

VANISHING OF SECOND COHOMOLOGY FOR TENSOR PRODUCTS OF TYPE II_1 VON NEUMANN ALGEBRAS

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Abstract

We show that the second cohomology group $H^2(M\bar{\otimes}N, M\bar{\otimes}N)$ is always zero for arbitrary type II_1 von Neumann algebras M and N .

Key Words: von Neumann algebra, Hochschild cohomology, tensor product

AMS Classification: 46L10, 46L06

1 Introduction

The theory of bounded Hochschild cohomology for von Neumann algebras was initiated by Johnson, Kadison and Ringrose in a series of papers [13, 16, 17], which laid the foundation for subsequent developments. These were a natural outgrowth of the theorem of Kadison [15] and Sakai [27] which established that every derivation $\delta: M \rightarrow M$ on a von Neumann algebra M is inner; $H^1(M, M) = 0$ in cohomological terminology. While cohomology groups can be defined for general M -bimodules (see Section 2 for definitions), this derivation result ensured special significance for M as a bimodule over itself. When M is represented on a Hilbert space H , then $B(H)$ is also an important M -bimodule, but here the known results are less definitive. For example, it is not known whether every derivation $\delta: M \rightarrow B(H)$ is inner, a problem known to be equivalent to the similarity problem [18]. In [17], it was shown the $H^n(M, M) = 0$ for all $n \geq 1$ when M is an injective von Neumann algebra, a class which includes the type I algebras. They conjectured that this should be true for all von Neumann algebras, now known as the Kadison–Ringrose conjecture. The purpose of this paper is to verify this for the second cohomology of tensor products of type II_1 von Neumann algebras.

The study of this conjecture reduces to four cases in parallel with the type decomposition of Murray and von Neumann. Three of these are solved. As noted above, the type I case was determined at the outset of the theory, while the types II_∞ and III cases were solved by Christensen and Sinclair [9] after they had developed the theory of completely bounded multilinear maps [8] and applied it, jointly with Effros [5], to cohomology into $B(H)$. They showed that $H_{cb}^n(M, M) = 0$, $n \geq 1$, for all von Neumann algebras, where the subscript indicates that all relevant multilinear maps are required to be completely bounded. Since then, all progress has hinged on reducing a given cocycle to one which is completely bounded and then quoting their result. In this paper we follow a different path, although complete boundedness will play an important role.

The one remaining open case is that of type II_1 von Neumann algebras. There are several positive results for special classes: the McDuff factors [9], those factors with Cartan subalgebras [22, 6, 29, 1], and those with property Γ [9, 4, 7]. While tensor products form a large class of type II_1 von Neumann algebras, the prime factors fall outside our scope. The best known examples are the free group factors, shown to be prime by Ge [12], and these do not lie in any of the classes already mentioned, so nothing is known about their cohomology.

Section 2 gives a brief review of definitions and some results that we will need subsequently. The heart of the paper is Section 3 where we prove that $H^2(M \overline{\otimes} N, M \overline{\otimes} N) = 0$ for separable type II_1 von Neumann algebras. This restriction is made in order to be able to choose certain special hyperfinite subalgebras that are only available in this setting. The proof proceeds through a sequence of lemmas which reduce a given cocycle to one with extra features, after which we can exhibit it as a coboundary. In this process, particular use is made of complete boundedness and the basic construction, [14], for containments of type II_1 algebras. Section 4 handles the general case by deducing it from the separable situation using several known techniques to be found in [28, §6.5]. However, Lemma 4.1 appears to be new.

For general background on cohomology we refer to the survey article [26] and the monograph [28]. The theory of complete boundedness is covered in several books [11, 19, 20], while [31] contains an introduction to the basic construction algebra.

2 Preliminaries and notation

Since this paper is only concerned with second cohomology, we will only give the definitions at this level, referring to [28] for the general case. Let A be a C^* -algebra with an A -bimodule V . For $n = 1, 2, 3$, $\mathcal{L}^n(A, V)$ denotes the space of bounded n -linear maps from $A \times \cdots \times A$ into V , while $\mathcal{L}^0(A, V)$ is defined to be V . For $v \in V$ and $\phi_n \in \mathcal{L}^n(A, V)$, $n = 1, 2$, the coboundary map $\partial: \mathcal{L}^n(A, V) \rightarrow \mathcal{L}^{n+1}(A, V)$ is defined as follows:

$$\partial v(a) = va - av, \quad a \in A, \quad (2.1)$$

$$\partial\phi_1(a_1, a_2) = a_1\phi_1(a_2) - \phi_1(a_1a_2) + \phi_1(a_1)a_2, \quad a_i \in A, \quad (2.2)$$

$$\partial\phi_2(a_1, a_2, a_3) = a_1\phi_2(a_2, a_3) - \phi_2(a_1a_2, a_3) + \phi_2(a_1, a_2a_3) - \phi_2(a_1, a_2)a_3, \quad a_i \in A, \quad (2.3)$$

An algebraic calculation gives $\partial\partial = 0$. Cocycles are those maps ϕ for which $\partial\phi = 0$, while coboundaries are maps of the form $\partial\xi$. The n^{th} cohomology group $H^n(A, V)$ is then the quotient space of n -cocycles modulo n -coboundaries. In particular, $H^1(A, V)$ is the space of derivations modulo inner derivations. Since we plan to prove that certain second cohomology groups are zero, this amounts to showing that each 2-cocycle is a 2-coboundary. There is a considerable theory of cohomology, much of which is summarized in [28]. We use this monograph as our standard reference, but include some results below which are not to be found there. The first two of these concern complete boundedness, the second of which is a small extension of the factor case of [9] (the results of this paper appear in [28]).

