b_i^* \theta(x) b_i$. For $y \in Y \cup Y^* \cup \{yy^* : y \in Y \cup Y^*\}$, we have

$$\begin{aligned} \|\phi_n(y) - \theta(y)\| &= \left\| \sum_{i=1}^m \lambda_i b_i^* \theta(y) b_i - \sum_{i=1}^m \lambda_i b_i^* b_i \theta(y) \right\| + \left\| (\sum_{i=1}^m \lambda_i b_i^* b_i - 1_A)\theta(y) \right\| \\ &< \gamma + 1/n. \end{aligned} \tag{7.19}$$

As θ is a (Y,γ)-approximate $*$-homomorphism, ϕ_n is a $(Y, 4\gamma + 3/n)$-approximate $*$-homomorphism. Therefore we can define a $(Y, 4\gamma)$-approximate $*$-homomorphism $\phi_Y : D \to A_\infty$ by $\phi_Y(x) = \pi((\phi_n(x)))$, where $\pi : \prod_{\mathbb{N}} A \to A_\infty$ is the quotient map. For each d in the unit ball of D, the map $x_1 \otimes x_2 \mapsto x_1\theta(d)x_2$ extends to contractive linear map from $A \widehat{\otimes} A$ into A. The estimate (7.18) then gives $\|a_j\phi_n(d) - \phi_n(d)a_j\| < 1/n$ for $j \le n$ and so ϕ_Y takes values in $A_\infty \cap A'$. Moreover, as θ is unital, we have $\|\phi_Y(1_D) - 1_{A_\infty}\| \le \gamma$. Now, by our choice of Y, Lemma 3.2 yields a (Z,ε)-approximate $*$-homomorphism $\psi_{Z,\varepsilon} : D \to A_\infty \cap A'$ such that $\|\psi_{Z,\varepsilon} - \phi_Y\| \le 12\gamma^{1/2}$.

Using separability of D, upon increasing Z and decreasing ε we obtain a $*$-homomorphism $\psi : D \to (A_\infty \cap A')_\infty$ such that $\|\psi(1_D) - 1_{(A_\infty)_\infty}\| \le 12\gamma^{1/2} + \gamma < 1$. The latter in particular implies that ψ is unital. As D is nuclear, we may use the Choi–Effros lifting theorem to obtain a ucp lift $\bar{\psi} : D \to \prod_{\mathbb{N}\times\mathbb{N}} A$. Now a standard diagonal argument yields a unital $*$-homomorphism $\tilde{\psi} : D \to A_\infty \cap A'$. By [52, Theorem 2.2] this shows that A is D-stable. □

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