THE COMPLETELY BOUNDED APPROXIMATION PROPERTY FOR EXTENDED CUNTZ–PIMSNER ALGEBRAS

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Abstract. The extended Cuntz–Pimsner algebra $E(H)$, introduced by Pimsner, is constructed from a Hilbert B,B–bimodule H over a C^*–algebra B. In this paper we investigate the Haagerup invariant $\Lambda(\cdot)$ for these algebras, the main result being that

$$\Lambda(E(H)) = \Lambda(B)$$

when H is full over B. In particular, $E(H)$ has the completely bounded approximation property if and only if the same is true for B.

1. Introduction

The Haagerup invariant for a C^*–algebra A is defined to be the smallest constant $\Lambda(A)$ for which there exists a net $\{\phi_\alpha \colon A \to A\}_{\alpha \in N}$ of finite rank maps satisfying

$$\lim_\alpha \|\phi_\alpha(x) - x\| = 0,\ x \in A, \quad \text{and} \quad \|\phi_\alpha\|_{cb} \le \Lambda(A),\ \alpha \in N. \tag{1}$$

If no such net exists then $\Lambda(A)$ is defined to be ∞, while if $\Lambda(A) < \infty$ then A is said to have the *completely bounded approximation property* (CBAP). The definition of $\Lambda(A)$ arose from [12, 8]. In the first of these it was shown that $C_r^*(\mathbb{F}_2)$, the reduced C^*–algebra of the free group on two generators, has such a net of contractions, and a stronger result using complete contractions was obtained in the second paper. Subsequently, many examples of different values of $\Lambda(\cdot)$ were given in [6]. An interesting problem is to investigate the behavior of $\Lambda(\cdot)$ under the standard constructions of C^*–algebra theory. In [19], the formula $\Lambda(A_1 \otimes A_2) = \Lambda(A_1)\Lambda(A_2)$ was established for the minimal tensor product, while $\Lambda(A \rtimes_\alpha G) = \Lambda(A)$ was proved for discrete amenable groups in [20] and for general amenable groups in [16]. Our objective in this paper is to show that $\Lambda(B) = \Lambda(E(H))$, where $E(H)$ is the extended Cuntz–Pimsner algebra arising from a Hilbert B,B–bimodule over a C^*–algebra B, [17].

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The C^*–algebras $E(H)$ appear in several areas of operator algebras, notably in the work of Muhly and Solel, [14, 15], and in work of the first author with Shlyakhtenko, [10]. It was shown in the latter paper that $E(H)$ is exact if and only if B is exact. Since exactness is a well known consequence of the CBAP, (see, for example, the argument in [16, Theorem 3.1(vii)]), this suggested the connection between $\Lambda(E(H))$ and $\Lambda(B)$. (See also the remarks at the end of this paper.) We now review briefly the definition of $E(H)$.

A right Hilbert B–module H has a B–valued inner product $\langle\cdot,\cdot\rangle_B$, conjugate linear and linear respectively in the first and second variables, and is said to be *full* if $\{\langle h_1,h_2\rangle_B\colon h_1,\, h_2\in H\}$ generates B. The C^*–algebra $\mathcal{L}(H)$ consists of the B–linear operators $T\colon H\to H$ for which there is a B–linear $T^*\colon H\to H$ satisfying $\langle Th_1,h_2\rangle_B=\langle h_1,T^*h_2\rangle_B$, and operators in $\mathcal{L}(H)$ are called *adjointable*. The C^*–algebra $\mathcal{L}(H)$ contains the closed ideal $\mathcal{K}(H)$, that is generated by the maps of the form

$$\theta_{x,y}(h)=x\langle y,h\rangle_B,\quad h\in H,$$

for arbitrary pairs $x,y\in H$, [13]. If there is an injective $*$–homomorphism $\rho\colon B\to\mathcal{L}(H)$, then there is a left action of B on H by $(b,h)\mapsto\rho(b)h$, and we say that H is a B,B–bimodule. The full Fock space $\mathcal{F}(H)$ is defined to be $B\oplus\bigoplus_{n\geq1}H^{(\otimes B)n}$, where $H^{(\otimes B)n}$ is the n–fold tensor product $H\otimes_B H\otimes_B\cdots\otimes_B H$, which is also a Hilbert B,B–bimodule. For $h\in H$, the operator $\ell(h)\colon\mathcal{F}(H)\to\mathcal{F}(H)$ is defined on generators by

$$\ell(h)h_1\otimes\cdots\otimes h_n=h\otimes h_1\otimes\cdots\otimes h_n,\quad h_i\in H,$$
$$\ell(h)b=hb.\quad b\in B.$$

