

VALUES OF THE PUKÁNSZKY INVARIANT IN FREE GROUP FACTORS AND THE HYPERFINITE FACTOR

KENNETH J. DYKEMA ^(*)	ALLAN M. SINCLAIR	ROGER R. SMITH ^(**)
Department of Mathematics	School of Mathematics	Department of Mathematics
Texas A&M University	University of Edinburgh	Texas A&M University
College Station, TX 77843	Edinburgh, EH9 3JZ	College Station, TX 77843
USA	SCOTLAND	USA
kdykema@math.tamu.edu	A.Sinclair@ed.ac.uk	rsmith@math.tamu.edu

Abstract

Let $A \subseteq M \subseteq B(L^2(M))$ be a maximal abelian self-adjoint subalgebra (masa) in a type II₁ factor M in its standard representation. The abelian von Neumann algebra \mathcal{A} generated by A and JAJ has a type I commutant which contains the projection $e_A \in \mathcal{A}$ onto $L^2(A)$. Then $\mathcal{A}'(1 - e_A)$ decomposes into a direct sum of type I _{n} algebras for $n \in \{1, 2, \dots, \infty\}$, and those n 's which occur in the direct sum form a set called the Pukánszky invariant, $\text{Puk}(A)$, also denoted $\text{Puk}_M(A)$ when the containing factor is ambiguous. In this paper we show that this invariant can take on the values $S \cup \{\infty\}$ when M is both a free group factor and the hyperfinite factor, and where S is an arbitrary subset of \mathbb{N} . The only previously known values for masas in free group factors were $\{\infty\}$ and $\{1, \infty\}$, and some values of the form $S \cup \{\infty\}$ are new also for the hyperfinite factor.

We also consider a more refined invariant (that we will call the measure–multiplicity invariant), which was considered recently by Neshveyev and Størmer and has been known to experts for a long time. We use the measure–multiplicity invariant to distinguish two masas in a free group factor, both having Pukánszky invariant $\{n, \infty\}$, for arbitrary $n \in \mathbb{N}$.

^(*)Research supported in part by NSF grant DMS-0300336, the Alexander von Humboldt Foundation, and the Edinburgh Mathematical Society.

^(**)Research supported in part by NSF grant DMS-0401043.

1 Introduction

The Pukánszky invariant $\text{Puk}(A)$ of a maximal abelian self-adjoint subalgebra (masa) A of a separable type II_1 factor N with normalized normal trace τ was introduced in [13]. If J denotes the canonical involution on $L^2(N, \tau)$, then the abelian von Neumann algebra \mathcal{A} generated by A and JAJ has a type I commutant \mathcal{A}' . The projection e_A onto $L^2(A)$ lies in \mathcal{A} and $\mathcal{A}'(1 - e_A)$ decomposes into a direct sum of type I_{n_i} algebras, where $1 \leq n_i \leq \infty$. Those n_i 's in this sum form $\text{Puk}(A)$, a subset of $\mathbb{N} \cup \{\infty\}$. This quantity is invariant under the action of any automorphism of N , and so serves as an aid to distinguishing pairs of masas. In [13, 10, 19], various values of the invariant were found for masas, primarily in the hyperfinite type II_1 factor R . In this paper we consider the possible values of the invariant for masas in the free group factors $L(\mathbb{F}_n)$ for $2 \leq n \leq \infty$. In this paper one of our main objectives is to show that in free group factors there exist strongly singular masas whose Pukánszky invariants are $S \cup \{\infty\}$ where S is an arbitrary subset of \mathbb{N} (this and other terminology will be explained in the next section). There are two standard examples of masas in the free group factors. One arises from a single generator of \mathbb{F}_k and the criteria of [3] show that it is singular, while its invariant is $\{\infty\}$, [16]. The other type is the radial or laplacian masa. Rădulescu, [14], has shown that this masa in $L(\mathbb{F}_2)$ has $\{\infty\}$ for its invariant, and it was shown in [17, 18] that both types of masas are strongly singular. On the other hand, the isomorphism between $L(\mathbb{F}_2)$ and $L(\mathbb{F}_5) \otimes \mathbb{M}_2$, [22], can be used to find a tensor product masa whose invariant is $\{1, \infty\}$, although this masa is not singular. These two possibilities, $\{\infty\}$ and $\{1, \infty\}$, were previously the only ones known, although it was shown in [19] that the invariant in a free group factor cannot be a finite set of integers, based on results from [7].

In the second section we investigate masas in free product algebras $M * Q$ where M is a diffuse finite von Neumann algebra while Q has a finite trace and dimension at least two. We show that any (singular) masa in M is also a (singular) masa in $M * Q$ by determining equality of the unitary normalizers of such a masa in the two algebras. This result was originally obtained in [12] by different methods. We then recall from [19] the construction of masas A_n , $1 \leq n < \infty$, with $\text{Puk}(A_n) = \{n, \infty\}$, and show that they are strongly singular in their containing factors M_n and in $M_n * Q$. These masas form the basis for the examples of the next section.

In the third section, for each set $S \subseteq \mathbb{N}$, we form the masa A as a direct sum of the

masas $\{A_i\}_{i \in S}$ inside the direct sum of $\{M_i\}_{i \in S}$. The free product with a suitably chosen von Neumann algebra Q gives a masa in $L(\mathbb{F}_2)$ and the main results of the section, Theorems 3.2 and 3.3, determine the Pukánszky invariant as $S \cup \{\infty\}$. A further free product with $L(\mathbb{F}_{n-2})$ then gives the corresponding result for each $L(\mathbb{F}_n)$, $3 \leq n \leq \infty$. More generally, we show that the same conclusions also hold for $L(\mathbb{F}_n * \Gamma)$, where Γ is an arbitrary countable discrete group.

In the fourth section we obtain the same existence result for masas having Pukánszky invariant $S \cup \{\infty\}$, but in the hyperfinite factor. This extends the set of known values from those obtained in [10, 19].

In the last two sections we consider a finer invariant which can distinguish masas with the same Pukánszky invariant. This has been used extensively in [10], where its origins are attributed to [8]. Since it involves a measure class and a multiplicity function, we will refer to it as the measure–multiplicity invariant. We use it to give examples in the sixth section of nonconjugate masas in a free group factor, both with Pukánszky invariant $\{n, \infty\}$. The groundwork for this result is laid in the fifth section.

We assume throughout, even when not stated explicitly, that all von Neumann algebras are separable in the sense that their preduals are norm-separable as Banach spaces. This is equivalent to the assumption of faithful representations on separable Hilbert spaces. Moreover, all groups will be discrete and countable, so that the associated group von Neumann algebras will be separable.

The results of this paper rely heavily on the theory of free products of von Neumann algebras. We refer the reader to [23] for the necessary background material, and also to [1, 11, 14] for related results on masas in free group factors. We note that *singularity* and *strong singularity* for masas were shown recently to be equivalent, [20]. We have retained the latter terminology in this paper, since showing strong singularity is often the most direct way of proving singularity of masas.

Acknowledgements Much of this work was accomplished while KJD was visiting the University of Münster during 2004-2005. The warm hospitality of the members of the Mathematics Institute is gratefully acknowledged. KJD and RRS thank Kunal Mukherjee for helpful comments about early versions of this paper.

2 Masas in free product factors

Throughout this section M and Q will be separable von Neumann algebras with faithful normal tracial states τ_M and τ_Q . We assume that M is diffuse and that Q has dimension at least 2. Moreover, M and Q are in standard form on separable Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 respectively so that there are distinguished unit vectors $\xi_i \in \mathcal{H}_i$, $i = 1, 2$, such that $\tau_M(\cdot) = \langle \cdot \xi_1, \xi_1 \rangle$ and $\tau_Q(\cdot) = \langle \cdot \xi_2, \xi_2 \rangle$. Recall from [23] that the free product von Neumann algebra $M * Q$ has a normal tracial state τ_{M*Q} whose restrictions to M and Q are respectively τ_M and τ_Q .

If A is a diffuse abelian von Neumann subalgebra of M then it is isomorphic to $L^\infty[0, 1]$ with a faithful state induced from the trace on M . This corresponds to a probability measure μ of the form $f(t) dt$ where $f \in L^1[0, 1]^+$ has unit norm. Then $(L^\infty[0, 1], dt)$ and $(L^\infty[0, 1], f dt)$ are continuous masas on $(L^2[0, 1], dt)$ and $(L^2[0, 1], f dt)$ respectively, and they are unitarily equivalent by a unitary which takes one separating and cyclic vector 1 to the other $f^{1/2}$, [9, Theorem 9.4.1]. The unitaries $\{e^{int}\}_{n \in \mathbb{Z}}$ then give rise to a set of unitaries $\{v^n\}_{n \in \mathbb{Z}}$ satisfying the orthogonality condition

$$\tau_M(v^{*m}v^n) = \tau_M(v^{n-m}) = 0, \quad m \neq n. \quad (2.1)$$

Such an operator v is called a *Haar unitary*.

Recall from [17] that for a map $\phi: N \rightarrow N$, where N is a von Neumann algebra with a faithful normal trace τ , $\|\phi\|_{\infty, 2}$ is defined by

$$\|\phi\|_{\infty, 2} = \sup\{\|\phi(x)\|_{2, \tau} : \|x\| \leq 1, x \in N\}. \quad (2.2)$$

The subscript τ indicates the trace used to calculate $\|\cdot\|_2$ when there could be ambiguity. We will require maps of the form $\Phi_{s,t}(x) = sxt$, $x \in N$, where s and t are fixed but arbitrary elements of N . The following simple lemma will be useful subsequently, and indicates that $\Phi_{s,t}$ depends continuously on s and t in an appropriate sense.

Lemma 2.1. *If $s, s', t, t' \in N$ are operators of norm at most 1, then*

$$\|\Phi_{s,s'} - \Phi_{t,t'}\|_{\infty, 2} \leq \|s - t\|_2 + \|s' - t'\|_2. \quad (2.3)$$

Proof. This is immediate from the algebraic identity

$$sxs' - txt' = (s - t)xs' + tx(s' - t'), \quad (2.4)$$

for $x \in N$. □

Lemma 2.2. *Let \mathbb{E}_M^{M*Q} be the unique trace preserving conditional expectation of $M*Q$ onto M and let v be a Haar unitary in M . Then*

$$\lim_{|k| \rightarrow \infty} \|\mathbb{E}_M^{M*Q}(xv^ky) - \mathbb{E}_M^{M*Q}(x)v^k\mathbb{E}_M^{M*Q}(y)\|_{2,\tau_M} = 0 \quad (2.5)$$

for all $x, y \in M*Q$.