Lemma 2.1. *Let $M \subseteq B(H)$ and $S \subseteq B(K)$ be type II₁ von Neumann algebras with S hyperfinite. If $\phi: M \overline{\otimes} S \rightarrow B(H \otimes_2 K)$ is bounded, normal and $(I \otimes S)$ -modular, then ϕ is completely bounded.*

Proof. We regard M and S as both represented on $B(H \otimes_2 K)$. The $(I \otimes S)$ -modularity implies that the restriction ψ of ϕ to $M \otimes I$ maps into S' , so ϕ maps the minimal tensor product $M \otimes_{\min} S$ into $C^*(S', S)$. Now S contains arbitrarily large matrix subfactors \mathbb{M}_n , $n \geq 1$, and $\phi|_{M \otimes \mathbb{M}_n}$ can be regarded as the composition of $\psi \otimes \text{id}_n: M \otimes \mathbb{M}_n \rightarrow S' \otimes \mathbb{M}_n$ with a $*$ -isomorphism $\pi_n: S' \otimes \mathbb{M}_n \rightarrow C^*(S', \mathbb{M}_n)$. The uniform bound $\|\phi\|$ on each of these restrictions then shows that ψ is completely bounded. Hyperfiniteness of S gives a $*$ -homomorphism $\rho: S' \otimes_{\min} S \rightarrow C^*(S', S)$ defined on elementary tensors by $s' \otimes s \mapsto s's$, [10, Proposition 4.5], so $\phi|_{M \otimes_{\min} S}$ is the composition $\rho \circ (\psi \otimes \text{id}_S)$, showing complete boundedness on $M \otimes_{\min} S$. The same conclusion on $M \overline{\otimes} S$ now follows from normality of ϕ and the Kaplansky density theorem applied to $M \otimes_{\min} S \otimes \mathbb{M}_n \subseteq M \overline{\otimes} S \otimes \mathbb{M}_n$. \square

The proof that we have given of this result relies on normality and hyperfiniteness, and it would be interesting to know if it holds without these restrictions. The next result is known for factors [9], but does not appear to be in the literature in the generality that we require.

The theory of multimodular maps plays a significant role in the study of cohomology, so we recall the definition for linear and bilinear maps. Let $A \subseteq B$ be C^* -algebras. A map $\psi: B \rightarrow B$ is A -modular if

$$a_1\psi(b)a_2 = \psi(a_1ba_2), \quad a_1, a_2 \in A, \quad b \in B. \quad (2.4)$$

A bilinear map $\phi: B \times B \rightarrow B$ is A -multimodular if

$$a_1\phi(b_1a_2, b_2)a_3 = \phi(a_1b_1, a_2b_2a_3), \quad a_i \in A, \quad b_i \in B. \quad (2.5)$$

Lemma 2.2. *Let M and S be type II_1 von Neumann algebras with S hyperfinite, and let $Q \subseteq M$ be a hyperfinite von Neumann subalgebra with $Q' \cap M = Z(M)$. Let $\phi: (M \overline{\otimes} S) \times (M \overline{\otimes} S) \rightarrow M \overline{\otimes} S$ be a bounded separately normal bilinear map which is $Q \overline{\otimes} S$ -multimodular. Then ϕ is completely bounded.*

Proof. The hypothesis of multimodularity allows us to apply Lemma 5.4.5 (ii) of [28] which yields the inequality

$$\left\| \sum_{i=1}^n \phi(x_i, y_i) \right\| \leq 2 \|\phi\| \left\| \sum_{i=1}^n x_i x_i^* \right\|^{1/2} \left\| \sum_{i=1}^n y_i^* y_i \right\|^{1/2} \quad (2.6)$$

for arbitrary finite sets of elements $x_i, y_i \in M \overline{\otimes} S$, $1 \leq i \leq n$. If ϕ_n denotes the n^{th} amplification of ϕ to $M \overline{\otimes} S \otimes \mathbb{M}_n$, then (2.6) says that

$$\|\phi_n(R, C)\| \leq 2 \|\phi\| \|R\| \|C\| \quad (2.7)$$

for operators R and C in the respective row and column spaces $\text{Row}_n(M \overline{\otimes} S)$ and $\text{Col}_n(M \overline{\otimes} S)$. Now S contains arbitrarily large matrix subfactors and so has no finite dimensional representations. From [21, Proposition 3.4], S norms $M \overline{\otimes} S$. Thus, for each pair $X, Y \in \mathbb{M}_n(M \overline{\otimes} S)$,

$$\begin{aligned} \|\phi_n(X, Y)\| &= \sup\{\|R\phi_n(X, Y)C\| : R \in \text{Row}_n(I \overline{\otimes} S), C \in \text{Col}_n(I \overline{\otimes} S), \|R\|, \|C\| \leq 1\} \\ &= \sup\{\|\phi_n(RX, YC)\| : R \in \text{Row}_n(I \overline{\otimes} S), C \in \text{Col}_n(I \overline{\otimes} S), \|R\|, \|C\| \leq 1\} \\ &\leq 2 \|\phi\| \|X\| \|Y\|, \end{aligned} \quad (2.8)$$

where the second equality uses $(I \overline{\otimes} S)$ -modularity and the final inequality is (2.7) applied to the row RX and the column YC . Since n was arbitrary in (2.8), complete boundedness of ϕ is established by this inequality. \square

In [25], it was shown that, for von Neumann algebras $M \subseteq B(H)$, every derivation $\delta: M \rightarrow B(H)$ is automatically bounded and ultraweakly continuous. We will require two further facts about derivations which we quote from the work of Christensen in the next two lemmas. In the first one, our statement is extracted from the proof of $[4 \Rightarrow 2]$ in the referenced theorem.

Lemma 2.3 (Theorem 3.1 in [3]). *Each completely bounded derivation $\delta: M \rightarrow B(H)$ is inner and is implemented by an operator in $B(H)$.*

Lemma 2.4 (special case of Theorem 5.1 in [2]). *If $M \subseteq N$ is an inclusion of finite von Neumann algebras, then each derivation $\delta: M \rightarrow N$ is inner and is implemented by an element of N .*

3 Separable algebras

In this section we will prove the vanishing of second cohomology for tensor products of type II_1 von Neumann algebras under the additional hypothesis that each algebra is separable. In this context, separability of a von Neumann algebra means the existence of

a countable ultraweakly dense subset or, equivalently, a faithful normal representation on a separable Hilbert space. If M is a separable type II_1 factor, then it was shown in [23] that M has a maximal abelian subalgebra (masa) A and a hyperfinite subfactor R such that $A \subseteq R \subseteq M$ and $R' \cap M = \mathbb{C}1$. This was generalized to separable type II_1 von Neumann algebras with the modifications that R is now a hyperfinite von Neumann subalgebra and that $R' \cap M$ is now the center $Z(M)$. This may be found in [24, Proof of 3.3] with a complete proof in [30]. Separability is essential for these results and this is the reason for restricting to separable algebras in this section. Throughout we assume that M and N are separable type II_1 von Neumann algebras with respective centers $Z(M)$ and $Z(N)$. We fix choices of masas A and B and hyperfinite type II_1 subalgebras R and S so that

$$A \subseteq R \subseteq M, \quad B \subseteq S \subseteq N, \quad (3.1)$$

and

$$R' \cap M = Z(M), \quad S' \cap N = Z(N). \quad (3.2)$$

We also note the trivial fact that centers are always contained in masas.