These are bounded adjointable operators on $\mathcal{F}(H)$ and satisfy

$$\ell(h_1)^*\ell(h_2)=\langle h_1,h_2\rangle_B,\quad h_i\in H,$$
$$b_1\ell(h)b_2=\ell(b_1hb_2),\quad b_i\in B.$$

Then $E(H)$ is the C^*–algebra generated in $\mathcal{L}(H)$ by $\{\ell(h)\colon h\in H\}$, and was introduced by Pimsner in [17].

Our approach to investigating $\Lambda(E(H))$ follows the methods of [10]. There, a sequence of operations was given to construct $E(H)$ from B in such a way that exactness was preserved at each step. Here we show that the Haagerup invariant is preserved, with the main technical device being Theorem 1 concerning quotients of C^*–algebras.

2. Results

For a short exact sequence

$$0 \to J \to A \to A/J \to 0$$

of C^*–algebras, we cannot expect, in general, a relationship between $\Lambda(A)$, $\Lambda(J)$ and $\Lambda(A/J)$, [16, Section 5]. However, our first result shows that these quantities are closely linked when the short exact sequence splits. The proof requires the notion of a *quasi–central approximate identity* for a closed ideal J contained in a C^*–algebra A, introduced in [1, 3]. This is an approximate identity $\{e_\alpha\}_{\alpha \in N}$, $0 \le e_\alpha \le 1$, which has the additional property of asymptotically commuting with the elements of A in the sense that

$$\lim_\alpha \|e_\alpha x - x e_\alpha\| = 0, \quad x \in A.$$

Such approximate identities always exist, [1, 3].

Theorem 1. *Let J be an ideal in a C^*–algebra A, let $\pi\colon\ A \to A/J$ be the quotient map, and suppose that there exists a completely contractive map $\rho\colon\ A/J \to A$ such that $\pi\rho = \mathrm{id}_{A/J}$. Then*

$$\Lambda(A) = \max\{\Lambda(J), \Lambda(A/J)\}.$$

Proof. Fix $a_1, \dots, a_n \in A$, $\|a_i\| \le 1$, and fix $\varepsilon > 0$. We will construct a finite rank map $\gamma\colon\ A \to A$ such that $\|\gamma(a_i) - a_i\| < \varepsilon$, $1 \le i \le n$, and $\|\gamma\|_{cb} \le \max\{\Lambda(J), \Lambda(A/J)\}$. This will then prove "$\le$" in the equality.

Fix $\delta > 0$, to be chosen later. Let $\{e_\alpha\}_{\alpha \in N}$ be a quasi-central approximate identity for J satisfying $0 \le e_\alpha \le 1$. By definition of $\Lambda(A/J)$, there exists a finite rank map $\psi\colon\ A/J \to A/J$, $\|\psi\|_{cb} \le \Lambda(A/J)$, such that

$$\|\psi(\pi(a_i)) - \pi(a_i)\| < \delta$$

for $1 \le i \le n$. Define a finite rank map $\widetilde{\psi}\colon\ A \to A$ by

$$\widetilde{\psi}(x) = \rho(\psi(\pi(x))), \qquad x \in A.$$

Clearly $\|\widetilde{\psi}\|_{cb} \leq \Lambda(A/J)$ and, since $\pi(\widetilde{\psi}(a_i)) = \psi(\pi(a_i))$, we may choose elements $j_i \in J$ such that

$$\|\widetilde{\psi}(a_i) - (a_i + j_i)\| < \delta, \qquad 1 \leq i \leq n. \tag{2}$$