Proof. In view of Lemma 2.1, it suffices to prove the result when x and y are words in $M*Q$. Each such word w can be expressed as $w = m + \sum_{i=1}^r \alpha_i w_i$ where $m \in M$, $\alpha_i \in \mathbb{C}$, and each w_i contains only letters of zero trace, and at least one letter from Q , [2, Lemma 1]. For such a representation, $\mathbb{E}_M^{M*Q}(w) = m$. Now (2.5) is immediate if x or y lies in M , by properties of the conditional expectation. Thus, it suffices to prove (2.5) when x and y are words whose letters have zero trace and contain at least one letter from Q . There are several cases to consider, depending on whether the last letter of x and the first letter of y are in M or Q . Suppose initially that both are in Q . Then xv^ky is a reduced word for $|k| \geq 1$, each letter has zero trace, and thus the two terms in (2.5) are both 0. Now suppose that x ends and y begins with letters from M . Then write $x = \tilde{x}m_1$, $y = m_2\tilde{y}$ where $m_1, m_2 \in M$, and the last letter of \tilde{x} and the first letter of \tilde{y} are in Q . Then we may express

$$xv^ky = \tilde{x}m_1v^km_2\tilde{y} = \tilde{x}(m_1v^km_2 - \tau_M(m_1v^km_2))\tilde{y} + \tau_M(m_1v^km_2)\tilde{x}\tilde{y}. \quad (2.6)$$

Thus

$$\mathbb{E}_M^{M*Q}(x) = \mathbb{E}_M^{M*Q}(y) = \mathbb{E}_M^{M*Q}(\tilde{x}(m_1v^km_2 - \tau_M(m_1v^km_2))\tilde{y}) = 0, \quad (2.7)$$

while $\lim_{|k| \rightarrow \infty} \tau_M(m_2m_1v^k) = 0$ since $\{v^k\}_{k \in \mathbb{Z}}$ is an orthonormal set of vectors in $L^2(M, \tau_M)$. This proves (2.5) in this case also. The remaining case, when exactly one of x and y^* ends with a letter from Q , is handled just as in the previous case but using only one of \tilde{x} or \tilde{y} . □

We denote by $\mathcal{N}_M(A)$ the set of unitaries $u \in M$ which normalize a given subalgebra A , in the sense that $uAu^* = A$. This group is called the *unitary normalizer* of A in M . The masa A is said to be *singular* if $\mathcal{N}_M(A)$ coincides with the unitary group of A , [3].

The following result is due to Popa, [12, Remark 6.3]. We present an alternative proof, based on conditional expectations, since this method is required below.

Theorem 2.3. *Let A be a diffuse von Neumann subalgebra of M . Then the following statements hold:*

(i) The unitary normalizers $\mathcal{N}_M(A)$ and $\mathcal{N}_{M*Q}(A)$ are equal.

(ii) If A is a (singular) masa in M , then it is also a (singular) masa in $M * Q$.

Proof. (i) Let $u \in M * Q$ be a unitary which normalizes A , and let v be a Haar unitary in A . Then $uv^k u^* \in A \subseteq M$ for $k \in \mathbb{Z}$, and so $\mathbb{E}_M^{M*Q}(uv^k u^*) = uv^k u^*$. Thus the $\|\cdot\|_{2,\tau_M}$ -norms of these elements are 1. From (2.5) in Lemma 2.2,

$$\lim_{|k| \rightarrow \infty} \|\mathbb{E}_M^{M*Q}(u)v^k \mathbb{E}_M^{M*Q}(u^*)\|_{2,\tau_M} = 1, \quad (2.8)$$

which is impossible unless $\|\mathbb{E}_M^{M*Q}(u)\|_{2,\tau_M} = 1$. But then Hilbert space orthogonality gives

$$1 = \|u\|_{2,\tau_{M*Q}}^2 = \|(I - \mathbb{E}_M^{M*Q})(u)\|_{2,\tau_{M*Q}}^2 + \|\mathbb{E}_M^{M*Q}(u)\|_{2,\tau_M}^2, \quad (2.9)$$

from which we conclude that $(I - \mathbb{E}_M^{M*Q})(u) = 0$. Thus $u \in M$, proving that $\mathcal{N}_{M*Q}(A) \subseteq \mathcal{N}_M(A)$. The reverse containment is obvious.

(ii) Any unitary $u \in M * Q$ which commutes with A lies in $\mathcal{N}_{M*Q}(A)$, and so in M , by part (i). If A is a masa in M , then $u \in A' \cap M = A$, and so A is a masa in $M * Q$. Further, suppose that A is singular. Then each unitary in $\mathcal{N}_M(A)$ lies in A and so the same is true for $\mathcal{N}_{M*Q}(A)$. Thus A is also singular in $M * Q$. \square

In [17], *strong singularity* of a masa $A \subseteq M$ was defined by the requirement that

$$\|\mathbb{E}_{uAu^*}^M - \mathbb{E}_A^M\|_{\infty,2} \geq \|u - \mathbb{E}_A^M(u)\|_2 \quad (2.10)$$

for all unitaries $u \in M$. The left-hand side of (2.10) vanishes when $u \in \mathcal{N}_M(A)$, and the inequality then shows that $u \in A$. Thus singularity of A is a consequence of strong singularity, and the reverse implication was established recently in [20]. The usefulness of strong singularity lies in the ease with which (2.10) can be verified in specific cases. There are two main criteria which establish strong singularity. If there is a unitary $v \in A$ such that

$$\lim_{|k| \rightarrow \infty} \|\mathbb{E}_A^M(xv^k y) - \mathbb{E}_A^M(x)v^k \mathbb{E}_A^M(y)\|_2 = 0 \quad (2.11)$$

for all $x, y \in M$, then we say that \mathbb{E}_A^M is an *asymptotic homomorphism* with respect to v . In [17, Theorem 4.7] it was shown that this property implies strong singularity for A when M is a type II₁ factor, but the proof is also valid for a general finite von Neumann algebra. There is a weaker form of (2.11), defining the *weak asymptotic homomorphism property (WAHP)*:

given $\varepsilon > 0$ and a finite set of elements $\{x_1, \dots, x_n, y_1, \dots, y_n\} \subseteq M$, there exists a unitary $u \in A$ such that

$$\|\mathbb{E}_A^M(x_i u y_j) - \mathbb{E}_A^M(x_i) u \mathbb{E}_A^M(y_j)\|_2 < \varepsilon, \quad 1 \leq i, j \leq n. \quad (2.12)$$

This property implies strong singularity, [15, Lemma 2.1], and it is also equivalent to singularity, [20, Theorem 2.3]. The previous theorem and these remarks lead to the following, which we include just to emphasize the methods employed for subsequent results.

Corollary 2.4. *Let A be a masa in M .*

(i) *If \mathbb{E}_A^M is an asymptotic homomorphism with respect to a Haar unitary $v \in A$ then this also holds for \mathbb{E}_A^{M*Q} ;*

(ii) *if A has the WAHP in M then it also has this property in $M * Q$.*

*In both cases A is strongly singular in both M and $M * Q$.*

Proof. (i) We first note that uniqueness of trace preserving conditional expectations implies that $\mathbb{E}_A^{M*Q} = \mathbb{E}_A^M \circ \mathbb{E}_M^{M*Q}$. From Lemma 2.2,

$$\lim_{|k| \rightarrow \infty} \|\mathbb{E}_M^{M*Q}(x v^k y) - \mathbb{E}_M^{M*Q}(x) v^k \mathbb{E}_M^{M*Q}(y)\|_2 = 0 \quad (2.13)$$

for all $x, y \in M * Q$. Since \mathbb{E}_A^M is a $\|\cdot\|_2$ -norm contraction, we may apply this operator to (2.13) to obtain

$$\lim_{|k| \rightarrow \infty} \|\mathbb{E}_A^{M*Q}(x v^k y) - \mathbb{E}_A^M(\mathbb{E}_M^{M*Q}(x) v^k \mathbb{E}_M^{M*Q}(y))\|_2 = 0 \quad (2.14)$$

for all $x, y \in M * Q$. The asymptotic homomorphism hypothesis, when applied to the second term in (2.14), leads to

$$\lim_{|k| \rightarrow \infty} \|\mathbb{E}_A^{M*Q}(x v^k y) - \mathbb{E}_A^{M*Q}(x) v^k \mathbb{E}_A^{M*Q}(y)\|_2 = 0 \quad (2.15)$$

for all $x, y \in M * Q$. Thus \mathbb{E}_A^{M*Q} is an asymptotic homomorphism with respect to v .

(ii) This is immediate from Theorem 2.3 and the equivalence of singularity and the WAHP.

In both cases, the (strong) singularity of A follows from the remarks preceding this corollary. \square

In [19], a family of masas inside group von Neumann factors was presented with various Pukánszky invariants. We recall these masas now since we wish to give some extra information about them. Consider an abelian subgroup H of an I.C.C. group G . In [19], the problem of describing the Pukánszky invariant of $L(H)$ in $L(G)$, when $L(H)$ is a masa, was solved in terms of double cosets and stabilizer subgroups. The double coset HgH of $g \in G \setminus H$ is $\{h g k : h, k \in H\}$. The stabilizer subgroup $K_g \subseteq H \times H$ is defined to be $\{(h, k) : h, k \in H, h g k = g\}$. Under the additional hypothesis that any two such subgroups K_c and K_d , for $c, d \in G \setminus H$, are either equal or satisfy the noncommensurability condition that $K_c K_d / (K_c \cap K_d)$ has infinite order, an equivalence relation was defined on the non-trivial double cosets by $HcH \sim HdH$ if and only if $K_c = K_d$. The numbers, including ∞ , in $\text{Puk}(L(H))$ are then the numbers of double cosets that occur in the various equivalence classes, [19, Theorem 4.1]. We use this now to discuss the Pukánszky invariant in certain free products of group factors. The groups that arise all satisfy the additional hypothesis above, so that [19, Theorem 4.1] applies to them.

Proposition 2.5. *Let Γ be a countable discrete group of order at least 2. For each $n \geq 1$, there exists a countable discrete amenable i.c.c. group G_n with an abelian subgroup H_n having the following properties:*

(i) $L(H_n)$ is a strongly singular masa in both $L(G_n)$ and $L(G_n) * L(\Gamma)$.

(ii) $\text{Puk}_{L(G_n)}(L(H_n)) = \text{Puk}_{L(G_n) * L(\Gamma)}(L(H_n)) = \{n, \infty\}$.