We wish to consider a bounded 2-cocycle $\phi: (M \overline{\otimes} N) \times (M \overline{\otimes} N) \rightarrow M \overline{\otimes} N$ and show that it is a coboundary. The general reduction results of [28, Ch. 3] allow us to impose the following extra conditions on ϕ :

(C1) ϕ is separately normal in each variable;

(C2) $\phi(x, y) = 0$ whenever x or y lies in $R \overline{\otimes} S$;

(C3) ϕ is $R \overline{\otimes} S$ -multimodular.

The latter condition is a consequence of (C2), from [28, Lemma 3.2.1], so (C2) is a slightly stronger requirement. We begin by making a further reduction.

Lemma 3.1. *Let ϕ be a 2-cocycle on $M \overline{\otimes} N$ which satisfies conditions (C1)–(C3). Then ϕ is equivalent to a 2-cocycle ψ on $M \overline{\otimes} N$ satisfying (C1)–(C3) and the additional condition*

$$\psi(m_1 \otimes I, m_2 \otimes I) = \psi(I \otimes n_1, I \otimes n_2) = 0 \quad (3.3)$$

for $m_1, m_2 \in M$, $n_1, n_2 \in N$.

Proof. Multimodularity with respect to $I \otimes S$ shows that

$$\begin{aligned} (I \otimes s)\phi(m_1 \otimes I, m_2 \otimes I) &= \phi(m_1 \otimes s, m_2 \otimes I) = \phi(m_1 \otimes I, m_2 \otimes s) \\ &= \phi(m_1 \otimes I, m_2 \otimes I)(I \otimes s), \quad m_1, m_2 \in M, s \in S, \end{aligned} \quad (3.4)$$

from which it follows that $\phi(m_1 \otimes I, m_2 \otimes I) \in (I \otimes S)' \cap (M \overline{\otimes} N) = M \overline{\otimes} Z(N)$ for all $m_1, m_2 \in M$. Note that $Z(N) \subseteq B \subseteq S$, and so $M \overline{\otimes} Z(N) \subseteq M \overline{\otimes} S$. Since

$$\phi(m_1 \otimes s_1, m_2 \otimes s_2) = \phi(m_1 \otimes I, m_2 \otimes I)(I \otimes s_1 s_2) \quad (3.5)$$

for $m_1, m_2 \in M$, $s_1, s_2 \in S$, we conclude that ϕ maps $(M \overline{\otimes} S) \times (M \overline{\otimes} S)$ to $M \overline{\otimes} S$. Thus the restriction of ϕ to $M \overline{\otimes} S$ is completely bounded by Lemma 2.2. It follows from [9] that there

is a normal $(R\overline{\otimes}S)$ -modular map $\alpha: M\overline{\otimes}S \rightarrow M\overline{\otimes}S$ such that $\phi|_{M\overline{\otimes}S} = \partial\alpha$, and a similar argument gives a normal $(R\overline{\otimes}S)$ -modular map $\beta: R\overline{\otimes}N \rightarrow R\overline{\otimes}N$ such that $\phi|_{R\overline{\otimes}N} = \partial\beta$. Using the normal conditional expectations $\mathbb{E}_{M\overline{\otimes}S}$ and $\mathbb{E}_{R\overline{\otimes}N}$ of $M\overline{\otimes}N$ onto $M\overline{\otimes}S$ and $R\overline{\otimes}N$ respectively, we now extend α and β to $(R\overline{\otimes}S)$ -modular maps $\tilde{\alpha}, \tilde{\beta}: M\overline{\otimes}N \rightarrow M\overline{\otimes}N$ by

$$\tilde{\alpha} = \alpha \circ \mathbb{E}_{M\overline{\otimes}S}, \quad \tilde{\beta} = \beta \circ \mathbb{E}_{R\overline{\otimes}N}. \quad (3.6)$$

Now define $\psi = \phi - \partial\tilde{\alpha} - \partial\tilde{\beta}$, a 2-cocycle equivalent to ϕ . We verify the desired properties for ψ . Separate normality is clear from the choices of α and β , so ψ satisfies (C1).

Since ϕ satisfies (C2), we have $\phi(I, I) = 0$. Then the cocycle identity

$$I\alpha(I) - \alpha(I) + \alpha(I)I = \phi(I, I) = 0 \quad (3.7)$$

gives $\alpha(I) = 0$, and modularity then implies that $\tilde{\alpha}|_{R\overline{\otimes}S} = \alpha|_{R\overline{\otimes}S} = 0$, with a similar result for $\tilde{\beta}$. A straightforward calculation then shows that $\partial\tilde{\alpha}(x, y)$ and $\partial\tilde{\beta}(x, y)$ are both 0 whenever at least one of x and y lies in $R\overline{\otimes}S$. Thus ψ satisfies (C2) and hence (C3). It remains to show that (3.3) is satisfied. We consider only the relation $\psi(m_1 \otimes I, m_2 \otimes I) = 0$ for $m_1, m_2 \in M$, since the argument for the second is identical.

For $m \in M$,

$$\tilde{\beta}(m \otimes I) = \beta(\mathbb{E}_R(m) \otimes I) = 0, \quad (3.8)$$

since β vanishes on $R\overline{\otimes}S$, and thus $\partial\tilde{\beta}|_{M\otimes I} = 0$. Consequently $\psi|_{M\otimes I} = \phi|_{M\otimes I} - \partial\tilde{\alpha}|_{M\otimes I}$, and we determine the latter term. For $m \in M$,

$$\tilde{\alpha}(m \otimes I) = \alpha(m \otimes \mathbb{E}_S(I)) = \alpha(m \otimes I), \quad (3.9)$$

so

$$\partial\tilde{\alpha}(m_1 \otimes I, m_2 \otimes I) = \partial\alpha(m_1 \otimes I, m_2 \otimes I) = \phi(m_1 \otimes I, m_2 \otimes I), \quad m_1, m_2 \in M, \quad (3.10)$$

since $\phi = \partial\alpha$ on $M\overline{\otimes}S$. This shows that (3.3) holds. \square

In light of this lemma, we may henceforth assume that the 2-cocycle ϕ on $M\overline{\otimes}N$ not only satisfies (C1)–(C3) but also condition (3.3). We will need to make use of the basic construction for an inclusion $P \subseteq Q$ of finite von Neumann algebras, where Q has a specified normal faithful trace τ . Then Q acts on the Hilbert space $L^2(Q, \tau)$, which we abbreviate to $L^2(Q)$, and its commutant is JQJ , where J is the canonical conjugation. We will use J for all such conjugations, which should be clear from the context. The Hilbert space projection of $L^2(Q)$ onto $L^2(P)$ is denoted by e_p , and the basic construction $\langle Q, e_p \rangle$ is the von Neumann algebra generated by Q and e_p . Since $\langle Q, e_p \rangle' = JPJ$, [14], it is clear that $\langle Q, e_p \rangle$ is hyperfinite precisely when P has this property.