Now $\{e_\alpha\}_{\alpha\in N}$ is an approximate identity for J so, for each $\alpha \in N$, we may choose $\beta(\alpha) \in N$, $\beta(\alpha) > \alpha$, such that

$$\|e_{\beta(\alpha)}e_\alpha^{1/2} - e_\alpha^{1/2}\| < \delta. \tag{3}$$

Note that taking adjoints in (3) gives

$$\|e_\alpha^{1/2}e_{\beta(\alpha)} - e_\alpha^{1/2}\| < \delta \tag{4}$$

also. For each $\alpha \in N$, choose a finite rank map $\phi_\alpha\colon\ J \to J$, $\|\phi_\alpha\|_{cb} \leq \Lambda(J)$ such that

$$\|\phi_\alpha(e_{\beta(\alpha)}a_ie_{\beta(\alpha)}) - e_{\beta(\alpha)}a_ie_{\beta(\alpha)}\| < \delta, \qquad 1 \leq i \leq n. \tag{5}$$

Then define $\widetilde{\phi}_\alpha\colon\ A \to A$ by

$$\widetilde{\phi}_\alpha(x) = \phi_\alpha(e_{\beta(\alpha)}xe_{\beta(\alpha)}), \qquad x \in A.$$

These maps are finite rank, and satisfy $\|\widetilde{\phi}_\alpha\|_{cb} \leq \Lambda(J)$. Then, for each $\alpha \in N$, define $\gamma_\alpha\colon\ A \to A$ by

$$\gamma_\alpha(x) = e_\alpha^{1/2}\widetilde{\phi}_\alpha(x)e_\alpha^{1/2} + (1-e_\alpha)^{1/2}\widetilde{\psi}(x)(1-e_\alpha)^{1/2}, \qquad x \in A. \tag{6}$$

Each of these maps can be expressed as a matrix product

$$\gamma_\alpha(x) = (e_\alpha^{1/2}, (1-e_\alpha)^{1/2}) \begin{pmatrix} \widetilde{\phi}_\alpha(x) & 0 \\ 0 & \widetilde{\psi}(x) \end{pmatrix} \begin{pmatrix} e_\alpha^{1/2} \\ (1-e_\alpha)^{1/2} \end{pmatrix}$$

from which it is clear that $\|\gamma_\alpha\|_{cb} \leq \max\{\Lambda(J), \Lambda(A/J)\}$. It remains to be shown that

$$\|\gamma_\alpha(a_i) - a_i\| < \varepsilon, \qquad 1 \leq i \leq n, \tag{7}$$

for a sufficiently large choice of $\alpha \in N$. We will estimate the two terms on the right hand side of (6) separately, when x is replaced by a_i.

The term $e_\alpha^{1/2}\widetilde{\phi}_\alpha(a_i)e_\alpha^{1/2}$ equals, by definition, $e_\alpha^{1/2}\phi_\alpha(e_{\beta(\alpha)}a_ie_{\beta(\alpha)})e_\alpha^{1/2}$ so, from (5),

$$\|e_\alpha^{1/2}\widetilde{\phi}_\alpha(a_i)e_\alpha^{1/2} - e_\alpha^{1/2}e_{\beta(\alpha)}a_ie_{\beta(\alpha)}e_\alpha^{1/2}\| < \delta$$

for $1 \leq i \leq n$. A simple triangle inequality argument, using (3) and (4), then shows that

$$\|e_\alpha^{1/2}\widetilde{\phi}_\alpha(a_i)e_\alpha^{1/2} - e_\alpha^{1/2}a_ie_\alpha^{1/2}\| < 3\delta, \tag{8}$$

for $1 \le i \le n$.

The second term, $(1-e_\alpha)^{1/2}\widetilde{\psi}(a_i)(1-e_\alpha)^{1/2}$, in (6) is within δ of $(1-e_\alpha)^{1/2}(a_i + j_i)(1-e_\alpha)^{1/2}$, from (2). Combining this estimate with (6) and (8) gives

$$\|\gamma_\alpha(a_i)-e_\alpha^{1/2}a_ie_\alpha^{1/2}-(1-e_\alpha)^{1/2}a_i(1-e_\alpha)^{1/2}\| \le \|(1-e_\alpha)^{1/2}j_i(1-e_\alpha)^{1/2}\|+4\delta, \quad 1 \le i \le n.$$

Thus

$$\begin{aligned}\|\gamma_\alpha(a_i)-a_i\| \le \|&a_i - e_\alpha^{1/2}a_ie_\alpha^{1/2} - (1-e_\alpha)^{1/2}a_i(1-e_\alpha)^{1/2}\| \\ &+ \|(1-e_\alpha)^{1/2}j_i(1-e_\alpha)^{1/2}\| + 4\delta, \qquad 1 \le i \le n. \end{aligned} \tag{9}$$