Proof. Fix an integer $n \in \mathbb{N}$. Let \mathbb{Q} denote the group of rationals under addition and let \mathbb{Q}^\times be the multiplicative group of nonzero rationals. Let $P_n \subseteq \mathbb{Q}^\times$ be the subgroup

$$P_n = \left\{ \frac{p}{q} 2^{nj} : j \in \mathbb{Z}, p, q \in \mathbb{Z}_{\text{odd}} \right\}, \quad 1 \leq n < \infty,$$

let $P_\infty \subseteq \mathbb{Q}^\times$ be the subgroup

$$P_\infty = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}_{\text{odd}} \right\},$$

and let G_n , $n \geq 1$, be the matrix group

$$G_n = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & f & 0 \\ 0 & 0 & g \end{pmatrix} : x, y \in \mathbb{Q}, f \in P_n, g \in P_\infty \right\}$$

with abelian subgroup H_n consisting of the diagonal matrices in G_n . Then $L(H_n)$ is a masa in the factor $L(G_n)$, and $\text{Puk}(L(H_n)) = \{n, \infty\}$, (see [19, Example 5.2], where it was also noted that G_n is amenable). There are three equivalence classes of double cosets whose sizes are n , ∞ and ∞ , and the corresponding stabilizer subgroups are respectively

$$\left\{ \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & g \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & g^{-1} \end{pmatrix} \right) : g \in P_\infty \right\}, \left\{ \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & f^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) : f \in P_n \right\},$$

$$\left\{ \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \right\}. \quad (2.16)$$

When H_n is viewed as a subgroup of $G_n * \Gamma$, where Γ is a countable discrete group of order at least 2, each element of $(G_n * \Gamma) \setminus G_n$ has a trivial stabilizer subgroup so the extra double cosets fall into the third equivalence class above, showing that

$$\text{Puk}_{L(G_n)*L(\Gamma)}(L(H_n)) = \text{Puk}_{L(G_n)}(L(H_n)) = \{n, \infty\}$$

in this case. Let v_n denote the group element $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & 0 \\ 0 & 0 & 5 \end{pmatrix} \in H_n$, viewed as a unitary in $L(H_n)$. If τ_n is the standard trace on $L(G_n)$, then $\tau_n(g) = 0$ for any group element $g \neq e$, so v_n is a Haar unitary in $L(H_n)$. A routine matrix calculation shows that $g_1 v_n^k g_2 \in G_n \setminus H_n$ for $|k|$ sufficiently large, when $g_1, g_2 \in G_n \setminus H_n$. Then, viewing these group elements as unitaries in $L(G_n)$, we have $\mathbb{E}_{L(H_n)}^{L(G_n)}(g_i) = 0$, $i = 1, 2$, and $\mathbb{E}_{L(H_n)}^{L(G_n)}(g_1 v_n^k g_2) = 0$ for $|k|$ sufficiently large, so that

$$\lim_{|k| \rightarrow \infty} \|\mathbb{E}_{L(H_n)}^{L(G_n)}(g_1 v_n^k g_2) - \mathbb{E}_{L(H_n)}^{L(G_n)}(g_1) v_n^k \mathbb{E}_{L(H_n)}^{L(G_n)}(g_2)\|_2 = 0. \quad (2.17)$$

Then a simple approximation argument gives (2.17) for g_1 and g_2 replaced by general elements $x, y \in L(G_n)$. This shows that $\mathbb{E}_{L(H_n)}^{L(G_n)}$ is an asymptotic homomorphism for v_n , where we have employed the methods of [15, Section 2]. Thus each $L(H_n)$ is strongly singular in both $L(G_n)$ and $L(G_n) * L(\Gamma)$, using Corollary 2.4. \square

Remark 2.6. The groups constructed above will form the basis for our results in the next section. Denote the von Neumann algebras $L(G_n)$ and $L(H_n)$ of Proposition 2.5 by M_n and A_n respectively. Now consider an arbitrary nonempty subset $S \subseteq \mathbb{N}$, let $M_S = \bigoplus_{n \in S} M_n$ and let $A_S = \bigoplus_{n \in S} A_n$. Choose numbers $\{\alpha_n\}_{n \in S}$ from $(0,1)$ such that $\sum_{n \in S} \alpha_n = 1$, and let τ_S

be the normalized trace on M_S given by $\tau_S = \sum_{n \in S} \alpha_n \tau_n$, where τ_n is the normalized trace on M_n , $n \in S$. Then let $v_S = \bigoplus_{n \in S} v_n$ be a unitary in M_S . Since arbitrary direct sums are approximable in $\|\cdot\|_{2, \tau_S}$ -norm by finitely nonzero ones, it is routine to verify that $\mathbb{E}_{A_S}^{M_S}$ is an asymptotic homomorphism for v_S and so, by Corollary 2.4, A_S is a strongly singular masa in $M_S * Q$ for any von Neumann algebra Q of dimension at least 2. These examples will be used below with $Q = L(\Gamma)$ for various choices of Γ . \square

We end this section with two observations which we include because they are simple deductions from our previous work. The first is known, [5, 6], and the second may be known but we do not have a reference.

Remark 2.7. (i) If M is a diffuse finite von Neumann algebra then any central unitary $u \in Z(M * Q)$ normalizes all masas in M so, by Theorem 2.3, $u \in M$. From properties of the free product, the only elements in M which commute with Q are scalar multiples of 1, so $Z(M * Q)$ is trivial and $M * Q$ is a factor.

(ii) With the same assumption on M , and Q any type II_1 factor, M embeds into the factor $M * Q$ with τ_M being the restriction to M of $\tau_{M * Q}$. Moreover, if u is a unitary in $M' \cap (M * Q)$, then u normalizes each masa in M , so must lie in M , by Theorem 2.3. It follows that, for this embedding, the relative commutant and the center coincide. Of course, $Z(M)$ will be present in the relative commutant for any embedding of M into a factor. \square

3 Construction of masas

In this section we construct strongly singular masas in the free group factors whose Pukánszky invariants are $S \cup \{\infty\}$ for arbitrary subsets S of \mathbb{N} . The construction is based on direct sums, and the following two results keep track of the contributions of the individual summands.

Lemma 3.1. *Suppose that B and D are separable von Neumann algebras with normal faithful traces τ_B and τ_D , respectively. Suppose that B and D are both of dimension at least two. Let λ_B and ρ_B denote the usual left and right actions of B on $L^2(B) := L^2(B, \tau_B)$. Let*

$$(N, \tau) = (B, \tau_B) * (D, \tau_D)$$

be their free product von Neumann algebra. Let λ_N and ρ_N denote as usual the left and right actions of N on $L^2(N) := L^2(N, \tau)$. Then there is a separable infinite dimensional Hilbert space \mathcal{K} and a unitary operator

$$U : L^2(N) \rightarrow L^2(B) \oplus L^2(B) \otimes \mathcal{K} \otimes L^2(B) \quad (3.1)$$

such that for all $b \in B$,

$$U\lambda_N(b)U^* = \lambda_B(b) \oplus \lambda_B(b) \otimes \text{id}_{\mathcal{K}} \otimes \text{id}_{L^2(B)} \quad (3.2)$$

$$U\rho_N(b)U^* = \rho_B(b) \oplus \text{id}_{L^2(B)} \otimes \text{id}_{\mathcal{K}} \otimes \rho_B(b). \quad (3.3)$$

Proof. This is very similar to part of Voiculescu's construction of the free product of von Neumann algebras [21], but for completeness we will describe it in some detail. Let us write $\mathcal{H}_D = L^2(D, \tau_D)$ and let \mathcal{H}_D° denote the orthocomplement of the specified vector $\hat{1}_D \in \mathcal{H}_D$. Similarly, let $\mathcal{H}_B = L^2(B)$ and let \mathcal{H}_B° denote the orthocomplement of the specified vector $\hat{1}_B \in \mathcal{H}_B$. Then by Voiculescu's construction,

$$L^2(N) = \mathbb{C}\xi \oplus \bigoplus_{\substack{n \in \mathbb{N} \\ i_1, \dots, i_n \in \{B, D\} \\ i_1 \neq i_2, i_2 \neq i_3, \dots, i_{n-1} \neq i_n}} \mathcal{H}_{i_1}^\circ \otimes \dots \otimes \mathcal{H}_{i_n}^\circ. \quad (3.4)$$

Let

$$\mathcal{K} = \mathcal{H}_D^\circ \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_D^\circ \otimes (\mathcal{H}_B^\circ \otimes \mathcal{H}_D^\circ)^{\otimes n}$$

and consider the unitary

$$U : L^2(N) \rightarrow L^2(B) \oplus L^2(B) \otimes \mathcal{K} \otimes L^2(B)$$

defined, using (3.4), as follows. The distinguished vector ξ is mapped to $\hat{1}_B \in L^2(B)$ and U acts as the identity on $\mathcal{H}_B^\circ \subseteq L^2(B)$. For vectors $\zeta_1 \otimes \cdots \otimes \zeta_n \in \mathcal{H}_{i_1}^\circ \otimes \cdots \otimes \mathcal{H}_{i_n}^\circ$, the action of U is given by

$$\zeta_1 \otimes \cdots \otimes \zeta_n \mapsto \begin{cases} \zeta_1 \otimes (\zeta_2 \otimes \cdots \otimes \zeta_{n-1}) \otimes \zeta_n, & n \geq 3, i_1 = i_n = B \\ \zeta_1 \otimes (\zeta_2 \otimes \cdots \otimes \zeta_n) \otimes \hat{1}_B, & i_1 = B, i_n = D \\ \hat{1}_B \otimes (\zeta_1 \otimes \cdots \otimes \zeta_{n-1}) \otimes \zeta_n, & i_1 = D, i_n = B \\ \hat{1}_B \otimes (\zeta_1 \otimes \cdots \otimes \zeta_n) \otimes \hat{1}_B, & i_1 = i_n = D \end{cases}$$

in $L^2(B) \otimes \mathcal{K} \otimes L^2(B)$. Since neither B nor D is equal to \mathbb{C} and both are separable, the Hilbert space \mathcal{K} is separable and infinite dimensional. The equality (3.2) is now easily obtained by verifying that

$$U\lambda_N(b) = (\lambda_B(b) \oplus \lambda_B(b) \otimes \text{id}_{\mathcal{K}} \otimes \text{id}_{L^2(B)})U. \quad (3.5)$$

The equation (3.3) follows from equation (3.2) and the fact (see [21]) that for $\zeta_1 \otimes \cdots \otimes \zeta_n \in \mathcal{H}_{i_1}^\circ \otimes \cdots \otimes \mathcal{H}_{i_n}^\circ$ as in (3.4),

$$J_N(\zeta_1 \otimes \cdots \otimes \zeta_n) = J_{i_n}\zeta_n \otimes \cdots \otimes J_{i_1}\zeta_1.$$

□

Theorem 3.2. *Let I be a finite or countable set containing at least two elements and, for each $i \in I$, let M_i be a diffuse separable von Neumann algebra with a normal faithful trace τ_i . Let A_i be a masa in M_i and let Q be a diffuse separable von Neumann algebra with a normal faithful trace τ_Q . Let $\alpha_i \in (0, 1)$ be such that $\sum_{i \in I} \alpha_i = 1$. Let*

$$M = \bigoplus_{i \in I} M_i$$

with trace τ given by

$$\tau((x_i)_{i \in I}) = \sum_{i \in I} \alpha_i \tau_i(x_i) \quad (3.6)$$

and let

$$A = \bigoplus_{i \in I} A_i \subseteq M. \quad (3.7)$$

Let

$$(N_i, \hat{\tau}_i) = (M_i, \tau_i) * (Q, \tau_Q) \quad (i \in I) \quad (3.8)$$

$$(N, \hat{\tau}) = (M, \tau) * (Q, \tau_Q) \quad (3.9)$$

be the free products of von Neumann algebras. Then, for all $i \in I$, N_i is a type II_1 factor and A_i is a masa in N_i . Moreover N is a type II_1 factor, A is a masa in N and the Pukánszky invariants satisfy

$$\text{Puk}_N(A) = \{\infty\} \cup \bigcup_{i \in I} \text{Puk}_{N_i}(A_i). \quad (3.10)$$

Proof. We first note that each N_i and N are factors in which respectively A_i and A are masas, by Theorem 2.3 and Remark 2.7.