For the inclusions $A\overline{\otimes}B \subseteq R\overline{\otimes}S \subseteq M\overline{\otimes}N$, we obtain an inclusion $\langle M\overline{\otimes}N, e_{R\overline{\otimes}S} \rangle \subseteq \langle M\overline{\otimes}N, e_{A\overline{\otimes}B} \rangle$ of hyperfinite von Neumann algebras. Since $J(A\overline{\otimes}B)J$ is a masa in $J(M\overline{\otimes}N)J$, the general theory of extended cobounding, [17] or [28], allows us to find a bounded $(R\overline{\otimes}S)$ -modular map $\lambda: M\overline{\otimes}N \rightarrow \langle M\overline{\otimes}N, e_{A\overline{\otimes}B} \rangle$ so that $\phi = \partial\lambda$. Hyperfiniteness gives a conditional expectation $\mathbb{E}: \langle M\overline{\otimes}N, e_{A\overline{\otimes}B} \rangle \rightarrow \langle M\overline{\otimes}N, e_{R\overline{\otimes}S} \rangle$, and the $(R\overline{\otimes}S)$ -modular composition $\gamma = \mathbb{E} \circ \lambda: M\overline{\otimes}N \rightarrow \langle M\overline{\otimes}N, e_{R\overline{\otimes}S} \rangle$ also has the property that $\phi = \partial\gamma$. Moreover, the results of [13] allow us to further assume that γ is normal.

We now introduce three auxiliary linear maps. At the outset these are not obviously bounded, and so can only be defined on the algebraic tensor product $M \otimes N$. We define $f, g: M \otimes N \rightarrow \langle M \overline{\otimes} N, e_{R \overline{\otimes} S} \rangle$ and $h: M \otimes N \rightarrow M \overline{\otimes} N$ on elementary tensors $m \otimes n \in M \otimes N$ by

$$f(m \otimes n) = \phi(m \otimes I, I \otimes n) + \gamma(m \otimes n), \quad (3.11)$$

$$g(m \otimes n) = \phi(I \otimes n, m \otimes I) + \gamma(m \otimes n), \quad (3.12)$$

$$h(m \otimes n) = g(m \otimes n) - f(m \otimes n) = \phi(I \otimes n, m \otimes I) - \phi(m \otimes I, I \otimes n). \quad (3.13)$$

The next lemma lists some basic properties of these maps.

Lemma 3.2. *The following properties hold:*

- (i) *The restrictions $\gamma|_{M \overline{\otimes} S}$ and $\gamma|_{R \overline{\otimes} N}$ are completely bounded derivations, spatially implemented by elements of $\langle M \overline{\otimes} N, e_{R \overline{\otimes} S} \rangle$.*
- (ii) *The restrictions $f|_{M \otimes I}$, $f|_{I \otimes N}$, $g|_{M \otimes I}$, and $g|_{I \otimes N}$ are equal to the respective restrictions of γ to these subalgebras, and are all bounded derivations spatially implemented by elements of $\langle M \overline{\otimes} N, e_{R \overline{\otimes} S} \rangle$.*
- (iii) *The restrictions $h|_{M \otimes I}$ and $h|_{I \otimes N}$ are both 0.*

Proof. (i) We consider only $\gamma|_{M \overline{\otimes} S}$, the other case being similar. Since $\phi|_{M \otimes I} = 0$, from (3.3), and $\phi = \partial\gamma$, we see that $\gamma|_{M \otimes I}$ is a derivation. The $(R \overline{\otimes} S)$ -modularity then implies that $\gamma|_{M \overline{\otimes} S}$ is a derivation, with the same conclusion for $\gamma|_{M \overline{\otimes} S}$ by normality of γ . Since γ is, in particular, $(I \otimes S)$ -modular, complete boundedness of $\gamma|_{M \overline{\otimes} S}$ follows from Lemma 2.1. Thus $\gamma|_{M \overline{\otimes} S}$ is implemented by an operator $t \in B(L^2(M \overline{\otimes} N))$, from Lemma 2.3. By hyperfiniteness of $\langle M \overline{\otimes} N, e_{R \overline{\otimes} S} \rangle$, there is a conditional expectation \mathbb{E} of $B(L^2(M \overline{\otimes} N))$ onto this subalgebra, and so $\gamma|_{M \overline{\otimes} S}$ is also implemented by $\mathbb{E}(t) \in \langle M \overline{\otimes} N, e_{R \overline{\otimes} S} \rangle$.

(ii) From (3.11) and (C2),

$$\begin{aligned} f(m \otimes I) &= \phi(m \otimes I, I \otimes I) + \gamma(m \otimes I) \\ &= \gamma(m \otimes I), \quad m \in M, \end{aligned} \quad (3.14)$$

so $f|_{M \otimes I} = \gamma|_{M \otimes I}$ is a derivation on $M \otimes I$ spatially implemented by an element of $\langle M \overline{\otimes} N, e_{R \overline{\otimes} S} \rangle$ from (i). The other three restrictions are handled similarly.