Using the Stone–Weierstrass Theorem and functional calculus to approximate square roots with polynomials, we see that the nets $\{e_\alpha^{1/2}\}_{\alpha\in N}$, and $\{(1-e_\alpha)^{1/2}\}_{\alpha\in N}$ asymptotically commute with each a_i, and

$$\|(1-e_\alpha)^{1/2}j_i\|^2 = \|j_i^*(1-e_\alpha)j_i\| \to 0$$

as $\alpha \to \infty$, so a sufficiently large choice of α in (9) gives

$$\|\gamma_\alpha(a_i)-a_i\| < 5\delta, \qquad 1 \le i \le n.$$

Now take δ to be $\varepsilon/5$, and (7) is proved.

We now turn to the inequality $\Lambda(J) \le \Lambda(A)$. Given $j_1, \dots, j_n \in J$, $\|j_i\| \le 1$, and $\varepsilon > 0$, we may choose a finite rank completely bounded map $\phi\colon\ A \to A$ such that $\|\phi\|_{cb} \le \Lambda(A)$ and

$$\|\phi(j_i)-j_i\| < \varepsilon, \qquad 1 \le i \le n.$$

For each $\alpha \in N$, define $\phi_\alpha\colon\ J \to J$ by

$$\phi_\alpha(j) = e_\alpha\phi(j)e_\alpha, \qquad 1 \le i \le n.$$

then each ϕ_α has finite rank, $\|\phi_\alpha\|_{cb} \le \Lambda(A)$, and, for $1 \le i \le n$,

$$\|\phi_\alpha(j_i)-j_i\| < \|e_\alpha j_ie_\alpha - j_i\| + \varepsilon. \tag{10}$$

A sufficiently large choice of α in (10) gives

$$\|\phi_\alpha(j_i)-j_i\| < \varepsilon, \qquad 1 \le i \le n,$$

showing that $\Lambda(J) \le \Lambda(A)$.

Finally we show that $\Lambda(A/J) \le \Lambda(A)$. Consider elements $\pi(a_1), \ldots, \pi(a_n) \in A/J$, and fix $\varepsilon > 0$. There exists $\phi\colon\ A \to A$ such that ϕ has finite rank, $\|\phi\|_{cb} \le \Lambda(A)$, and

$$\|\phi(\rho\pi(a_i)) - \rho\pi(a_i)\| < \varepsilon \tag{11}$$

for $1 \le i \le n$. Apply π to (11) to obtain

$$\|\pi\phi\rho(\pi(a_i)) - \pi(a_i)\| < \varepsilon$$

for $1 \le i \le n$. Now define $\widetilde{\phi}\colon\ A/J \to A/J$ by

$$\widetilde{\phi} = \pi\phi\rho.$$

Clearly $\widetilde{\phi}$ has finite rank, $\|\widetilde{\phi}\|_{cb} \le \Lambda(A)$, and

$$\|\widetilde{\phi}(\pi(a_i)) - \pi(a_i)\| < \varepsilon$$

for $1 \le i \le n$. This shows that $\Lambda(A/J) \le \Lambda(A)$, proving the result. □

Remark 2. If we loosen the hypotheses of Theorem 1 and require only that ρ be completely bounded, then the same proof yields the inequalities

$$\Lambda(A) \le \max\{\Lambda(J), \|\rho\|_{cb}\,\Lambda(A/J)\}, \qquad \Lambda(J) \le \Lambda(A), \qquad \Lambda(A/J) \le \|\rho\|_{cb}\,\Lambda(A).$$

The next three results are preparatory for Theorem 6, and will handle some technical points arising there. The first of these is a spceial case of [5, Cor. 4.4(ii)]. For completeness, we include a proof.

Proposition 3. *Let A and B be C^*–algebras. Let E be a right Hilbert A–module and let F be a right Hilbert B–module with a $*$–homomorphism $\phi : A \to \mathfrak{L}(F)$. Consider the internal tensor product $E \otimes_\phi F$, which is a right Hilbert B–module. Let $S \in \mathfrak{L}(E)$ and suppose $T \in \mathfrak{L}(F)$ is such that T and $\phi(a)$ commute for all $a \in A$. Then there is $R \in \mathfrak{L}(E \otimes_\phi F)$ satisfying $R(e \otimes f) = (Se) \otimes (Tf)$; we will write $R = S \otimes T$.*

Proof. The operator $S \otimes \mathrm{id}_F$ is well known to belong to $\mathfrak{L}(E \otimes_\phi F)$; (see [13, p. 42]). It will suffice to show $\mathrm{id}_E \otimes T \in \mathfrak{L}(E \otimes_\phi F)$, for in general we will have $S \otimes T = (S \otimes \mathrm{id}_F) \circ (\mathrm{id}_E \otimes T)$. Hence, without loss of generality, we assume $S = \mathrm{id}_E$.