Let $p_i \in A$ be the projection with entry 1 in the i th component A_i and entry 0 in every other component of the direct sum (3.7). These are orthogonal, central projections in M . For $i, j \in I$, let

$$q_{ij} = \lambda_N(p_i)\rho_N(p_j) \in \mathcal{B}(L^2(N)).$$

Then q_{ij} is an element of $\mathcal{A} = (\lambda_N(A) \cup \rho_N(A))''$, which is the center of \mathcal{A}' . Since the strong operator sum $\sum_{i,j \in I} q_{ij}$ is equal to the identity, it follows that the Pukánszky invariant $\text{Puk}_N(A)$ is equal to the union over all i and j of those $n \in \mathbb{N} \cup \{\infty\}$ such that $q_{ij}\mathcal{A}'(1 - e_A)$ has a nonzero part of type I_n . Implicitly using the unitary of Lemma 3.1 for the free product construction (3.9), we identify $L^2(N)$ with

$$L^2(M) \oplus L^2(M) \otimes \mathcal{K} \otimes L^2(M)$$

and we make the identifications

$$\begin{aligned} \lambda_N(p_i) &= \lambda_M(p_i) \oplus \lambda_M(p_i) \otimes \text{id}_{\mathcal{K}} \otimes \text{id}_{L^2(M)} \\ \rho_N(p_j) &= \rho_M(p_j) \oplus \text{id}_{L^2(M)} \otimes \text{id}_{\mathcal{K}} \otimes \rho_M(p_j). \end{aligned}$$

If $i \neq j$, then $\lambda_M(p_i)\rho_M(p_j) = 0$ and consequently $q_{ij}\mathcal{A}$ is identified with the algebra

$$(\lambda_M(A_i) \otimes \text{id}_{\mathcal{K}} \otimes \rho_M(A_j))'' \subseteq \mathcal{B}(L^2(M_i) \otimes \mathcal{K} \otimes L^2(M_j)),$$

which commutes with $\text{id}_{L^2(M_i)} \otimes \mathcal{B}(\mathcal{K}) \otimes \text{id}_{L^2(M_j)}$. Therefore, $q_{ij}\mathcal{A}'$ is purely of type I_∞ . Thus, $q_{ij}\mathcal{A}'$ contributes only ∞ to $\text{Puk}_N(A)$. On the other hand, in the case $i = j$, since $Ap_i = A_i$ and $Mp_i = M_i$, we see that $q_{ii}\mathcal{A}$ is identified with the von Neumann algebra generated by

$$\begin{aligned} &\{\lambda_{M_i}(a) \oplus \lambda_{M_i}(a) \otimes \text{id}_{\mathcal{K}} \otimes \text{id}_{L^2(M_i)} \mid a \in A_i\} \\ &\cup \{\rho_{M_i}(a) \oplus \text{id}_{L^2(M_i)} \otimes \text{id}_{\mathcal{K}} \otimes \rho_{M_i}(a) \mid a \in A_i\} \end{aligned}$$

in $\mathcal{B}(L^2(M_i) \oplus L^2(M_i) \otimes \mathcal{K} \otimes L^2(M_i))$. Using the unitary U from Lemma 3.1 in the case of the free product (3.8), we thereby identify $q_{ii}\mathcal{A}$ with the abelian von Neumann algebra \mathcal{A}_i used to define $\text{Puk}_{N_i}(A_i)$. Thus, $q_{ii}\mathcal{A}'$ contributes exactly $\text{Puk}_{N_i}(A_i)$ to $\text{Puk}_N(A)$. Taking the union over all i and j yields (3.10). \square

We now come to the main result of the paper. We let \mathbb{F}_k , $k \in \{1, 2, \dots, \infty\}$, denote the free group on k generators, where \mathbb{F}_1 is identified with \mathbb{Z} . To avoid discussion of separate cases below, we adopt the convention that \mathbb{F}_{k-r} means \mathbb{F}_k when $k = \infty$ and $r \in \mathbb{N}$.

Theorem 3.3. *Let S be an arbitrary subset of \mathbb{N} , let $k \in \{2, 3, \dots, \infty\}$ be arbitrary, and let Γ be an arbitrary countable discrete group. Then there exists a strongly singular masa $A \subseteq L(\mathbb{F}_k * \Gamma)$ whose Pukánszky invariant is $S \cup \{\infty\}$.*

Proof. We first consider the case when Γ is trivial. If S is empty, then we may take A to be the masa corresponding to one of the generators of \mathbb{F}_k . This masa is strongly singular, [17] and has Pukánszky invariant $\{\infty\}$, [19]. Thus we may assume that S is nonempty. For each $n \geq 1$, let G_n be the I.C.C. group of Proposition 2.5 with abelian subgroup H_n . The subgroup

$$\left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{Q} \right\}$$

is abelian and normal in G_n , and the quotient is isomorphic to H_n . As was noted in [19], each G_n is amenable, and $L(G_n)$ is the hyperfinite type II₁ factor R . If we write $M_n = L(G_n) \cong R$, then we have the masa $A_n := L(H_n)$ inside the hyperfinite type II₁ factor. Then define $M_S = \bigoplus_{i \in S} M_i$ and $A_S = \bigoplus_{i \in S} A_i$. We see that M_S is a direct sum of copies of R and so is hyperfinite. Consequently, $M_S * L(\mathbb{F}_1)$ is isomorphic to $L(\mathbb{F}_2)$, [5], while, for $3 \leq k \leq \infty$, we have the isomorphisms

$$M_S * L(\mathbb{F}_{k-1}) \cong M_S * (L(\mathbb{F}_1) * L(\mathbb{F}_{k-2})) \cong (M_S * L(\mathbb{F}_1)) * L(\mathbb{F}_{k-2}) \cong L(\mathbb{F}_2) * L(\mathbb{F}_{k-2}) \cong L(\mathbb{F}_k).$$

The results of Section 2 imply that A_S is a strongly singular masa in both M_S and $M_S * L(\mathbb{F}_{k-1})$. From Theorem 3.2,

$$\text{Puk}_{L(\mathbb{F}_k)}(A_S) = \{\infty\} \cup \bigcup_{i \in S} \text{Puk}_{M_i}(A_i) = S \cup \{\infty\}, \quad (3.11)$$

where the last equality comes from Proposition 2.5.

The case where the countable discrete group Γ is included is essentially the same. Simply observe that $M_S * (L(\mathbb{F}_{k-1}) * L(\Gamma)) \cong (M_S * L(\mathbb{F}_{k-1})) * L(\Gamma) \cong L(\mathbb{F}_k) * L(\Gamma)$. \square

This theorem has an interesting parallel in [10], where it was shown that any subset of $\mathbb{N} \cup \{\infty\}$ which contains 1 can be $\text{Puk}_R(A)$ for some masa A in the hyperfinite factor R .

When M and Q are type II_1 factors and A is a masa in M then it is also a masa in $M * Q$ by Theorem 2.3. In view of our results to this point, it is natural to ask whether

$$\text{Puk}_{M*Q}(A) = \text{Puk}_M(A) \cup \{\infty\}. \quad (3.12)$$

This is not true in general, as we now show. We follow Proposition 2.5, but replace the groups there by the ones below which come from [19, Example 5.1]:

$$G_n = \left\{ \begin{pmatrix} 1 & x \\ 0 & f \end{pmatrix} : f \in P_n, \quad x \in \mathbb{Q} \right\}, \quad H_n = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix} : f \in P_n \right\}. \quad (3.13)$$

The double cosets form a single equivalence class of n elements and each stabilizer subgroup is trivial (see [19] for details). When the masa $L(H_n)$ in $L(G_n)$ is viewed as a masa in $L(G_n * \Gamma)$ for a nontrivial countable discrete group Γ , the elements of $(G_n * \Gamma) \setminus G_n$ have trivial stabilizer subgroups, and so we still have one equivalence class of double cosets but now with infinitely many elements. We conclude that $\text{Puk}_{L(G_n)}(L(H_n)) = \{n\}$, while $\text{Puk}_{L(G_n)*L(\Gamma)}(L(H_n)) = \{\infty\}$. Thus (3.12) fails in general.

4 Masas in the hyperfinite factor

In [10], it was shown that any subset of $\mathbb{N} \cup \{\infty\}$ containing 1 could be the Pukánszky invariant of a masa in the hyperfinite type II₁ factor R . Subsequently many other admissible subsets were found in [19]. Building on the examples of the latter paper, we now show that any subset of $\mathbb{N} \cup \{\infty\}$ containing $\{\infty\}$ is a possible value of the invariant, exactly like the result obtained for the free group factors in the previous section.

Example 4.1. We will need below the group P_∞ , defined by

$$P_\infty = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}_{\text{odd}} \right\},$$

as in the proof of Proposition 2.5.

Let S be an arbitrary subset of \mathbb{N} . We construct a strongly singular masa A in the hyperfinite factor R with $\text{Puk}(A) = S \cup \{\infty\}$ as follows. We will assume that S is nonempty since $\{\infty\}$ is already a known value (see [19, Example 5.1] with $n = \infty$). Let $\{n_1, n_2, \dots\}$ be a listing of the numbers in S , where each is repeated infinitely often to ensure that the list is infinite. Then define a matrix group G by specifying the general group elements to be

$$\begin{pmatrix} 1 & x_1 & x_2 & x_3 & \cdots \\ 0 & f_1 2^{n_1 k} & 0 & 0 & \cdots \\ 0 & 0 & f_2 2^{n_2 k} & 0 & \cdots \\ 0 & 0 & 0 & f_3 2^{n_3 k} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (4.1)$$

where $k \in \mathbb{Z}$, $x_j \in \mathbb{Q}$, $f_j \in P_\infty$, and the relations $x_i \neq 0$ and $f_j \neq 1$ occur only finitely often. This makes G a countable group which is easily checked to be I.C.C. Moreover G is amenable since there is an abelian normal subgroup N (those matrices with only 1's on the diagonal) so that the quotient G/N is isomorphic to the abelian subgroup H consisting of the diagonal matrices in G . Thus $R := L(G)$ is the hyperfinite factor and $A := L(H)$ is a masa in R , as in the examples of [19, Section 5]. For any finite nonempty subset T of \mathbb{N} , let H_T be the subgroup of H obtained by the requirements that $k = 0$ and that $f_i = 1$ for $i \in T$.