(iii) From (ii)

$$h|_{M \otimes I} = g|_{M \otimes I} - f|_{M \otimes I} = \gamma|_{M \otimes I} - \gamma|_{M \otimes I} = 0, \quad (3.15)$$

with a similar result for $h|_{I \otimes N}$. □

Proposition 3.3. *The map f of (3.11) is a derivation on $M \otimes N$.*

Proof. For $m \in M$ and $n \in N$, Lemma 3.2 (ii) gives

$$\begin{aligned} (m \otimes I)f(I \otimes n) + f(m \otimes I)(I \otimes n) &= (m \otimes I)\gamma(I \otimes n) + \gamma(m \otimes I)(I \otimes n) \\ &= [(m \otimes I)\gamma(I \otimes n) - \gamma(m \otimes n) + \gamma(m \otimes I)(I \otimes n)] \\ &\quad + \gamma(m \otimes n) \\ &= \phi(m \otimes I, I \otimes n) + \gamma(m \otimes n) \\ &= f(m \otimes n), \end{aligned} \quad (3.16)$$

using $\phi = \partial\gamma$. A similar calculation leads to

$$(I \otimes n)g(m \otimes I) + g(I \otimes n)(m \otimes I) = g(m \otimes n). \quad (3.17)$$

We now use (3.16) and (3.17) to calculate ∂f on pairs of elementary tensors, noting that f is a derivation on $M \otimes I$ and $I \otimes N$. For $m_1, m_2 \in M, n_1, n_2 \in N$,

$$\begin{aligned} \partial f(m_1 \otimes n_1, m_2 \otimes n_2) &= (m_1 \otimes n_1)f(m_2 \otimes n_2) - f(m_1 m_2 \otimes n_1 n_2) + f(m_1 \otimes n_1)(m_2 \otimes n_2) \\ &= (m_1 \otimes n_1)[(m_2 \otimes I)f(I \otimes n_2) + f(m_2 \otimes I)(I \otimes n_2)] \\ &\quad - [(m_1 m_2 \otimes I)f(I \otimes n_1 n_2) + f(m_1 m_2 \otimes I)(I \otimes n_1 n_2)] \\ &\quad + [(m_1 \otimes I)f(I \otimes n_1) + f(m_1 \otimes I)(I \otimes n_1)](m_2 \otimes n_2) \\ &= (m_1 m_2 \otimes n_1)f(I \otimes n_2) + (m_1 \otimes n_1)f(m_2 \otimes I)(I \otimes n_2) \\ &\quad - (m_1 m_2 \otimes I)[(I \otimes n_1)f(I \otimes n_2) + f(I \otimes n_1)(I \otimes n_2)] \\ &\quad - [(m_1 \otimes I)f(m_2 \otimes I) + f(m_1 \otimes I)(m_2 \otimes I)](I \otimes n_1 n_2) \\ &\quad + (m_1 \otimes I)f(I \otimes n_1)(m_2 \otimes n_2) + f(m_1 \otimes I)(m_2 \otimes n_1 n_2) \\ &= (m_1 m_2 \otimes n_1)f(I \otimes n_2) + (m_1 \otimes n_1)f(m_2 \otimes I)(I \otimes n_2) \\ &\quad - (m_1 m_2 \otimes n_1)f(I \otimes n_2) - (m_1 m_2 \otimes I)f(I \otimes n_1)(I \otimes n_2) \\ &\quad - (m_1 \otimes I)f(m_2 \otimes I)(I \otimes n_1 n_2) - f(m_1 \otimes I)(m_2 \otimes n_1 n_2) \\ &\quad + (m_1 \otimes I)f(I \otimes n_1)(m_2 \otimes n_2) + f(m_1 \otimes I)(m_2 \otimes n_1 n_2) \\ &= (m_1 \otimes I)[(I \otimes n_1)f(m_2 \otimes I) + f(I \otimes n_1)(m_2 \otimes I)](I \otimes n_2) \\ &\quad - (m_1 \otimes I)[(m_2 \otimes I)f(I \otimes n_1) + f(m_2 \otimes I)(I \otimes n_1)](I \otimes n_2). \end{aligned} \quad (3.18)$$

Recalling that f, g and γ agree on $M \otimes I$ and $I \otimes N$, while $\phi = \partial\gamma$, (3.18) becomes

$$\begin{aligned} \partial f(m_1 \otimes n_1, m_2 \otimes n_2) &= (m_1 \otimes I)[(I \otimes n_1)\gamma(m_2 \otimes I) + \gamma(I \otimes n_1)(m_2 \otimes I)](I \otimes n_2) \\ &\quad - (m_1 \otimes I)[(m_2 \otimes I)\gamma(I \otimes n_1) + \gamma(m_2 \otimes I)(I \otimes n_1)](I \otimes n_2) \\ &= (m_1 \otimes I)[\phi(I \otimes n_1, m_2 \otimes I) + \gamma(m_2 \otimes n_1)](I \otimes n_2) \\ &\quad - (m_1 \otimes I)[\phi(m_2 \otimes I, I \otimes n_1) + \gamma(m_2 \otimes n_1)](I \otimes n_2) \\ &= (m_1 \otimes I)[g(m_2 \otimes n_1) - f(m_2 \otimes n_1)](I \otimes n_2) \\ &= (m_1 \otimes I)h(m_2 \otimes n_1)(I \otimes n_2). \end{aligned} \quad (3.19)$$

Here we have used the relations (3.11)–(3.13). Now define $F = \partial f$. Then (3.19) is

$$F(m_1 \otimes n_1, m_2 \otimes n_2) = (m_1 \otimes I)h(m_2 \otimes n_1)(I \otimes n_2). \quad (3.20)$$

The identity $\partial F = 0$ for the triple $(I \otimes n_1, m_2 \otimes I, m_3 \otimes I)$ yields

$$(m_2 \otimes n_1)h(m_3 \otimes I) - (m_2 \otimes I)h(m_3 \otimes n_1) + h(m_2 m_3 \otimes n_1) - h(m_2 \otimes n_1)(m_3 \otimes I) = 0, \quad (3.21)$$

and so

$$h(m_2 m_3 \otimes n_1) = (m_2 \otimes I)h(m_3 \otimes n_1) + h(m_2 \otimes n_1)(m_3 \otimes I) \quad (3.22)$$

since $h|_{M \otimes I} = 0$. It follows from (3.22) that, for each fixed $n_1 \in N$, the map $\delta(m \otimes I) = h(m \otimes n_1)$, $m \in M$, defines a derivation of $M \otimes I$ into $M \overline{\otimes} N$. Since $M \overline{\otimes} N$ is finite, δ is implemented by an element $a \in M \overline{\otimes} N$ by Lemma 2.4. For $r \in R$,

$$\delta(r \otimes I) = h(r \otimes n_1) = \phi(I \otimes n_1, r \otimes I) - \phi(r \otimes I, I \otimes n_1) = 0, \quad (3.23)$$

from (3.13) and (C2). Thus $a \in (R \otimes I)' \cap (M \overline{\otimes} N) = Z(M) \overline{\otimes} N$, so a commutes with $M \otimes I$. We conclude that $h(m \otimes n_1) = 0$ for $m \in M$. Since $n_1 \in N$ was arbitrary, $h = 0$ and, from (3.19), $\partial f = 0$. This shows that f is a derivation on the algebraic tensor product $M \otimes N$. \square