The $\mathbf{C}$–linear map from the algebraic tensor product (over $\mathbf{C}$) $E \otimes F$ to itself defined by $e \otimes f \mapsto e \otimes Tf$ is a right B–module map and sends the submodule

$$N = \mathrm{span}\,\{ea \otimes f - e \otimes \phi(a)f \mid e \in E,\ f \in F,\ a \in A\}$$

into itself. In order to see that the resulting map $(E \otimes F)/N \to (E \otimes F)/N$ gives rise to a bounded B–linear map $E \otimes_\phi F \to E \otimes_\phi F$, let $n \in \mathbf{N}$, $e_1, \dots, e_n \in E$ and $f_1, \dots, f_n \in F$, and let $g = \sum_{i=1}^n e_i \otimes f_i$. Then

$$\langle g, g\rangle_B = \sum_{i,j} \langle f_i, \phi(\langle e_i, e_j\rangle_A) f_j\rangle_B = \langle f, \phi_n(X) f\rangle_{F^n} = \langle \phi_n(X^{1/2}) f, \phi_n(X^{1/2}) f\rangle_{F^n},$$

where f is the column vector $(f_1, \dots, f_n)^t$ in the Hilbert B–module F^n, where X is the matrix $X = (\langle e_i, e_j\rangle_A)_{1\le i,j\le n} \in \mathbb{M}_n(A)$, which, by [13, Lemma 4.2], is positive, and where $\phi_n : \mathbb{M}_n(A) \to \mathbb{M}_n(\mathcal{L}(F)) = \mathcal{L}(F^n)$ is the $*$–homomorphism obtained by application of ϕ to each element of a matrix. On the other hand, letting $h = \sum_{i=1}^n e_i \otimes T f_i$, we have

$$\begin{aligned}\langle h, h\rangle_B &= \sum_{i,j} \langle T f_i, \phi(\langle e_i, e_j\rangle_A) T f_j\rangle_B \\ &= \langle \phi_n(X^{1/2}) \widetilde{T} f, \phi_n(X^{1/2}) \widetilde{T} f\rangle_{F^n} = \langle \widetilde{T} \phi_n(X^{1/2}) f, \widetilde{T} \phi_n(X^{1/2}) f\rangle_{F^n},\end{aligned}$$

where $\widetilde{T} = \operatorname{diag}(T, \dots, T) \in \mathbb{M}_n(\mathcal{L}(F))$. Therefore, (*cf* [13, Prop. 1.2]),

$$\langle h, h\rangle_B = |\widetilde{T} \phi_n(X^{1/2}) f|^2 \le \|\widetilde{T}\|^2 |\phi_n(X^{1/2}) f|^2 = \|T\|^2 \langle g, g\rangle_B$$

and $\|h\| \le \|T\|\,\|g\|$. Consequently, we get $\mathrm{id}_E \otimes T : E \otimes_\phi F \to E \otimes_\phi F$ with $\|\mathrm{id}_E \otimes T\| \le \|T\|$. An easy calculation shows that $\mathrm{id}_E \otimes T^*$ is the adjoint of $\mathrm{id}_E \otimes T$. Thus $\mathrm{id}_E \otimes T \in \mathcal{L}(E \otimes_\phi F)$. □

Recall that a conditional expectation of a C*–algebra A onto a C*–subalgebra B is by definition a projection of norm 1. It follows from [22] that a conditional expectation is both completely contractive and completely positive.

Proposition 4. *Let $A_1 \subseteq A_2 \subseteq \cdots \subseteq A$ be an increasing chain of C^*–subalgebras of a C^*–algebra A, such that $\bigcup_{n=1}^\infty A_n$ is dense in A. Suppose that there are conditional expectations $\rho_n : A_{n+1} \to A_n$ onto A_n, for all $n \ge 1$. Then*

$$\Lambda(A) = \sup_{n \in \mathbf{N}} \Lambda(A_n). \tag{12}$$

Proof. From the family $(\rho_n)_{n\ge1}$ we obtain conditional expectations $\psi_n : A \to A_n$ onto A_n, $n \ge 1$. It then follows that $\Lambda(A) \ge \Lambda(A_n)$ for all $n \ge 1$.

