Fair Division Problems
When demands or desires of one party are in conflict with those of another; however, objects must be divided or contents must be shared in such a way that no one is treated unfairly.

Examples include divorce, inheritance, the liquidation of a business, labor-management negotiation, or an international dispute.

Fair Division Schemes, or Procedures
The objective in fair division problems is to devise a scheme for dividing objects (divisible or indivisible) in such a way that each of the parties (or individual players) involved obtains a share that he or she considers to be fair.

There are several schemes that we will analyze, namely:

The Adjusted Winner Procedure

The Knaster Inheritance Procedure

Divide and Choose

Cake-Division Procedure: Proportionality
The Adjusted Winner Procedure

The Adjusted Winner Procedure – Developed in the mid-1990s, this procedure allows two parties to settle any dispute involving issues or objects by each party quantifying the importance he/she attaches to getting its own way on each of the objects or issues.

Four Basic Steps for Adjusted Winner Procedure

1) Each party distributes 100 points over the items in such a way that reflects their relative worth. Essentially, the party quantifies the importance for an item by placing a higher “bid” on that item.

2) Each item is initially given to the party that assigned it more points. Each party then assesses how many of his or her own points he or she has received. The party with the fewest points is now given each item on which both parties place the same points.

3) Since the point totals are not likely to be equal, let A be the party with the higher point total and B be the other party. Start transferring items from A to B in a certain order (as in step 4) until the point totals are equal.

4) The order is determined by going through the items in order of increasing point ratio. An item’s point ratio, fraction, is = \((A’s \ point \ value) \div (B’s \ point \ value)\), where A is the party with the higher point total.
Examples

1) Suppose that Will and Ana place the following valuations major assets which will be divided up. Using the adjusted winner procedure determine how the items are distributed.

<table>
<thead>
<tr>
<th>Asset</th>
<th>Will</th>
<th>Ana</th>
</tr>
</thead>
<tbody>
<tr>
<td>Artwork</td>
<td>30</td>
<td>40</td>
</tr>
<tr>
<td>Business</td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>House</td>
<td>50</td>
<td>30</td>
</tr>
</tbody>
</table>

Solution

a) Assigning items

<table>
<thead>
<tr>
<th>Asset</th>
<th>Will</th>
<th>Ana</th>
</tr>
</thead>
<tbody>
<tr>
<td>Artwork</td>
<td>30</td>
<td>40</td>
</tr>
<tr>
<td>Business</td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>House</td>
<td>50</td>
<td>30</td>
</tr>
</tbody>
</table>

Total points: 50 for Will, 70 for Ana

b) Point ratios

<table>
<thead>
<tr>
<th>Asset</th>
<th>Will</th>
<th>Ana</th>
</tr>
</thead>
<tbody>
<tr>
<td>Artwork</td>
<td>30</td>
<td>40</td>
</tr>
<tr>
<td>Business</td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>House</td>
<td>50</td>
<td>30</td>
</tr>
</tbody>
</table>
c) Items retained

Since the point ratio for Artwork is less than the point ratio for Business we start with that item. If \( x \) is the fraction of Artwork that Ana will retain, then \((1-x)\) is the fraction of Artwork that Will will retain. Therefore the balance equation is:

\[
50 + 30(1-x) = 40x + 30 \\
50 + 30 - 30x = 40x + 30 \\
50 = 70x \\
x = 50/70 \\
x = 5/7
\]

Thus:

Will retains House and (2/7) of the Artwork.
Ana retains Business and (5/7) of the Artwork.

2) Suppose that Louis and Adam place the following valuations major assets which will be divided up. Using the adjusted winner procedure determine how the items are distributed.

<table>
<thead>
<tr>
<th>Asset</th>
<th>Louis</th>
<th>Adam</th>
</tr>
</thead>
<tbody>
<tr>
<td>Land</td>
<td>10</td>
<td>38</td>
</tr>
<tr>
<td>Mansion</td>
<td>40</td>
<td>20</td>
</tr>
<tr>
<td>Apartment</td>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>Cars</td>
<td>38</td>
<td>10</td>
</tr>
<tr>
<td>Cash</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>
Solution

a) Assigning items

<table>
<thead>
<tr>
<th>Asset</th>
<th>Louis</th>
<th>Adam</th>
</tr>
</thead>
<tbody>
<tr>
<td>Land</td>
<td>10</td>
<td>38</td>
</tr>
<tr>
<td>Mansion</td>
<td>40</td>
<td>20</td>
</tr>
<tr>
<td>Apartment</td>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>Cars</td>
<td>38</td>
<td>10</td>
</tr>
<tr>
<td>Cash</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Total points</td>
<td>78</td>
<td>68</td>
</tr>
</tbody>
</table>