Proposition 3.4. *There exists a bounded normal map $\xi: M \overline{\otimes} N \rightarrow M \overline{\otimes} N$ such that*

$$\xi(m \otimes n) = \phi(m \otimes I, I \otimes n), \quad m \in M, n \in N. \quad (3.24)$$

Proof. From Proposition 3.3, f is a derivation on $M \otimes N$ with values in $\langle M \overline{\otimes} N, e_{R \overline{\otimes} S} \rangle = \langle M, e_R \rangle \overline{\otimes} \langle N, e_S \rangle$. By Lemma 3.2 (ii), $f|_{M \otimes I}$ is a completely bounded derivation implemented by an element $t \in \langle M, e_R \rangle \overline{\otimes} \langle N, e_S \rangle$. Define a derivation $\delta: M \otimes N \rightarrow \langle M, e_R \rangle \overline{\otimes} \langle N, e_S \rangle$ by

$$\delta(m \otimes n) = f(m \otimes n) - [t(m \otimes n) - (m \otimes n)t], \quad m \in M, n \in N. \quad (3.25)$$

Then $\delta|_{M \otimes I} = 0$ from (3.25), so δ is $(M \otimes I)$ -modular. From Lemma 3.2 (ii), $f|_{1 \otimes N}$ is a derivation implemented by an element of $\langle M, e_R \rangle \overline{\otimes} \langle N, e_S \rangle$, so from (3.25) there is an element b in this algebra such that

$$\delta(I \otimes n) = b(I \otimes n) - (I \otimes n)b, \quad n \in N. \quad (3.26)$$

The $(M \otimes I)$ -modularity of δ shows that

$$(m \otimes I)\delta(I \otimes n) = \delta(m \otimes n) = \delta(I \otimes n)(m \otimes I), \quad m \in M, n \in N, \quad (3.27)$$

and we conclude that the range of $\delta|_{1 \otimes N}$ lies in $(M \otimes I)' \cap \langle M, e_R \rangle \overline{\otimes} \langle N, e_S \rangle$. This algebra is $(M' \cap \langle M, e_R \rangle) \overline{\otimes} \langle N, e_S \rangle$, equal to $(JMJ \cap (JRJ)') \overline{\otimes} \langle N, e_S \rangle$, and in turn equal to $(JZ(M)J) \overline{\otimes} \langle N, e_S \rangle$. The latter algebra is hyperfinite, so if we take a conditional expectation onto it and apply this to (3.26), then we conclude that the element b of (3.26) may be assumed to lie in $(JZ(M)J) \overline{\otimes} \langle N, e_S \rangle$. Then b commutes with $M \otimes I$, so

$$\begin{aligned} \delta(m \otimes n) &= (m \otimes I)\delta(I \otimes n) = (m \otimes I)[b(I \otimes n) - (I \otimes n)b] \\ &= b(m \otimes n) - (m \otimes n)b, \quad m \in M, n \in N. \end{aligned} \quad (3.28)$$

Thus δ has a unique bounded normal extension to $M \overline{\otimes} N$, and (3.25) shows that the same is then true for f . Since $\xi = f - \gamma$ on $M \otimes N$ from (3.11), and γ is already bounded and normal on $M \overline{\otimes} N$, this gives a bounded normal extension of ξ to $M \overline{\otimes} N$. \square

Remark 3.5. Equation (3.25) shows that the derivation f on $M \otimes N$ has a unique normal extension to $M \overline{\otimes} N$. Taking ultraweak limits in the equation

$$f(xy) = xf(y) + f(x)y, \quad x, y \in M \otimes N, \quad (3.29)$$

shows that this extension is also a derivation on $M \overline{\otimes} N$. \square

We now come to the main result of this section.

Theorem 3.6. *Let M and N be separable type II₁ von Neumann algebras. Then*

$$H^2(M\overline{\otimes}N, M\overline{\otimes}N) = 0. \quad (3.30)$$

Proof. We have already reduced consideration of a general cocycle ϕ to one which satisfies (C1)–(C3) and (3.3). With the previously established notation, Proposition 3.4 and Remark 3.5 show that ξ and f have bounded normal extensions from $M \otimes N$ to $M\overline{\otimes}N$. Using the same letters for the extensions, we see that ξ maps $M\overline{\otimes}N$ to itself, while f is a derivation on $M\overline{\otimes}N$ from Remark 3.5. Thus

$$\phi = \partial\gamma = \partial f - \partial\xi = \partial(-\xi) \quad (3.31)$$

on $(M\overline{\otimes}N) \times (M\overline{\otimes}N)$. This shows that ϕ is a coboundary with respect to the bounded linear map $-\xi$, proving the result. \square

Remark 3.7. We will require one more piece of information about maps ξ on $M\overline{\otimes}N$ for which $\phi = \partial\xi$, namely that they can be chosen so that $\|\xi\| \leq C\|\phi\|$ for an absolute constant C . The argument is already essentially in [28, Lemma 6.5.1], so we only sketch it here.

If no such C existed, then it would be possible to find separable type II₁ algebras M_n and N_n for $n \geq 1$, and cocycles ϕ_n on $M_n\overline{\otimes}N_n$ of norm 1 so that any ξ_n satisfying $\phi_n = \partial\xi_n$ necessarily had norm at least n . Form separable algebras $M = \bigoplus_{n=1}^{\infty} M_n$ and $N = \bigoplus_{n=1}^{\infty} N_n$ and define a cocycle ϕ on $M\overline{\otimes}N$ by

$$\phi(m_i \otimes n_j, \tilde{m}_k \otimes \tilde{n}_\ell) = \phi_i(m_i \otimes n_i, \tilde{m}_i \otimes \tilde{n}_i)$$

when $i = j = k = \ell$, and 0 otherwise. By Theorem 3.6 there exists a bounded map ξ on $M\overline{\otimes}N$ so that $\phi = \partial\xi$ (which can be assumed to be $Z(M\overline{\otimes}N)$ -modular), but this would then contradict the lower bounds on $\|\xi_n\|$ by restricting ξ to the component algebras. \square

4 The general case

The techniques of Section 3 relied heavily on the existence of hyperfinite subalgebras whose relative commutants are the center, and these are only guaranteed to exist in the separable case. We will use Theorem 3.6 and Remark 3.7 to derive the general result, but we require some preliminary lemmas. A complication for a general type II₁ von Neumann algebra M is that it need not have a faithful normal trace. However, a standard maximality argument gives a family of central projections p_λ with sum I such that each Mp_λ has such a trace. Until we reach Theorem 4.5, we restrict attention to those algebras which do have faithful normal traces.