To see the reverse inequality in (12), let $n \in \mathbf{N}$, let $F \subseteq A_n$ be a finite subset and let $\varepsilon > 0$. Then there is a finite rank map $\phi : A_n \to A_n$ such that $\|\phi(x) - x\| < \varepsilon$ for all $x \in F$ and $\|\phi\|_{cb} \le \Lambda(A_n)$. But $\phi \circ \psi_n : A \to A_n \subseteq A$ satisfies $\|\phi \circ \psi_n(x) - x\| < \varepsilon$

for all $x \in F$ and $\|\phi \circ \psi_n\|_{cb} = \|\phi\|_{cb}$. By indexing over all finite subsets F of $\bigcup_n A_n$ and all $\epsilon > 0$, the corresponding finite rank maps $\phi \circ \psi_n$ yield a net approximating the identity map, with completely–bounded–norm uniformly bounded by the right–hand–side of (12). $\square$

Proposition 5. *Let B be a C^*–algebra and let H be a right Hilbert B–module such that $\{\langle h_1, h_2\rangle_B \colon h_1, h_2 \in H\}$ generates B. Then $\Lambda(\mathcal{K}(H)) = \Lambda(B)$.*

Proof. This result follows immediately from the fact, mentioned on [4, p. 391], that $\mathcal{K}(H)$ is an inductive limit of matrix algebras over B, in the sense of the definition on [4, pp. 380–381].

For the reader's convenience, we will describe a related proof based on another construction of D. Blecher [5]. Following the notation of [5], let $C_n(B)$ denote the right Hilbert B–module which consists of columns over B of length n with the B–valued inner product

$$\langle (a_1, \dots, a_n)^t, (b_1, \dots, b_n)^t \rangle_B = \sum_{i=1}^n a_i^* b_i, \qquad a_i, b_i \in B.$$

Then, by [5, Theorem 3.1] and the part of the proof found on [5, p. 266], there exist nets of adjointable contractive maps

$$H \xrightarrow{\phi_\alpha} C_{n(\alpha)}(B) \xrightarrow{\psi_\alpha} H, \qquad \alpha \in N,$$

such that

$$\lim_\alpha \|\psi_\alpha(\phi_\alpha(h)) - h\| = 0, \qquad h \in H.$$

These maps induce a diagram

$$\mathcal{K}(H) \xrightarrow{\widetilde{\phi}_\alpha} \mathcal{K}(C_{n(\alpha)}(B)) \xrightarrow{\widetilde{\psi}_\alpha} \mathcal{K}(H), \qquad \alpha \in N, \tag{13}$$

of complete contractions given by

$$\widetilde{\phi}_\alpha(S) = \phi_\alpha S \phi_\alpha^*, \qquad \widetilde{\psi}_\alpha(T) = \psi_\alpha T \psi_\alpha^*, \qquad \alpha \in N,$$

for $S \in \mathcal{K}(H)$ and $T \in \mathcal{K}(C_{n(\alpha)}(B))$, whose compositions converge to $\mathrm{id}_{\mathcal{K}(H)}$ in the point norm topology. The relations

$$\phi_\alpha \theta_{h_1,h_2} \phi_\alpha^* = \theta_{\phi_\alpha(h_1),\phi_\alpha(h_2)}, \qquad \psi_\alpha \theta_{k_1,k_2} \psi_\alpha^* = \theta_{\psi_\alpha(k_1),\psi_\alpha(k_2)},$$

for $h_i \in H$, $k_i \in C_{n(\alpha)}(B)$, are easy to check and show that $\widetilde{\phi}_\alpha$ and $\widetilde{\psi}_\alpha$ have the appropriate ranges in (13). Now $\mathcal{K}(C_{n(\alpha)}(B))$ is the matrix algebra $\mathbb{M}_{n(\alpha)}(B)$ over B, for which $\Lambda(\mathcal{K}(C_{n(\alpha)}(B))) = \Lambda(\mathbb{M}_{n(\alpha)}(B)) = \Lambda(B)$.