Now, since Adam has less points than Louis we assign the value 2 to Adam

<table>
<thead>
<tr>
<th>Asset</th>
<th>Louis</th>
<th>Adam</th>
</tr>
</thead>
<tbody>
<tr>
<td>Land</td>
<td>10</td>
<td>38</td>
</tr>
<tr>
<td>Mansion</td>
<td>40</td>
<td>20</td>
</tr>
<tr>
<td>Apartment</td>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>Cars</td>
<td>38</td>
<td>10</td>
</tr>
<tr>
<td>Cash</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Total points</td>
<td>78</td>
<td>70</td>
</tr>
</tbody>
</table>
b) Point Ratios

<table>
<thead>
<tr>
<th>Asset</th>
<th>Louis</th>
<th>Adam</th>
</tr>
</thead>
<tbody>
<tr>
<td>Land</td>
<td>10</td>
<td>38</td>
</tr>
<tr>
<td>Mansion</td>
<td>40</td>
<td>20</td>
</tr>
<tr>
<td>Apartment</td>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>Cars</td>
<td>38</td>
<td>10</td>
</tr>
<tr>
<td>Cash</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Total points</td>
<td>78</td>
<td>70</td>
</tr>
</tbody>
</table>

c) items retained

Since the point ratio for Mansion is less than the point ratio for Cars we start with that item. If x is the fraction of mansion that Louis will retain, then (1– x) is the fraction of Mansion that Adam will retain. Therefore the balance equation is:

\[
38 + 40x = 70 + 20(1– x)
\]

\[
38 + 40x = 70 + 20 - 20x
\]

\[
60x = 90 - 38
\]

\[
60x = 52
\]

\[
x = 52 / 60 = 13 / 15
\]

Thus:

Louis retains the Cars and (13 / 15) of the Mansion.

Adam retains Land, Apartment, Cash, and (13 / 15) of the Artwork.
3) Suppose that Mike and Phil place the following valuations items which will be divided up. Using the adjusted winner procedure determine how the items are distributed.

<table>
<thead>
<tr>
<th>Asset</th>
<th>Mike</th>
<th>Phil</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chair</td>
<td>50</td>
<td>2</td>
</tr>
<tr>
<td>Mattress</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>Sofa</td>
<td>15</td>
<td>35</td>
</tr>
<tr>
<td>Washer</td>
<td>5</td>
<td>40</td>
</tr>
<tr>
<td>Dryer</td>
<td>10</td>
<td>3</td>
</tr>
</tbody>
</table>

**Total points**

<table>
<thead>
<tr>
<th>Asset</th>
<th>Mike</th>
<th>Phil</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>60</td>
<td>75</td>
</tr>
</tbody>
</table>
Now, since Mike has less points than Phil we assign the value 20 to Mike

<table>
<thead>
<tr>
<th>Asset</th>
<th>Mike</th>
<th>Phil</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chair</td>
<td>50</td>
<td>2</td>
</tr>
<tr>
<td>Mattress</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>Sofa</td>
<td>15</td>
<td>35</td>
</tr>
<tr>
<td>Washer</td>
<td>5</td>
<td>40</td>
</tr>
<tr>
<td>Dryer</td>
<td>10</td>
<td>3</td>
</tr>
</tbody>
</table>

Total points 80 75

b) Point ratios

<table>
<thead>
<tr>
<th>Asset</th>
<th>Mike</th>
<th>Phil</th>
<th>point ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chair</td>
<td>50</td>
<td>2</td>
<td>50/2 = 25</td>
</tr>
<tr>
<td>Mattress</td>
<td>20</td>
<td>20</td>
<td>20/20 = 1</td>
</tr>
<tr>
<td>Sofa</td>
<td>15</td>
<td>35</td>
<td></td>
</tr>
<tr>
<td>Washer</td>
<td>5</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>Dryer</td>
<td>10</td>
<td>3</td>
<td>10/3 = 3.3</td>
</tr>
</tbody>
</table>
c) items retained

Since the point ratio for Mattress is less than the point ratio for Chair and Dryer we start with that item. If $x$ is the fraction of Mattress that Mikes will retain, then $(1-x)$ is the fraction of Mattress that Phil will retain. Therefore the balance equation is:

\[
50 + 10 + 20x = 35 + 40 + 20(1-x)
\]
\[
60 + 20x = 75 + 20 - 20x
\]
\[
40x = 95 - 60 \Rightarrow x = \frac{35}{40} = \frac{7}{8}
\]

Thus:

Mike retains the Chair, Dryer, and $(7/8)$ of the Mattress.

Phil retains Sofa, Washer, and $(1/8)$ of the Mansion

Properties of the Adjusted Winner Allocation

For two parties, the adjusted winner procedure produces an allocation, based on each player’s assignment of 100 points over the items to be divided, that has the following properties:

The allocation is **Equitable**: Both players believe that he or she received the same fractional part of the total value.