Lemma 4.1. *Let M and N be type II₁ von Neumann algebras with faithful normal unital traces τ_M and τ_N respectively, and let $Q \subseteq M\overline{\otimes}N$ be a separable von Neumann subalgebra. Then there exist separable type II₁ von Neumann subalgebras $M_0 \subseteq M$ and $N_0 \subseteq N$ such that $Q \subseteq M_0\overline{\otimes}N_0$.*

Proof. We may certainly assume that Q contains arbitrarily large matrix subalgebras of $M \otimes I$ and $I \otimes N$, and this will guarantee that the M_0 and N_0 that we construct are type II_1 .

Let $\tau = \tau_M \otimes \tau_N$ be a faithful normal unital trace on $M \overline{\otimes} N$, and fix a countable ultraweakly dense sequence $\{q_n\}_{n=1}^\infty$ in the unit ball of Q . The ultrastrong and $\|\cdot\|_2$ -norm topologies are equivalent on the unit ball of $M \overline{\otimes} N$, and so the q_n 's are the $\|\cdot\|_2$ -limits of sequences from $M \otimes N$, each element of which is a finite sum of elementary tensors. Let M_0 (respectively N_0) be the von Neumann algebra generated by the first (respectively second) entries in all of these elementary tensors. Each is separable. Then $L^2(Q) \subseteq L^2(M_0 \overline{\otimes} N_0)$ and so $Q \subseteq M_0 \overline{\otimes} N_0$ by considering the conditional expectation of $M \overline{\otimes} N$ onto $M_0 \overline{\otimes} N_0$, which also defines the Hilbert space projection of $L^2(M \overline{\otimes} N)$ onto $L^2(M_0 \overline{\otimes} N_0)$ (see the proof of [1, Lemma 2.2]). \square

Lemma 4.2. *Let M and N be type II_1 von Neumann algebras with faithful normal unital traces τ_M and τ_N respectively. Let ϕ be a separately normal bounded bilinear map from $(M \overline{\otimes} N) \times (M \overline{\otimes} N)$ to $M \overline{\otimes} N$. Given a finite set $F \subseteq M \overline{\otimes} N$, there exist separable type II_1 von Neumann subalgebras $M_F \subseteq M$ and $N_F \subseteq N$ such that $F \subseteq M_F \overline{\otimes} N_F$ and ϕ maps $(M_F \overline{\otimes} N_F) \times (M_F \overline{\otimes} N_F)$ to $(M_F \overline{\otimes} N_F)$.*

Proof. We apply Lemma 4.1 repeatedly. Let Q_0 be the von Neumann generated by F and choose separable von Neumann algebras $M_0 \subseteq M$ and $N_0 \subseteq N$ so that $Q_0 \subseteq M_0 \overline{\otimes} N_0$. Then let Q_1 be the von Neumann algebra generated by $M_0 \overline{\otimes} N_0$ and the range of $\phi|_{M_0 \overline{\otimes} N_0}$; the separate normality of ϕ ensures that Q_1 is separable. Now choose separable von Neumann algebras so that $Q_1 \subseteq M_1 \overline{\otimes} N_1$. By construction, ϕ maps $(M_0 \overline{\otimes} N_0) \times (M_0 \overline{\otimes} N_0)$ into $M_1 \overline{\otimes} N_1$. Continuing in this way, we obtain an ascending sequence $\{M_i \overline{\otimes} N_i\}_{i=0}^\infty$ of separable von Neumann algebras so that ϕ maps $(M_i \overline{\otimes} N_i) \times (M_i \overline{\otimes} N_i)$ into $M_{i+1} \overline{\otimes} N_{i+1}$. Define M_F and N_F as the respective ultraweak closures of $\bigcup_{i=0}^\infty M_i$ and $\bigcup_{i=0}^\infty N_i$. Then separate normality shows that ϕ maps $(M_F \overline{\otimes} N_F) \times (M_F \overline{\otimes} N_F)$ into $M_F \overline{\otimes} N_F$ as required. \square

The next result is a special case of the subsequent main result.

Proposition 4.3. *Let M and N be type II_1 von Neumann algebras with faithful normal unital traces. Then*

$$H^2(M \overline{\otimes} N, M \overline{\otimes} N) = 0. \quad (4.1)$$

Proof. Theorem 3.3.1 of [28] allows us to restrict attention to a separately normal 2-cocycle ϕ on $M \overline{\otimes} N$. For each finite subset F of $M \overline{\otimes} N$, let M_F and N_F be the separable von Neumann subalgebras constructed in Lemma 4.2, so that ϕ maps $(M_F \overline{\otimes} N_F) \times (M_F \overline{\otimes} N_F)$ to $M_F \overline{\otimes} N_F$. Let ϕ_F be the restriction of ϕ to this subalgebra. By Theorem 3.6, there is a bounded linear map $\xi_F: M_F \overline{\otimes} N_F \rightarrow M_F \overline{\otimes} N_F$ so that $\phi_F = \partial \xi_F$, and Remark 3.7 allows us to assume a uniform bound on $\|\xi_F\|$ independent of F . The construction of a bounded map $\xi: M \overline{\otimes} N \rightarrow M \overline{\otimes} N$ such that $\phi = \partial \xi$ now follows the proof of [28, Theorem 6.5.3]. \square

Remark 4.4. An examination of the proof of [28, Theorem 6.5.3] combined with Remark 3.7 shows the existence of an absolute constant K so that, under the hypotheses of Proposition 4.3, to each 2-cocycle ϕ on $M \overline{\otimes} N$ there corresponds a bounded map ξ on $M \overline{\otimes} N$ satisfying $\phi = \partial \xi$ and $\|\xi\| \leq K \|\phi\|$. \square

The final step is to remove the hypothesis of faithful traces from Proposition 4.3.