Each nontrivial double coset is generated by a nontrivial element

$$x = \begin{pmatrix} 1 & x_1 & x_2 & x_3 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix},$$

of N , and we split into two cases according to whether the number of nonzero x_i is exactly 1 or is greater than 1. In the first case, suppose that x_1 is the sole nonzero value. We obtain n_1 distinct cosets generated by the elements

$$\begin{pmatrix} 1 & 2^k & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ \vdots & \vdots & & & \ddots \end{pmatrix}, \quad 1 \leq k \leq n_1, \quad (4.2)$$

respectively, whose stabilizer subgroups are all $\{(h, h^{-1}) : h \in H_{\{1\}}\}$. A similar result holds when the nonzero entry occurs in the i^{th} position: n_i distinct cosets with stabilizer subgroups $\{(h, h^{-1}) : h \in H_{\{i\}}\}$.

If T is a finite subset of \mathbb{N} with $|T| > 1$, then two elements of N , having nonzero entries respectively x_i and y_i for $i \in T$, generate the same double coset precisely when there exist $k \in \mathbb{Z}$ and $f_i \in P_\infty$ such that $x_i = y_i f_i 2^{n_i k}$, $i \in T$. The stabilizer subgroup in this case is $\{(h, h^{-1}) : h \in H_T\}$. Thus the distinct stabilizer subgroups are pairwise noncommensurable and [19, Theorem 4.1] allows us to determine the Pukánszky invariant by counting the equivalence classes of double cosets. In the first case we obtain the integers $n_i \in S$. In the second case each equivalence class has infinitely many elements and so the contribution is $\{\infty\}$, showing that $\text{Puk}(A) = S \cup \{\infty\}$, as required. \square

5 Representations of abelian C^* -algebras.

We use the approach to direct integrals found in Kadison and Ringrose [9, §14.1], because it fits well with our computations that will follow in Section 6. Suppose X is a compact Hausdorff space and $\pi : C(X) \rightarrow \mathcal{B}(\mathcal{H})$ is a unital $*$ -representation. Then there is a Borel measure μ on X so that \mathcal{H} can be written as a direct integral

$$\mathcal{H} = \int_X^{\oplus} \mathcal{H}_x d\mu(x)$$

and so that π is a diagonal representation with, for all $f \in C(X)$ and $x \in X$,

$$\pi(f)_x = f(x)\text{id}_{\mathcal{H}_x}.$$

The multiplicity function $m(x) = \dim(\mathcal{H}_x)$ is μ -measurable. Of course such results are known in much greater generality. The pair $([\mu], m)$, where m is taken up to redefinition on sets of μ -measure zero, is a conjugacy invariant for π , and every such pair arises from some unital $*$ -representation of $C(X)$. We will use Proposition 5.8 to find μ and m in concrete examples.

Definition 5.1. We call $[\mu]$ the *measure class* of π and m the *multiplicity function* of π .

We now consider what happens to the measure class and multiplicity function under certain natural constructions.

Proposition 5.2. *For $i = 1, 2$ let $A_i = C(X_i)$ be an abelian, unital C^* -algebra and let $\pi_i : A_i \rightarrow \mathcal{B}(\mathcal{H}_i)$ be a unital $*$ -representation. Let $[\mu_i]$ be the measure class and m_i the multiplicity function of π_i .*

(i) *Let X be the disconnected union of X_1 and X_2 , so that we identify $A = C(X)$ with $A_1 \oplus A_2$. Let $\pi : A \rightarrow \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ be the unital $*$ -representation given by*

$$\pi(f_1 \oplus f_2) = \pi_1(f_1) \oplus \pi_2(f_2), \quad (f_i \in A_i).$$

Then the measure class of π is $[\mu]$, where

$$\mu(E_1 \cup E_2) = \mu_1(E_1) + \mu_2(E_2), \quad (E_i \subseteq X_i)$$

and the multiplicity function m of π is such that the restriction of m to X_i is m_i .

(ii) Let $\pi = \pi_1 \otimes \pi_2 : A_1 \otimes A_2 \rightarrow \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ be the tensor product representation of the tensor product C^* -algebra $A_1 \otimes A_2$, which we identify with $C(X_1 \times X_2)$. Then the measure class of π is $[\mu_1 \otimes \mu_2]$ and the multiplicity function m of π is given by $m(x_1, x_2) = m_1(x_1)m_2(x_2)$.

Proof. Part (i) is obvious.

For part (ii), by Lemma 14.1.23 of [9], we may without loss of generality assume

$$\begin{aligned}\mu_i &= \sum_{1 \leq n \leq \infty} \mu_{i,n} \\ \mathcal{H}_i &= \bigoplus_{1 \leq n \leq \infty} L^2(\mu_{i,n}) \otimes \mathcal{K}_n, \\ \pi_i(f) &= \bigoplus_{1 \leq n \leq \infty} M_f^{(i,n)} \otimes \text{id}_{\mathcal{K}_n} \quad (f \in A_i),\end{aligned}$$

for the family of mutually singular measures $(\mu_{i,n})_{1 \leq n \leq \infty}$, where \mathcal{K}_n is a Hilbert space of dimension n and where $M_f^{(i,n)}$ is multiplication by f on $L^2(\mu_{i,n})$. Now the required formulas are easily proved. \square

Corollary 5.3. *Let $\pi : C(X) \rightarrow \mathcal{B}(\mathcal{H})$ be a unital $*$ -representation of an abelian C^* -algebra with measure class $[\mu]$ and multiplicity function m . Let \mathcal{V} be a Hilbert space with dimension k . Then the representation $\pi \otimes \text{id} : C(X) \rightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{V})$ given by $\pi \otimes \text{id}(a) = \pi(a) \otimes \text{id}_{\mathcal{V}}$ has measure class $[\mu]$ and multiplicity function $x \mapsto m(x)k$.*

Proposition 5.4. *Suppose that $\phi : C(X) \rightarrow C(Y)$ is a surjective $*$ -homomorphism of unital, abelian C^* -algebras. We identify Y with a closed subset of X and ϕ with the restriction of functions. Let $\pi : C(Y) \rightarrow \mathcal{B}(\mathcal{H})$ be a unital $*$ -representation with measure class $[\mu]$ and multiplicity function m . Then the $*$ -representation $\tilde{\pi} = \pi \circ \phi : C(X) \rightarrow \mathcal{B}(\mathcal{H})$ has measure class $[\tilde{\mu}]$ and multiplicity function \tilde{m} , where $\tilde{\mu}(E) = \mu(E \cap Y)$ and the restriction of \tilde{m} to Y is m .*

Below, \widehat{H} will denote the dual of a locally compact abelian group H . We recall the well known fact that \widehat{H} is compact when H is discrete.

Lemma 5.5. *Let H be a countable abelian group and let X be a set on which H acts transitively. Let π be the $*$ -representation of $C^*(H) \cong C(\widehat{H})$ on the Hilbert space $\ell^2(X)$ that*

results from this action. Let $K \subseteq H$ be the stabilizer subgroup of any element of X under the action of H , and let

$$K^\circ = \{\gamma \in \widehat{H} \mid \gamma(K) = \{1\}\}$$

be the annihilator of K , a closed subgroup of the compact group \widehat{H} . Then the measure class of π is supported on K° and is equal there to the class of Haar measure on K° . The multiplicity function of π is the constant function 1.

Proof. We identify X with the quotient group H/K equipped with the obvious action of H . Then K° is the dual group of the quotient group H/K , so via the Fourier transform yields $\ell^2(X) \cong L^2(K^\circ, \lambda)$, where λ is Haar measure on K° . This isomorphism intertwines the given representation of $C^*(H) = C(\widehat{H})$ on $\ell^2(X)$ with the representation σ of $C(\widehat{H})$ on $L^2(K^\circ, \lambda)$ given by

$$(\sigma(f)\xi)(\gamma) = f(\gamma)\xi(\gamma), \quad (f \in C(\widehat{H}), \xi \in L^2(K^\circ, \lambda), \gamma \in K^\circ).$$

□

A special case of the above lemma is the following one.

Lemma 5.6. *Let H be an abelian subgroup of a discrete I.C.C. group G and let $\pi : C^*(H) \otimes C^*(H) \rightarrow \mathcal{B}(\ell^2(G))$ be the left-right representation of the C^* -tensor product, given by*

$$\pi(\lambda_h \otimes \lambda_{h'})\delta_g = \delta_{hgh'} \quad (h, h' \in H, g \in G).$$

We identify $C^(H) \otimes C^*(H)$ with $C(\widehat{H} \times \widehat{H})$. Let $a \in G$ and let*

$$K_a = \{(h_1, h_2) \in H \times H \mid h_1 a h_2 = a\}.$$

Then the cyclic subrepresentation of π on $\ell^2(HaH)$ has measure class $[\mu]$, where μ is concentrated on the annihilator subgroup $(K_a)^\circ \subset \widehat{H} \times \widehat{H}$ and is equal to Haar measure there. The multiplicity function of π is constantly equal to 1.

The next lemma shows how to write the direct sum of direct integrals of Hilbert space as a direct integral.