Given $S_1, \ldots, S_r \in \mathcal{K}(H)$ and $\varepsilon > 0$, choose $\alpha \in N$ so large that

$$\|\widetilde{\psi}_\alpha(\widetilde{\phi}_\alpha(S_i)) - S_i\| < \varepsilon/2, \qquad 1 \leq i \leq r, \tag{14}$$

and then choose a finite rank map $\mu\colon \mathcal{K}(C_{n(\alpha)}(B)) \to \mathcal{K}(C_{n(\alpha)}(B))$ such that $\|\mu\|_{cb} \leq \Lambda(B)$ and

$$\|\mu(\widetilde{\phi}_\alpha(S_i)) - \widetilde{\phi}_\alpha(S_i)\| < \varepsilon/2, \qquad 1 \leq i \leq r. \tag{15}$$

Then $\widetilde{\psi}_\alpha\mu\widetilde{\phi}_\alpha\colon \mathcal{K}(H) \to \mathcal{K}(H)$, the composition of the diagram

$$\mathcal{K}(H) \xrightarrow{\widetilde{\phi}_\alpha} \mathcal{K}(C_{n(\alpha)}(B)) \xrightarrow{\mu} \mathcal{K}(C_{n(\alpha)}(B)) \xrightarrow{\widetilde{\psi}_\alpha} \mathcal{K}(H),$$

has finite rank with completely bounded norm at most $\Lambda(B)$, and the triangle inequality gives

$$\|\widetilde{\psi}_\alpha\mu\widetilde{\phi}_\alpha(S_i) - S_i\| < \varepsilon, \qquad 1 \leq i \leq r,$$

from (14) and (15). This shows that $\Lambda(\mathcal{K}(H)) \leq \Lambda(B)$.

To establish the reverse inequality, we can consider H as a left $\mathcal{K}(H)$–module. Then [18, Prop. 3.8] shows that the roles of $\mathcal{K}(H)$ and B are reversed, and the above argument gives $\Lambda(\mathcal{K}(H)) \geq \Lambda(B)$. We note that a similar result for modules over von Neumann algebras appears in [2, Lemma 4.8]. □

We are now able to state and prove the main result of the paper.

Theorem 6. *Let B be a C^*–algebra and let H be a Hilbert B,B–bimodule such that $\{\langle h_1, h_2\rangle_B \colon h_1, h_2 \in H\}$ generates B.*

Consider the extended Cuntz–Pimsner C^*–algebra $E(H)$. Then $\Lambda(E(H)) = \Lambda(B)$. In particular, $E(H)$ has the completely bounded approximation property if and only if B does.

Proof. We will follow quite closely the proof of [10, Thm. 3.1]. Since B is contained as a C^*–subalgebra of $E(H)$ which is the image of a conditional expectation $E(H) \to B$, we have $\Lambda(B) \leq \Lambda(E(H))$.

Let $\widetilde{H} = H \oplus B$ and identify H with the submodule $H \oplus 0 \subseteq \widetilde{H}$. Then (see [17]), $E(H)$ is contained in $E(\widetilde{H})$ in the obvious way. There is a projection $P : \widetilde{H} \to H$ that commutes with the left action of B, and using Proposition 3 to take tensor products of copies of P yields a projection $F(P) : \mathcal{F}(\widetilde{H}) \to \mathcal{F}(H)$, compression with respect to which gives a conditional expectation $E(\widetilde{H}) \to E(H)$. Therefore, $\Lambda(E(H)) \leq \Lambda(E(\widetilde{H}))$.

We will show that $\Lambda(E(\widetilde{H})) \leq \Lambda(B)$. By [10, Claim 3.4], $E(\widetilde{H})$ is isomorphic to the crossed product $A \rtimes_\Psi \mathbf{N}$ of a certain C^*–subalgebra $A \subseteq E(\widetilde{H})$ by an injective endomorphism Ψ, that is given by $\Psi(a) = LaL^*$ for an isometry $L \in E(\widetilde{H})$. As described by Cuntz [7] and Stacey [21] (see also the discussion on p. 432 of [10]), the crossed product $A \rtimes_\Psi \mathbf{N}$ is isomorphic to a corner $p(\widetilde{A} \rtimes_\alpha \mathbf{Z})p$ of the crossed product of a C^*–algebra $\widetilde{A}$ by an automorphism α, where $\widetilde{A}$ is the inductive limit C^*–algebra of the system

$$A \xrightarrow{\Psi} A \xrightarrow{\Psi} A \xrightarrow{\Psi} \cdots . \tag{16}$$