The allocation is **Envy-free**: Neither player would be happier with what the other received.

The allocation is **Pareto-optimal**: No other allocation, arrived at by any means, can make one party better off without making the other party worse off.

*Pareto optimality (also named after Vilfredo Pareto) is an extremely important property to economists. The order of transfer in step 4 of the adjusted winner procedure is so important because it guarantees that the outcome is Pareto-optimal.*
Section 13.2

The Knaster Inheritance Procedure

Proposed by Bronislaw Knaster in 1945, this is an *auction scheme* designed for estate settlement with many heirs, assuming that each heir has a large amount of cash at his/her disposal.

Basic Steps in Knaster Inheritance Procedure with $n$ Heirs.

For each object, the following steps are performed:

1) The heirs, independently and simultaneously, submits monetary bids for the object.

2) The high bidder is awarded the object, and he or she places all but $1/n$ of his or her bid in a kitty.

3) Each of the other heirs withdraws from the kitty $1/n$ of his or her bid.

4) The money remaining in the kitty is divided equally among the $n$ heirs.
Examples

1) Suppose there is a house and four heirs Bob, Carol, Ted, and Alice. To decide who gets the house they use Knaster inheritance procedure. Bob bids $120,000, Carol bids $200,000, Ted bids $140,000, and Alice bids $180,000. What are the results of the fair division?

Solution

<table>
<thead>
<tr>
<th>Bob</th>
<th>Carol</th>
<th>Ted</th>
<th>Alice</th>
</tr>
</thead>
<tbody>
<tr>
<td>120,000</td>
<td>200,000</td>
<td>140,000</td>
<td>180,000</td>
</tr>
</tbody>
</table>

a) Carol obtains the house because she was the highest bidder.
b) She places \( \frac{3}{4} \times 200,000 = 150,000 \) into a temporary kitty.
c) 
Bob withdraws from the kitty \( \frac{120,000}{4} = 30,000 \)
Ted withdraws from the kitty \( \frac{140,000}{4} = 35,000 \)
Alice withdraws from the kitty \( \frac{180,000}{4} = 45,000 \)

After this, there is extra money \( 150,000 - 110,000 = 40,000 \), which is now divided equally among the four heirs (\( 10,000 \) to each one) giving as result:

Bob gets $30,000 + $10,000 = $40,000
Carol gets the house and paid $140,000 (= $40,000 + $45,000 + $55,000)
Ted gets $35,000 + $10,000 = $45,000
Alice gets $45,000 + $10,000 = $55,000
2) Albert, Robert, and Danny want to get the Peter Frampton's famous guitar. Albert bids $48,000, Robert bids $80,000, and Danny bids $60,000. Use Knaster inheritance procedure to find what the results are for a fair division.

Solution

a) Robert gets the guitar because he was the highest bidder.

b) He places \( \frac{2}{3} \cdot 80,000 = 53,333 \) into a temporary kitty.

c) 
Albert withdraws from the kitty \( \frac{48,000}{3} = 16,000 \)
Danny withdraws from the kitty \( \frac{60,000}{3} = 20,000 \)

After this, there is extra money \( 53,333 - 36,000 = 17,333 \), which is now divided equally among the three heirs (\( 5,777 \) to each one) giving as result:

Albert gets \( 16,000 + 5,777 = 21,777 \)
Robert gets the house and paid \( 47,554 \) (\( = 21,777 + 25,777 \))
Danny gets \( 20,000 + 5,777 = 25,777 \)
3) Suppose that the four heirs Bob, Carol, Ted, and Alice have to divide and estate consisting of a house, cabin, and a boat. If the bids are as follows

<table>
<thead>
<tr>
<th></th>
<th>Bob</th>
<th>Carol</th>
<th>Ted</th>
<th>Alice</th>
</tr>
</thead>
<tbody>
<tr>
<td>House</td>
<td>120,000</td>
<td>200,000</td>
<td>140,000</td>
<td>180,000</td>
</tr>
<tr>
<td>Cabin</td>
<td>60,000</td>
<td>40,000</td>
<td>90,000</td>
<td>50,000</td>
</tr>
<tr>
<td>Boat</td>
<td>30,000</td>
<td>24,000</td>
<td>20,000</td>
<td>20,000</td>
</tr>
</tbody>
</table>

For the House we are did it in example one, how about the other two:

**Cabin**

Ted gets the Cabin and puts $\frac{3}{4} \times 90,000 = 67,500$ in the kitty

Bob withdraws from the kitty $\frac{60,000}{4} = 15,000$

Carol withdraws from the kitty $\frac{40,000}{4} = 10,000$

Alice withdraws from the kitty $\frac{50,000}{4} = 12,500$

After this, there is extra money $67,500 - 37,000 = 30,000$, which is now divided equally among the four heirs ($7,500 to each one) giving as result:

Ted gets the cabin and paid $60,000$ ($= 22,500 + 17,500 + 20,000$) 

Bob gets $15,000 + 7500 = 22,500$

Carol gets $10,000 + 7500 = 17,500$

Alice gets $12,500 + 7500 = 20,000$
**Boat**

Bob gets the Boat and puts \((\frac{3}{4}) 30,000 = $22,500\) in the kitty

Carol withdraws from the kitty \(\frac{24,000}{4} = $6,000\)

Ted withdraws from the kitty \(\frac{20,000}{4} = $5,000\)

Alice withdraws from the kitty \(\frac{20,000}{4} = $5,500\)

After this, there is extra money \(22,500 - 16,000 = $6,500\), which is now divided equally among the four heirs (\$1,625 to each one) giving as result:

Bob gets the boat and paid \(20,875 = ($7,625 + $6,625 + $6,625)\)

Carol gets \($6,000 + $1,625 = $7,625\)

Ted gets \($5,000 + $1,625 = $6,625\)

Alice gets \($5,000 + $1,625 = $6,625\)

**Summarizing**

Bob \(\text{Boat} + ($40,000 + $22,500 - $20,875 = $41,625)\)

Carol \(\text{House} + ( -140,000 + 17,500 + $7,625 = -$114,875)\)

Ted \(\text{Cabin} + ($45,000 - $60,000 + $6,625 = -$8375)\)

Alice \($55,000 + $20,000 + $6,625\)
Section 13.5

Divide and Choose

Divide-and-Choose Procedure

The origins of the divide and choose procedure date back thousands of years.

The rules to the procedure are simple: Someone divides the object into two parts and the other person chooses first.

The natural strategies of the divide and choose procedure are quite obvious:

- The divider makes the two parts equal in his estimation.
- The chooser selects whichever piece he feels is more valuable.

Other strategic considerations may be relevant, such as the question: Would you rather be the divider or the chooser? Not knowing the others' preference, one would want to be the chooser.

This is particular case of the cake-division procedure which is for $n$ players.
Example:

Ana and James will split a cake. If Ana divides and James chooses,

Ana's view  
James' view

(a) how many units of cake does Ana think she gets?
(b) how many units of cake does James think he gets?

Solution

From the Ana's point of view the cake is cut as

Ana's view  
James' view

4 pieces for Ana  
5 pieces for James
Cake division procedure

Cake-division procedure

A cake-division procedure for \( n \) players is a procedure that the players can use to allocate a cake among themselves (no outside arbitrators) so that each player has a strategy that will guarantee that player a piece with which he or she is “satisfied”.

A cake-division procedure for \( n \) players is called proportional if each player's strategy guarantees that player a piece of size or value at least \( 1/n \) of the whole in his or her own estimation.

Lone-Divider Method – (Hugo Steinhaus)

A cake-division procedure that works for three players and produces an allocation that is proportional but not, in general, envy-free.

One cuts the cake and the other two players must “approve a piece” if they think it is the size of at least one-third.

If the other two players “disapprove of a piece,” then that piece can go to the cutter, or some pieces can be put back together and recut.
Consider the following steps to follow in this process

1) Assume we have three players X, Y, and Z. Let X be the divider.

2) Player X divides the cake into three equal pieces, a, b, and c.

3) If players Y and Z each like different pieces, they those pieces and X gets the remaining piece.

4) If players Y and Z both want the same piece, they give one of the other pieces to X. The remaining two pieces are combined and then Y divides and Z choose

Example
Suppose that players 1, 2, and 3 view a cake as follows.

![Player 1](image1.png) ![Player 2](image2.png) ![Player 3](image3.png)

If player 1 cuts the cake into what she perceives as three equal pieces, draw three diagrams to show how each player will view the division. Can all players be satisfied by this division?
Solution

Each player needs to have a piece that they see as having at least 6 square units. Player 1 must choose piece $A$ in order for the other players to be satisfied. Both players 2 and 3 would be satisfied with either of the remaining pieces. Since player 2 views both $B$ and $C$ the same, it would seem reasonable that player 3 would prefer piece $B$ to piece $C$, but would be satisfied with either. Thus, all players can be satisfied.

Example

Suppose that players 1, 2, and 3 view a cake as follows.

If player 2 cuts the cake into what she perceives as three equal pieces, draw three diagrams to show how each player will view the division. Can all players be satisfied by this division?
Last-Diminisher Method – (Banach and Knaster)  

A cake division procedure for any number of players that produces an allocation that is proportional but not, in general, envy-free.

For four or more players, one person cuts a piece of the cake and hands it to another player.

The player examines it. If it looks fair, he passes it along to the next player. If it looks too big, he trims it first (puts the trimmed part back on the cake) and then passes it on (either unaltered or diminished).

This continues down the line until all everyone has had a chance to trim. The last person to trim the piece receives that piece and exits the game.