Lemma 5.7. *Let X be a σ -compact, locally compact Hausdorff space and let μ and μ' be completions of σ -finite Borel measures on X . Let separable Hilbert spaces \mathcal{H} and \mathcal{H}' be*

direct integrals of $\{\mathcal{H}_p\}_{p \in X}$ and $\{\mathcal{H}'_p\}_{p \in X}$ over (X, μ) and (X, μ') , respectively. The Lebesgue decompositions yield

$$\begin{aligned} \mu &= \mu_0 + \mu_1 \quad \text{with } \mu_0 \perp \mu', \quad \mu_1 \ll \mu' \\ \mu' &= \mu'_0 + \mu'_1 \quad \text{with } \mu'_0 \perp \mu, \quad \mu'_1 \ll \mu. \end{aligned} \tag{5.1}$$

Let $X = X_0 \cup X_1 = X'_0 \cup X'_1$ be measurable partitions of X such that

$$\mu_0(X_1) = \mu'(X_0) = \mu'_0(X'_1) = \mu(X'_0) = 0. \tag{5.2}$$

Thus, μ_0 is concentrated on X_0 , μ_1 on X_1 , μ'_0 on X'_0 and μ'_1 on X'_1 . Let $\nu = \mu + \mu'$, namely

$$\nu(A) = \mu(A) + \mu'(A \cap X'_0). \tag{5.3}$$

For $p \in X$ consider the Hilbert spaces

$$\mathcal{K}_p = \begin{cases} 0, & p \in X_0 \cap X'_0 \\ \mathcal{H}'_p, & p \in X_1 \cap X'_0 \\ \mathcal{H}_p, & p \in X_0 \cap X'_1 \\ \mathcal{H}_p \oplus \mathcal{H}'_p, & p \in X_1 \cap X'_1. \end{cases}$$

Then $\mathcal{K} := \mathcal{H} \oplus \mathcal{H}'$ is the direct integral of $\{\mathcal{K}_p\}_{p \in X}$ over (X, ν) . Furthermore, if $a \in \mathcal{B}(\mathcal{H})$ and $a' \in \mathcal{B}(\mathcal{H}')$ are decomposable with respect to the direct integrals $\mathcal{H} = \int_X^\oplus \mathcal{H}_p d\mu(p)$ and $\mathcal{H}' = \int_X^\oplus \mathcal{H}'_p d\mu'(p)$, respectively, then $a \oplus a' \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}')$ is decomposable with respect to the direct integral

$$\mathcal{H} \oplus \mathcal{H}' = \int_X^\oplus \mathcal{K}_p d\nu(p), \tag{5.4}$$

with

$$(a \oplus a')_p = \begin{cases} 0, & p \in X_0 \cap X'_0 \\ a'_p, & p \in X_1 \cap X'_0 \\ a_p, & p \in X_0 \cap X'_1 \\ a_p \oplus a'_p, & p \in X_1 \cap X'_1. \end{cases} \tag{5.5}$$

Proof. Let f be the Radon–Nikodym derivative $d\mu'_1/d\mu$. We may without loss of generality assume $f > 0$ everywhere on $X_1 \cap X'_1$. Let $x \in \mathcal{H}$ and $x' \in \mathcal{H}'$ and set $\tilde{x} = x \oplus x' \in \mathcal{K}$. For

$p \in X$, we set

$$\mathcal{K}_p \ni \tilde{x}(p) = \begin{cases} 0, & p \in X_0 \cap X'_0 \\ x'(p), & p \in X_1 \cap X'_0 \\ x(p), & p \in X_0 \cap X'_1 \\ x(p) \oplus f(p)^{1/2}x'(p), & p \in X_1 \cap X'_1 \end{cases}$$

A straightforward calculation shows

$$\int_X \|\tilde{x}(p)\|^2 d\nu(p) = \|x\|^2 + \|x'\|^2 = \|\tilde{x}\|^2.$$

Suppose that for all $p \in X$, $\tilde{u}(p) \in \mathcal{K}_p$ is such that for all $\tilde{x} \in \mathcal{K}$, the function $p \mapsto \langle \tilde{u}(p), \tilde{x}(p) \rangle$ is ν -integrable, and let us show there is $\tilde{y} \in \mathcal{K}$ such that $\tilde{y}(p) = \tilde{u}(p)$ for ν -a.e. $p \in X$. For every $p \in X_1 \cap X'_1$, let $Q_p : \mathcal{K}_p \rightarrow \mathcal{H}_p$ be the orthogonal projection onto the first direct summand; we denote by $I - Q_p : \mathcal{K}_p \rightarrow \mathcal{H}'_p$ the orthogonal projection onto the second direct summand. Set

$$\mathcal{H}_p \ni u(p) = \begin{cases} 0, & p \in X'_0 \\ \tilde{u}(p), & p \in X_0 \cap X'_1 \\ Q_p \tilde{u}(p), & p \in X_1 \cap X'_1. \end{cases}$$

Let $x \in \mathcal{H}$ and let $\tilde{x} = x \oplus 0 \in \mathcal{K}$. Then for all $p \in X_1 \cap X'_1$ we have

$$\langle \tilde{u}(p), \tilde{x}(p) \rangle = \langle \tilde{u}(p), Q_p \tilde{x}(p) \rangle = \langle u(p), x(p) \rangle,$$

while if $p \in X_0 \cap X'_1$, then also $\langle \tilde{u}(p), \tilde{x}(p) \rangle = \langle u(p), x(p) \rangle$. Since the restrictions of μ and ν to X'_1 agree and since $\mu(X'_0) = 0$, it follows that the map $p \mapsto \langle u(p), x(p) \rangle$ is μ -integrable. Therefore, there is $y \in \mathcal{H}$ such that $y(p) = u(p)$ for μ -a.e. $p \in X$.

Now let

$$\mathcal{H}'_p \ni u'(p) = \begin{cases} 0, & p \in X_0 \\ \tilde{u}(p), & p \in X_1 \cap X'_0 \\ f(p)^{-1/2}(I - Q_p)\tilde{u}(p), & p \in X_1 \cap X'_1. \end{cases}$$

Let $x' \in \mathcal{H}'$ and let $\tilde{x}' = 0 \oplus x' \in \mathcal{K}$, so that

$$\tilde{x}'(p) = \begin{cases} 0, & p \in X_0 \\ x'(p), & p \in X_1 \cap X'_0 \\ 0 \oplus f(p)^{1/2}x'(p), & p \in X_1 \cap X'_1. \end{cases}$$

Then for all $p \in X_1 \cap X'_1$ we have

$$\langle \tilde{u}(p), \tilde{x}'(p) \rangle = f(p)^{1/2} \langle (I - Q_p)\tilde{u}(p), x'(p) \rangle = f(p) \langle u'(p), x'(p) \rangle,$$

while for $p \in X_1 \cap X'_0$ we have $\langle \tilde{u}(p), \tilde{x}'(p) \rangle = \langle u'(p), x'(p) \rangle$. Thus,

$$\begin{aligned} \int_X |\langle u'(p), x'(p) \rangle| d\mu'(p) &= \int_{X_1 \cap X'_1} |\langle u'(p), x'(p) \rangle| d\mu'(p) + \int_{X_1 \cap X'_0} |\langle u'(p), x'(p) \rangle| d\mu'(p) \\ &= \int_{X_1 \cap X'_1} |\langle u'(p), x'(p) \rangle| f(p) d\mu(p) + \int_{X_1 \cap X'_0} |\langle u'(p), x'(p) \rangle| d\mu'(p) \\ &= \int_{X_1 \cap X'_1} |\langle \tilde{u}(p), \tilde{x}'(p) \rangle| d\mu(p) + \int_{X_1 \cap X'_0} |\langle \tilde{u}(p), \tilde{x}'(p) \rangle| d\mu'(p) \\ &= \int_X |\langle \tilde{u}(p), \tilde{x}'(p) \rangle| d\nu(p) < \infty. \end{aligned}$$

Therefore, the function $p \mapsto \langle u'(p), x'(p) \rangle$ is μ' -integrable and there is $y' \in \mathcal{H}'$ such that $u'(p) = y'(p)$ for μ' -a.e. $p \in X$. Let $\tilde{y} = y \oplus y' \in \mathcal{K}$. Then $\tilde{y}(p) = \tilde{u}(p)$ for ν -a.e. $p \in X$. This proves the direct integral formula (5.4).

The decomposability of $a \oplus a'$ with respect to (5.4) and the validity of (5.5) are now clear. \square

A particular case of the above lemma is the following proposition, which we will frequently use in computations.

Proposition 5.8. *Let $\pi : C(X) \rightarrow \mathcal{B}(\mathcal{H})$ and $\pi' : C(X) \rightarrow \mathcal{B}(\mathcal{H}')$ be unital $*$ -representations whose measure classes are $[\mu]$ and $[\mu']$ and whose multiplicity functions are m and m' , respectively. Consider the Lebesgue decomposition as in (5.1) and the partitions so that we have (5.2). Then the representation $\pi \oplus \pi' : C(X) \rightarrow \mathcal{B}(\mathcal{H} \oplus \mathcal{H}')$ has measure class $[\nu]$ with $\nu = \mu + \mu'_0$ given by (5.3), and has multiplicity function \tilde{m} given by*

$$\tilde{m}(x) = \begin{cases} m'(x), & x \in X_1 \cap X'_0 \\ m(x), & x \in X_0 \cap X'_1 \\ m(x) + m'(x), & x \in X_1 \cap X'_1. \end{cases}$$

Before we state and prove the next result, we need to make some remarks which will justify the calculations below. Consider an inclusion $M \subseteq N$ of finite von Neumann algebras where N has a faithful finite normal trace τ . Then $L^1(N)$ denotes the completion of N when equipped with the norm $\|x\|_1 = \tau(|x|)$, $x \in N$. The polar decomposition shows that

$$\|x\|_1 = \sup\{|\tau(xy)| : y \in N, \|y\| \leq 1\}, \quad (5.6)$$

from which it follows that $|\tau(x)| \leq \|x\|_1$. Thus τ has a bounded extension, also denoted τ , to a linear functional on $L^1(N)$. The N -bimodule properties of N extend by continuity to $L^1(N)$ and the relation $\tau(x\zeta) = \tau(\zeta x)$ for $x \in N$, $\zeta \in L^1(N)$, follows by boundedness of τ on $L^1(N)$. Similarly, τ defines a continuous linear functional on $L^2(N)$ by $\tau(\zeta) = \langle \zeta, 1 \rangle$, for $\zeta \in L^2(N)$. If \mathbb{E} is the unique trace preserving conditional expectation of N onto M then it is also a contraction when viewed as a map of $L^2(N)$ onto $L^2(M)$. If $x \in N$, then

$$\begin{aligned} \|\mathbb{E}(x)\|_1 &= \sup\{|\tau(\mathbb{E}(x)y)| : y \in M, \|y\| \leq 1\} = \sup\{|\tau(\mathbb{E}(xy))| : y \in M, \|y\| \leq 1\} \\ &= \sup\{|\tau(xy)| : y \in M, \|y\| \leq 1\} \leq \|x\|_1, \end{aligned} \quad (5.7)$$

using the module properties of \mathbb{E} . Thus \mathbb{E} has a bounded extension to a contraction of $L^1(N)$ to $L^1(M)$. The module property $\mathbb{E}(m_1\zeta m_2) = m_1\mathbb{E}(\zeta)m_2$, for $\zeta \in L^1(N)$ and $m_1, m_2 \in M$, follows by $\|\cdot\|_1$ -continuity of \mathbb{E} . The bilinear map $\Psi : N \times N \rightarrow N$, defined by $\Psi(x, y) = xy$, satisfies

$$\begin{aligned} \|\Psi(x, y)\|_1 &= \sup\{|\tau(xyz)| : z \in N, \|z\| \leq 1\} \\ &= \sup\{|\langle zx, y^* \rangle| : z \in N, \|z\| \leq 1\} \\ &\leq \|x\|_2 \|y\|_2, \end{aligned} \quad (5.8)$$

and so Ψ extends to a jointly continuous map, also denoted Ψ , from $L^2(N) \times L^2(N)$ to $L^1(N)$. Since Ψ is the product map at the level of N , we will write $\zeta\eta$ for $\Psi(\zeta, \eta) \in L^1(N)$ when $\zeta, \eta \in L^2(N)$. Moreover, the adjoint on N extends to an isometric conjugate linear map $\zeta \mapsto \zeta^*$ on both $L^1(N)$ and $L^2(N)$, agreeing with J in the latter case. Any relation that holds on N will extend by continuity to the appropriate $L^p(N)$, where $p = 1$ or 2 . For example, we have

$$\langle \zeta, \eta \rangle = \tau(\eta^*\zeta) = \tau(\zeta\eta^*), \quad \zeta, \eta \in L^2(N). \quad (5.9)$$

We will apply these remarks with $N = \overline{B}$ and $M = \overline{A}$ below.