Theorem 4.5. *Let M and N be type II_1 von Neumann algebras. Then*

$$H^2(M\overline{\otimes}N, M\overline{\otimes}N) = 0. \quad (4.2)$$

Proof. As noted earlier, there are orthogonal sets of central projections $p_\lambda \in Z(M)$ and $q_\mu \in Z(N)$, each summing to I , so that Mp_λ and Nq_μ have faithful normal unital traces. Given a separately normal 2-cocycle ϕ on $M\overline{\otimes}N$, [28, Theorem 3.2.7] allows us to assume that it is $Z(M\overline{\otimes}N)$ -multimodular. Thus the restriction $\phi_{\lambda,\mu}$ of ϕ to $Mp_\lambda\overline{\otimes}Nq_\mu$ maps back to this algebra. By Proposition 4.3 and Remark 4.4, there are maps $\xi_{\lambda,\mu}: Mp_\lambda\overline{\otimes}Nq_\mu \rightarrow Mp_\lambda\overline{\otimes}Nq_\mu$ so that $\phi_{\lambda,\mu} = \partial\xi_{\lambda,\mu}$ with a uniform bound on $\|\xi_{\lambda,\mu}\|$. This allows us to define a bounded map $\xi: M\overline{\otimes}N \rightarrow M\overline{\otimes}N$ by $\xi = \bigoplus_{\lambda,\mu} \xi_{\lambda,\mu}$, and then $\phi = \partial\xi$. \square

References

- [1] J. Cameron, Hochschild Cohomology of II_1 factors with Cartan masas, *Proc. Edinburgh Math. Soc.* **52** (2009), 287–295.
- [2] E. Christensen, Extension of derivations, *J. Funct. Anal.* **27** (1978), 234–247.
- [3] E. Christensen, Extensions of derivations II, *Math. Scand.* **50** (1982), 111–122.
- [4] E. Christensen, Finite von Neumann algebra factors with property Γ , *J. Funct. Anal.* **186** (2001), 366–380.
- [5] E. Christensen, E.G. Effros and A.M. Sinclair, Completely bounded multilinear maps and C^* -algebraic cohomology, *Invent. Math.* **90** (1987), 279–296.
- [6] E. Christensen, F. Pop, A.M. Sinclair and R.R. Smith, On the cohomology groups of certain finite von Neumann algebras, *Math. Ann.* **307** (1997), 71–92.
- [7] E. Christensen, F. Pop, A.M. Sinclair and R.R. Smith, Hochschild cohomology of factors with property Γ , *Ann. of Math. (2)* **158** (2003), 635–659.
- [8] E. Christensen and A.M. Sinclair, Representations of completely bounded multilinear operators, *J. Funct. Anal.* **72** (1987), 151–181.
- [9] E. Christensen and A.M. Sinclair, On the Hochschild cohomology for von Neumann algebras, unpublished manuscript.
- [10] E.G. Effros and E.C. Lance, Tensor products of operator algebras, *Adv. Math.* **25** (1977), 1–34.
- [11] E.G. Effros and Z.-J. Ruan, *Operator spaces*, London Math. Soc. Monographs New Series, Vol. 23, The Clarendon Press, Oxford University Press, New York, 2000.
- [12] L. Ge, Applications of free entropy to finite von Neumann algebras. II, *Ann. of Math. (2)* **147** (1998), 143–157.

- [13] B.E. Johnson, R.V. Kadison and J.R. Ringrose, Cohomology of operator algebras. III. Reduction to normal cohomology, *Bull. Soc. Math. France* **100** (1972), 73–96.
- [14] V.F.R. Jones, Index for subfactors, *Invent. Math.* **72** (1983), 1–25.
- [15] R.V. Kadison, Derivations of operator algebras, *Ann. of Math. (2)* **83** (1966), 280–293.
- [16] R.V. Kadison and J.R. Ringrose, Cohomology of operator algebras I. Type I von Neumann algebras, *Acta Math.* **126** (1971), 227–243.
- [17] R.V. Kadison and J.R. Ringrose, Cohomology of operator algebras II. Extended cobounding and the hyperfinite case, *Ark. Mat.* **9** (1971), 55–63.
- [18] E. Kirchberg, The derivation problem and the similarity problem are equivalent, *J. Operator Theory* **36** (1996), 59–62.
- [19] V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, Vol. 78, Cambridge Univ. Press, Cambridge, UK, 2002.
- [20] G. Pisier, *Introduction to operator space theory*, London Math. Soc. Lecture Note Series, Vol. 294, Cambridge Univ. Press, Cambridge, UK, 2003.
- [21] F. Pop, A.M. Sinclair and R.R. Smith, Norming C^* -algebras by C^* -subalgebras, *J. Funct. Anal.*, **175** (2000), 168–196.
- [22] F. Pop and R.R. Smith, Cohomology for certain finite factors, *Bull. London Math. Soc.* **26** (1994), 303–308.
- [23] S. Popa, On a problem of R.V. Kadison on maximal abelian $*$ -subalgebras in factors, *Invent. Math.* **65** (1981), 269–281.
- [24] S. Popa, Hyperfinite subalgebras normalized by a given automorphism and related problems, In *Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983)*, pp. 421–433, Lecture Notes in Math. vol. 1132, Springer, Berlin, 1985.
- [25] J.R. Ringrose, Automatic continuity of derivations of operator algebras, *J. London Math. Soc. (2)* **5** (1972), 432–438.
- [26] J.R. Ringrose, Cohomology of operator algebras, *Lectures on operator algebras (Tulane Univ. Ring and Operator Theory Year, 1970–1971, Vol. II; dedicated to the memory of David M. Topping)* pp. 355–434, Lecture Notes in Math., Vol. 247, Springer, Berlin, 1972.
- [27] S. Sakai, Derivations of W^* -algebras, *Ann. of Math. (2)* **83** (1966), 273–279.
- [28] A.M. Sinclair and R.R. Smith, *Hochschild cohomology of von Neumann algebras*, London Math. Soc. Lecture Note Series, Vol. 203, Cambridge Univ. Press, Cambridge, UK, 1995.

- [29] A.M. Sinclair and R.R. Smith, Hochschild cohomology for von Neumann algebras with Cartan subalgebras, *Amer. J. Math.* **120** (1998), 1043–1057.
- [30] A.M. Sinclair and R.R. Smith, Cartan subalgebras of finite von Neumann algebras, *Math. Scand.*, **85** (1999), 105–120.
- [31] A.M. Sinclair and R.R. Smith, *Finite von Neumann algebras and masas*, London Math. Soc. Lecture Note Series, Vol. 351, Cambridge Univ. Press, Cambridge, UK, 2008.