We have $\Lambda(E(\widetilde{H})) = \Lambda(p(\widetilde{A} \rtimes_\alpha \mathbf{Z})p) \leq \Lambda(\widetilde{A} \rtimes_\alpha \mathbf{Z})$. From [20, Thm. 3.4], we have $\Lambda(\widetilde{A} \rtimes_\alpha \mathbf{Z}) = \Lambda(\widetilde{A})$. Letting $\sigma : A \to A$ be $\sigma(a) = L^*aL$, we have that σ is a completely positive left inverse of Ψ. Thus $\Psi \circ \sigma$ is a conditional expectation from A onto $\Psi(A)$ and Proposition 4 applies to the inductive limit (16). We conclude that $\Lambda(\widetilde{A}) = \Lambda(A)$.

From the proof of [10, Claim 3.5], we have an increasing chain

$$B = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A \tag{17}$$

of C^*–subalgebras of A, such that $\bigcup_{n=1}^\infty A_n$ is dense in A. Moreover, for each $n \geq 1$ there is an ideal $I_n \subseteq A_n$ such that A_n/I_n is isomorphic to A_{n-1}; this isomorphism followed by the inclusion $A_{n-1} \hookrightarrow A_n$ splits the short exact sequence

$$0 \to I_n \to A_n \to A_n/I_n \to 0.$$

Finally, the ideal I_n is isomorphic to $\mathcal{K}(H^{\otimes_B n})$. By Proposition 5, $\Lambda(I_n) \leq \Lambda(B)$. Using Theorem 1 and proceeding by induction, one shows that $\Lambda(A_n) \leq \Lambda(B)$ for all $n \geq 0$. The quotient map $A_n \to A_n/I_n$ followed by the isomorphism $A_n/I_n \to A_{n-1}$ is a left inverse for the inclusion $A_{n-1} \hookrightarrow A_n$. Thus, Proposition 4 applies to the chain (17) and we obtain $\Lambda(A) \leq \Lambda(B)$.

These inequalities, applied sequentially, lead to $\Lambda(E(H)) \leq \Lambda(B)$. $\square$

3. Concluding remarks

An interesting open problem is whether taking reduced (amalgamated) free products preserves the class of C^*–algebras that possess the CBAP, or, for example, the class of C^*–algebras having Haagerup invariant equal to 1. It was shown in [10, Prop. 4.2] that a reduced free product C^*–algebra A,

$$(A, \phi) = (A_1, \phi_1) * (A_2, \phi_2), \tag{18}$$

can be embedded into $E(H)$ for a particular Hilbert bimodule H over $A_1 \otimes_{\min} A_2$. It was also shown in [10, Prop. 5.1] that a reduced amalgamated free product C^*–algebra A,

$$(A, \phi) = (A_1, \phi_1) *_B (A_2, \phi_2),$$

can be realized as a quotient of a subalgebra of $E(H')$ for a particular Hilbert bimodule H' over $A_1 \oplus A_2$. This, together with the result from [10] that $E(H)$ is exact when H is a Hilbert bimodule over an exact C^*–algebra, yielded a new proof that the class of exact C^*–algebras is closed under taking reduced (amalgamated) free products. (See [9] for the first proof of this fact.)

However, this paper's Theorem 6 does not answer in a similar way the question about the CBAP for free products, (at least not obviously), because the CBAP does not automatically pass to subalgebras or to quotients. Moreover, the copy of the free product C^*–algebra A from (18) in $E(H)$ exhibited in [10] is not in general the image of a conditional expectation $E(H) \to A$. To see this, first note from [11] that $E(H)$ is a nuclear C^*–algebra whenever H is a Hilbert bimodule over a nuclear C^*–algebra. (This also follows readily from the proof of exactness found in [10], combined with the conditional expectation $E(\widetilde{H}) \to E(H)$ used in the proof of Theorem 6.)

There are many known examples when A_1 and A_2 in (18) are nuclear, but their free product A is not nuclear. In these cases, $E(H)$ is nuclear and A, therefore, cannot be the image of a conditional expectation $E(H) \to A$.

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