Proposition 5.9. *Let $A = C(X)$ be embedded as a unital C^* -subalgebra of a separable C^* -algebra B and let τ be a faithful, tracial state on B . Let ν be the Borel measure on X such that*

$$\tau(a) = \int_X a(x) d\nu(x), \quad (a \in A).$$

Let λ denote the representation of A on $L^2(B, \tau)$ by left multiplication. Then the measure class of λ is $[\nu]$.

Proof. Let $[\sigma]$ denote the measure class of λ . Let λ_A denote the left action of A on $L^2(A, \tau)$. Now λ_A is a direct summand of λ and the measure class of λ_A is easily seen to be $[\nu]$. By Proposition 5.8, $\nu \ll \sigma$. In order to show $\sigma \ll \nu$, it will suffice to show that whenever π is a cyclic subrepresentation of λ , then the measure class of π is absolutely continuous with respect to ν . Indeed, if $\sigma \not\ll \nu$, then letting $X_0 \subset X$ be a set of positive measure such that the restriction of σ to X_0 is singular to ν , there is $\zeta \in L^2(B, \tau)$ such that $\int_{X_0} \|\zeta(x)\|^2 d\nu(x) > 0$.

Let $\zeta \in L^2(B, \tau)$. Let \overline{B} denote the s.o.-closure of B acting via the Gelfand–Naimark–Segal representation on $L^2(B, \tau)$, and let \overline{A} denote the s.o.-closure of A acting via λ on $L^2(B, \tau)$. Let $\mathbb{E} : \overline{B} \rightarrow \overline{A}$ denote the τ -preserving conditional expectation and also the extension

$$\mathbb{E} : L^1(\overline{B}, \tau) \rightarrow L^1(\overline{A}, \tau),$$

which exists by the preceding remarks. The measure class of π is $[\rho]$, where for all $a \in C(X)$, $\langle a\zeta, \zeta \rangle = \int_X a(x) d\rho(x)$. Assume $a \geq 0$. Then, using the discussion before this proposition, we have

$$\langle a\zeta, \zeta \rangle = \tau(\zeta^* a \zeta) = \tau(a \zeta \zeta^*) = \tau(\mathbb{E}(a \zeta \zeta^*)) = \tau(a \mathbb{E}(\zeta \zeta^*)).$$

Since $0 \leq \mathbb{E}(\zeta \zeta^*) \in L^1(A, \tau) \cong L^1(\nu)$, we have

$$\langle a\zeta, \zeta \rangle = \int_X a(x) f(x) d\nu(x)$$

for some $f \in L^1(\nu)$, $f \geq 0$. Consequently, $d\rho = f d\nu$ and ρ is absolutely continuous with respect to ν . \square

Proposition 5.10. *For $i = 1, 2$, let B_i be a unital, separable C^* -algebra having faithful, tracial states τ_i and with $\dim(B_i) \geq 2$. Let $A = C(X)$ be unitaly embedded as a C^* -subalgebra of B_1 . Let ν be the measure on X such that*

$$\tau_1(a) = \int_X a(x) d\nu(x), \quad (a \in C(X)).$$

Let $\lambda_1, \rho_1 : A \rightarrow \mathcal{B}(L^2(B_1))$ be the left and right actions of A on $L^2(B_1) := L^2(B_1, \tau_1)$. Let $\pi_1 : A \otimes A \rightarrow \mathcal{B}(L^2(B_1))$ be the $$ -representation of the C^* -tensor product $A \otimes A$ given by $\pi_1(a_1 \otimes a_2) = \lambda_1(a_1) \rho_1(a_2)$. Let $(B, \tau) = (B_1, \tau_1) * (B_2, \tau_2)$ be the reduced free product of C^* -algebras. Let $\lambda, \rho : A \rightarrow \mathcal{B}(L^2(B))$ be the left and right actions of A on $L^2(B) := L^2(B, \tau)$ and let $\pi : A \otimes A \rightarrow \mathcal{B}(L^2(B))$ be given by $\pi(a_1 \otimes a_2) = \lambda(a_1) \rho(a_2)$. Then $L^2(B_1) \subseteq L^2(B)$ is*

a reducing subspace for $\pi(A \otimes A)$. Let $\tilde{\pi}$ be the representation of $A \otimes A$ on $L^2(B) \ominus L^2(B_1)$ obtained from π by restriction, so that

$$\pi = \pi_1 \oplus \tilde{\pi}.$$

We identify $A \otimes A$ with $C(X \times X)$. Then the measure class of $\tilde{\pi}$ is $[\nu \otimes \nu]$ and the multiplicity function of $\tilde{\pi}$ is constantly ∞ .

Proof. This proof is at bottom quite similar to the proof of Lemma 3.1, and just as in that case, we begin by decomposing the free product Hilbert space. From the construction of the reduced free product C^* -algebra, we have

$$L^2(B) = \mathbb{C}\hat{1} \oplus \bigoplus_{\substack{n \in \mathbb{N} \\ i_1, \dots, i_n \in \{1, 2\} \\ i_j \neq i_{j+1}}} \mathcal{H}_{i_1}^\circ \otimes \dots \otimes \mathcal{H}_{i_n}^\circ,$$

where $\mathcal{H}_i^\circ = L^2(B_i) \ominus \mathbb{C}\hat{1}$. Therefore,

$$L^2(B) \ominus L^2(B_1) = \bigoplus_{k=0}^{\infty} \left(\begin{aligned} &\mathcal{H}_2^\circ \otimes (\mathcal{H}_1^\circ \otimes \mathcal{H}_2^\circ)^{\otimes k} \oplus \mathcal{H}_2^\circ \otimes (\mathcal{H}_1^\circ \otimes \mathcal{H}_2^\circ)^{\otimes k} \otimes \mathcal{H}_1^\circ \\ &\oplus \mathcal{H}_1^\circ \otimes \mathcal{H}_2^\circ \otimes (\mathcal{H}_1^\circ \otimes \mathcal{H}_2^\circ)^{\otimes k} \\ &\oplus \mathcal{H}_1^\circ \otimes \mathcal{H}_2^\circ \otimes (\mathcal{H}_1^\circ \otimes \mathcal{H}_2^\circ)^{\otimes k} \otimes \mathcal{H}_1^\circ \end{aligned} \right). \quad (5.10)$$

Writing $L^2(B_1) = \mathcal{H}_1^\circ \oplus \mathbb{C}$, for each k we make the obvious identification of the direct sum of the four direct summands on the right hand side of (5.10) with

$$L^2(B_1) \otimes \mathcal{H}_2^\circ \otimes (\mathcal{H}_1^\circ \otimes \mathcal{H}_2^\circ)^{\otimes k} \otimes L^2(B_1).$$

Let

$$\mathcal{K} = \bigoplus_{k=0}^{\infty} \mathcal{H}_2^\circ \otimes (\mathcal{H}_1^\circ \otimes \mathcal{H}_2^\circ)^{\otimes k}.$$

Then we have the unitary

$$U : L^2(B) \ominus L^2(B_1) \rightarrow L^2(B_1) \otimes \mathcal{K} \otimes L^2(B_1)$$

that gives

$$U(\tilde{\pi}(a_1 \otimes a_2))U^* = \lambda_1(a_1) \otimes \text{id}_{\mathcal{K}} \otimes \rho_1(a_2), \quad (a_1, a_2 \in A).$$

By Corollary 5.3, it will suffice to show that the representation

$$\lambda_1 \otimes \rho_1 : A \otimes A \rightarrow \mathcal{B}(L^2(B_1) \otimes L^2(B_1))$$

has measure class $[\nu \otimes \nu]$. By Proposition 5.9, λ_1 has measure class ν . Since ρ_1 is unitarily equivalent to λ_1 it also has measure class ν , so by Proposition 5.2, $\lambda_1 \otimes \rho_1$ has measure class $[\nu \otimes \nu]$. □

6 Computations of invariants

6.1. Neshveyev and Størmer [10] considered the conjugacy invariant for a masa A in a II_1 factor M derived from writing a direct integral decomposition of the left–right action,

$$a \otimes b \mapsto aJb^*J, \tag{6.1}$$

where J is the anti-unitary conjugation on $L^2(M)$ given by $J\hat{a} = (a^*)^\wedge$, of the C^* -tensor product $A \otimes A$ on $L^2(M)$. We will now review this invariant. Choosing a separable and weakly dense C^* -subalgebra $\mathfrak{A} = C(Y)$ of A , we may write $A = L^\infty(Y, \nu)$ for a compact Hausdorff space Y and a completion of a Borel measure ν on Y . Let $\pi : \mathfrak{A} \otimes \mathfrak{A} \rightarrow \mathcal{B}(L^2(M))$ denote the restriction of the left–right action (6.1) to the C^* -tensor product $\mathfrak{A} \otimes \mathfrak{A}$, which we identify with $C(Y \times Y)$ in the usual way. Let $[\eta]$ be the measure class and m the multiplicity function of π . We will always take η to be a finite measure. Then $[\eta]$ is invariant under the flip $(x, y) \mapsto (y, x)$ of $Y \times Y$ and the projection of $[\eta]$ onto the first and second coordinates is $[\nu]$. Neshveyev and Størmer [10] observed that $(Y, [\eta], m)$ is a conjugacy invariant of $A \subseteq M$, in the sense that if $A \subseteq M$ and $B \subseteq N$ are masas and if there is an isomorphism $M \rightarrow N$ taking A onto B , then (for any choice of separable, weakly dense C^* -subalgebras of A and B), there is a transformation of measure spaces, $F : (Y_A, [\nu_A]) \rightarrow (Y_B, [\nu_B])$ such that $(F \times F)_*([\eta_A]) = [\eta_B]$ and $m_B \circ (F \times F) = m_A$ (η_A -almost everywhere). We will refer to the equivalence class of $(Y, [\eta], m)$ under such transformations as the *measure–multiplicity* invariant of the masa $A \subseteq M$. In fact, Neshveyev and Størmer showed more, namely that the equivalence class of $(Y, [\eta], m)$ is a complete invariant for the pair (A, J) acting on $L^2(M)$. They also showed that the Pukánszky invariant of $A \subseteq M$ is precisely the set of essential values of the multiplicity function m taken on the complement of the diagonal $\Delta(Y)$ in $Y \times Y$.

Example 6.2. Let $n \in \mathbb{N} \cup \{\infty\}$ and consider the matrix groups

$$G_n = \left\{ \begin{pmatrix} f & x \\ 0 & 1 \end{pmatrix} \mid f \in P_n, x \in \mathbb{Q} \right\}, \quad H_n = \left\{ \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \mid f \in P_n \right\} \subseteq G_n,$$

where we have the subgroups of the multiplicative group of nonzero rational numbers

$$P_\infty = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}^*, p, q \text{ odd} \right\}$$

and, for n finite,

$$P_n = \{f2^{kn} \mid f \in P_\infty, k \in \mathbb{Z}\}.$$

Then, as shown in [19, Ex. 5.1], $L(G_n)$ is the hyperfinite II_1 factor and $L(H_n)$ is a strongly singular masa in $L(G_n)$ with Pukánszky invariant $\{n\}$. Moreover, the measure–multiplicity invariant of $L(H_n) \subseteq L(G_n)$ is the equivalence class of $(\widehat{H}_n, [\mu_n], m)$, where μ_n is the sum of Haar measure on $\widehat{H}_n \times \widehat{H}_n$ and Haar measure on the diagonal subgroup $\Delta(\widehat{H}_n)$, and where the multiplicity function m takes value 1 on $\Delta(\widehat{H}_n)$ and n on its complement.

Proof. This follows from the double decomposition of G_n as double cosets over H_n (see [19, Ex. 5.1]), Lemma 5.6 and Proposition 5.8. \square

Example 6.3. Let $n \in \mathbb{N}$. With $H_n \subseteq G_n$ and $H_\infty \subseteq G_\infty$ as in Example 6.2, $L(H_n \times H_\infty)$ is a strongly singular masa in $L(G_n \times G_\infty)$ whose measure–multiplicity invariant is the equivalence class of

$$(\widehat{H}_n \times \widehat{H}_\infty, [\eta], m),$$

where η is the sum of

- (i) Haar measure on $\widehat{H}_n \times \widehat{H}_\infty \times \widehat{H}_n \times \widehat{H}_\infty$
- (ii) Haar measure on the subgroup

$$D_n = \{(\alpha, \beta_1, \alpha, \beta_2) \mid \alpha \in \widehat{H}_n, \beta_1, \beta_2 \in \widehat{H}_\infty\} \quad (6.2)$$

- (iii) Haar measure on the subgroup

$$D_\infty = \{(\alpha_1, \beta, \alpha_2, \beta) \mid \alpha_1, \alpha_2 \in \widehat{H}_n, \beta \in \widehat{H}_\infty\}$$

- (iv) Haar measure on the diagonal subgroup $\Delta(\widehat{H}_n \times \widehat{H}_\infty)$

and where the multiplicity function m is given by

$$m(\gamma) = \begin{cases} 1, & \gamma \in \Delta(\widehat{H}_n \times \widehat{H}_\infty) \\ n, & \gamma \in D_n \setminus \Delta(\widehat{H}_n \times \widehat{H}_\infty) \\ \infty, & \text{else.} \end{cases}$$

Proof. This follows from Example 6.2 and Proposition 5.2. \square

Example 6.4. Let $n \in \mathbb{N}$ and let Γ be any nontrivial finite or countably infinite group. Let

$$H_n \times H_\infty \subseteq G_n \times G_\infty \subset (G_n \times G_\infty) * \Gamma,$$

with $H_n \times H_\infty \subseteq G_n \times G_\infty$ as in Example 6.3 above. Then $L(H_n \times H_\infty)$ is a singular masa in $L((G_n \times G_\infty) * \Gamma)$, whose measure–multiplicity invariant is the equivalence class of

$$(\widehat{H}_n \times \widehat{H}_\infty, [\eta], m),$$

where η and m are exactly as in Example 6.3.

Proof. This follows from Example 6.3 and applications of Propositions 5.10 and 5.8. \square

If Γ is taken to be infinite amenable, then $L((G_n \times G_\infty) * \Gamma)$ is isomorphic to the free group factor $L(\mathbb{F}_2)$, by [4] (see also [5]).

Example 6.5. Let

$$A = L(H_\infty) \oplus L(H_n \times H_\infty) \subseteq N = L(G_\infty) \oplus L(G_n \times G_\infty).$$

Consider the normal, faithful, tracial state

$$\tau_N(x_1 \oplus x_2) = \frac{1}{2}(\tau_{G_\infty}(x_1) + \tau_{G_n \times G_\infty}(x_2))$$

on N . Let Q be any diffuse von Neumann algebra with separable predual and a normal faithful state τ_Q and let

$$(M, \tau_M) = (N, \tau_N) * (Q, \tau_Q)$$

be the free product. By Theorem 2.3, M is a II_1 factor and $A \subseteq M$ is a strongly singular masa. The measure–multiplicity invariant of $A \subseteq M$ is the equivalence class of $(X, [\sigma], m)$, where X is the disconnected sum of $X_1 = \widehat{H}_\infty$ and $X_2 = \widehat{H}_n \times \widehat{H}_\infty$, where σ is the sum of the measures

- (i) μ_∞ , as described in Example 6.2, supported on $X_1 \times X_1 \subseteq X \times X$
- (ii) η , as described in Example 6.3, supported on $X_2 \times X_2 \subseteq X \times X$
- (iii) $\nu \otimes \nu$ on $X \times X$, where the measure ν on X is the sum of Haar measure on the (dual) group X_1 and Haar measure on X_2 ,

and where m takes the value 1 on the diagonal $\Delta(X)$, the value n on $D_n \subseteq X_2 \times X_2 \subseteq X \times X$, with D_n as given in equation (6.2), and is equal to ∞ elsewhere.

Proof. Let $\mathfrak{A} = C^*(H_\infty) \oplus C^*(H_n \times H_\infty) \subseteq A$. We find the measure class and multiplicity function of the left–right representation of the C^* –algebra $\mathfrak{A} \otimes \mathfrak{A}$ on $L^2(N, \tau_N)$, by using Propositions 5.2 and 5.4. Then we find the measure class of the left–right representation of $\mathfrak{A} \otimes \mathfrak{A}$ on $L^2(M, \tau_M)$ by using Propositions 5.10 and 5.8. \square

If Q is taken to be the hyperfinite II_1 factor, then M in Example 6.5 is isomorphic to the free group factor $L(\mathbb{F}_2)$, by [5]. Thus, Examples 6.4 and 6.5 provide two constructions of masas in the free group factor $L(\mathbb{F}_2)$, both having Pukánszky invariant $\{n, \infty\}$. We will distinguish these two masas using the measure–multiplicity invariant, or actually a formally weaker invariant derived from it.

6.6. Let $(Y, [\eta], m)$ arise as in the definition of the measure–multiplicity invariant of a masa $A \subseteq M$. As already mentioned, m takes the value 1 on the diagonal $\Delta(Y)$, and one easily sees that the restriction of η to $\Delta(Y)$ is equivalent to the measure ν as in 6.1, when $\Delta(Y)$ is identified with Y in the obvious way. Therefore, the restrictions of m and η to the complement of $\Delta(Y)$ contain the same information as $(Y, [\eta], m)$.

Lemma 6.7. *Let Q be a von Neumann algebra having normal faithful traces τ_1 and τ_2 and let $A \subseteq Q$ be a von Neumann subalgebra. Let $E_i : Q \rightarrow A$ denote the τ_i –preserving conditional expectation onto A , ($i = 1, 2$). If $x \in Q$ and $x \geq 0$, then the support projections of $E_1(x)$ and $E_2(x)$ agree.*

Proof. Let $p_i \in A$ be such that the support projection of $E_i(x)$ is $1 - p_i$. Then $0 = p_1 E_1(x) p_1 = E_1(p_1 x p_1)$, so $p_1 x p_1 = 0$. But then $p_1 E_2(x) p_1 = E_2(p_1 x p_1) = 0$, so $p_1 \leq p_2$. By symmetry, $p_2 \leq p_1$. \square

6.8. Let M be a II_1 factor and $A \subseteq M$ a masa. Choose a triple $(Y, [\eta], m)$ belonging to the measure–multiplicity invariant of $A \subseteq M$, with η finite, as considered in 6.1. Recall that $A = L^\infty(Y, \nu)$ is embedded in $L^\infty(Y \times Y, \eta)$ as functions constant in the second coordinate. Let $E : L^\infty(Y \times Y, \eta) \rightarrow A$ denote the conditional expectation that preserves integration with respect to η . Given $n \in \mathbb{N} \cup \{\infty\}$, let $P_n \in L^\infty(Y \times Y, \eta)$ be the characteristic function of the set where the multiplicity function m takes the value n off of the diagonal $\Delta(Y)$, and let $q_n(A) = q_n(A, M)$ be the support projection of the conditional expectation $E(P_n)$. By

Lemma 6.7, $q_n(A)$ is independent of the choice of η in the measure class $[\eta]$. Moreover, if $F : (Y_A, [\nu_A]) \rightarrow (Y_B, [\nu_B])$ is the transformation of measure spaces considered in 6.1, then we have $q_n(A, M) = q_n(B, N) \circ F$ for the corresponding support projections. Therefore, if $A \subset M$ and $B \subset N$ are masas that are conjugate by an isomorphism from M to N , then it induces an isomorphism from A to B that sends $q_n(A)$ to $q_n(B)$ for all n .

Fix $n \in \mathbb{N}$. In Example 6.4, take $\Gamma = \mathbb{Z}$, so that $M = L((G_n \times G_\infty) * \Gamma) = L(\mathbb{F}_2)$ and let $A_{6.4} = L(H_n \times H_\infty)$ be the masa of $L(\mathbb{F}_2)$ obtained there. In Example 6.5, take Q to be the hyperfinite II_1 factor so that $M = L(\mathbb{F}_2)$ and let $A_{6.5}$ be the masa of $L(\mathbb{F}_2)$ obtained there.

Theorem 6.9. *The masas $A_{6.4}$ and $A_{6.5}$ in $L(\mathbb{F}_2)$ both have Pukánszky invariant $\{n, \infty\}$, but are non-conjugate.*

Proof. The values of the Pukánszky invariant can be read off from the measure–multiplicity invariants, which were computed in Examples 6.4 and 6.5. The derived invariant q_n from 6.8 above in these cases becomes $q_n(A_{6.4}) = 1$ while $q_n(A_{6.5}) = 1_{X_2}$, the characteristic function of $X_2 \subseteq X$, which is not the identity of $A_{6.5}$. \square

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