REPRESENTATIONS OF GROUP ALGEBRAS IN SPACES OF COMPLETELY BOUNDED MAPS

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Abstract. Let G be a locally compact group, $\pi : G \to \mathcal{U}(\mathcal{H})$ be a strongly continuous unitary representation, and $\mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H}))$ the space of normal completely bounded maps on $\mathcal{B}(\mathcal{H})$. We study the range of the map

$$\Gamma_\pi : \mathrm{M}(G) \to \mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H})), \quad \Gamma_\pi(\mu) = \int_G \pi(s) \otimes \pi(s)^* d\mu(s)$$

where we identify $\mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H}))$ with the extended Haagerup tensor product $\mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$. We use the fact that the C*-algebra generated by integrating π to $\mathrm{L}^1(G)$ is unital exactly when π is norm continuous, to show that $\Gamma_\pi(\mathrm{L}^1(G)) \subset \mathcal{B}(\mathcal{H}) \otimes^h \mathcal{B}(\mathcal{H})$ exactly when π is norm continuous. For the case that G is abelian, we study $\Gamma_\pi(\mathrm{M}(G))$ as a subset of the Varopoulos algebra. We also characterise positive definite elements of the Varopoulos algebra in terms of completely positive operators.

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1. Introduction

In [24], Størmer conducted an interesting study of spaces of completely bounded maps on $\mathcal{B}(\mathcal{H})$. For subalgebras $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{B}(\mathcal{H})$ he defined what is now known as the Haagerup tensor product $\mathcal{A}\otimes^h\mathcal{B}$, as a completion of the set of elementary operators of the form $x \mapsto \sum_{i=1}^n a_i x b_i$ where each $a_i \in \mathcal{A}$ and each $b_i \in \mathcal{B}$. This approach gives the same tensor product norm as that in the more standard approach (see [8], for example), as shown in [21].

If G is an abelian group and $\pi : G \to \mathcal{U}(\mathcal{H})$ is a strongly continuous unitary representation, the homomorphism Γ_π from the measure algebra $\mathrm{M}(G)$ to the space $\mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H}))$ of normal completely bounded maps on $\mathcal{B}(\mathcal{H})$, defined by

$$\Gamma_\pi(\mu) = \int_G \pi(s) \otimes \pi(s)^* d\mu(s) \tag{1.1}$$

was studied by Størmer. (We identify $\mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H}))$ with the extended Haagerup tensor product $\mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$ from [4] and [9].) He used this homomorphism to generate many examples of regular and non-regular Banach subalgebras of $\mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H}))$. It was shown in [24, Lem. 5.6] that if π is norm continuous (i.e. continuous when the norm topology is placed on $\mathcal{U}(\mathcal{H})$) then for any f in $\mathrm{L}^1(G)$

$$\Gamma_\pi(f) = \int_G f(s)\pi(s) \otimes \pi(s)^* ds \in \mathrm{C}^*_\pi \otimes^h \mathrm{C}^*_\pi \tag{1.2}$$

where C^*_π is the C*-algebra generated by $\left\{\int_G f(s)\pi(s)ds : f \in \mathrm{L}^1(G)\right\}$.

We note that for an arbitrary locally compact group G, the map Γ_λ as in (1.1), where λ is the left regular representation, was studied in [11] and [16].

In this paper we will make use of the theory of completely bounded normal maps on $\mathcal{B}(\mathcal{H})$ from [21] to study the range of Γ_π. We show that, for a general locally compact group G,

$$\Gamma_\pi(\mathrm{L}^1(G)) \subset \mathrm{C}^*_\pi \otimes^{eh} \mathrm{C}^*_\pi$$

where $\otimes^{eh}$ denotes the extended Haagerup tensor product from [9], [7] and [4]. Moreover, using the fact that C^*_π is unital exactly when the representation $\pi : G \to \mathcal{U}(\mathcal{H})$ is norm continuous, we show that the validity of (1.2) for every f in $\mathrm{L}^1(G)$ gives a characterisation of the norm continuity of π.

In the case that G is abelian, we develop the "Fourier-Stieltjes transform" for $\Gamma_\pi(\mathrm{M}(G))$. The range of this transform is a Varopoulos type algebra $\mathrm{V}^b(E_\pi)$, which will be defined below. We use some general results on completely positive maps to characterise complete positivity of elements of $\mathrm{V}^b(E_\pi)$, as operators on $\mathcal{B}(\mathcal{H})$, extending some results from [24]. In particular, we characterise those μ in $\mathrm{M}(G)$ for which $\Gamma_\pi(\mu)$ is completely positive.

2. Spaces of Normal Completely Bounded Maps

Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{B}(\mathcal{H})$ be the space of bounded operators on $\mathcal{H}$ and let $\mathcal{V}$ and $\mathcal{W}$ be closed subspaces of $\mathcal{B}(\mathcal{H})$. The *Haagerup tensor product* $\mathcal{V} \otimes^h \mathcal{W}$ is defined in [13] and [6]. The *extended Haagerup tensor product* $\mathcal{V}\otimes^{eh}\mathcal{W}$ is developed in [9] and [7]; and also in [4], but in the context of dual spaces where it is called the "weak* Haagerup tensor product" and denoted $\mathcal{V} \otimes^{w^*h} \mathcal{W}$. It is shown in [22] that the approach of [4] can be modified to develop the extended Haagerup tensor product in general.

Following [22], we thus define $\mathcal{V}\otimes^{eh}\mathcal{W}$ to be the space of all (formal) series $\sum_{i\in I} v_i\otimes w_i$ where each $v_i \in \mathcal{V}$, each $w_i \in \mathcal{W}$, and each of the series $\sum_{i\in I} v_i v_i^*$ and $\sum_{i\in I} w_i^* w_i$ converges weak* in $\mathcal{B}(\mathcal{H})$. The index set I is established to have cardinality $|I| = \dim \mathcal{H}$. Two series $\sum_{i\in I} v_i \otimes w_i$ and $\sum_{i\in I} v_i' \otimes w_i'$ define the same element of $\mathcal{V} \otimes^{eh} \mathcal{W}$ provided $\sum_{i\in I} v_i x w_i = \sum_{i\in I} v_i' x w_i'$ for each x in $\mathcal{B}(\mathcal{H})$. Then $\mathcal{V} \otimes^{eh} \mathcal{W}$ is a Banach space when endowed with the norm

$$\|T\|_{eh} = \inf\left\{ \left\|\sum_{i\in I} v_i v_i^*\right\|^{1/2} \left\|\sum_{i\in I} w_i^* w_i\right\|^{1/2} : T = \sum_{i\in I} v_i \otimes w_i \right\}$$

and the infimum is attained. As in [4], note that the Haagerup tensor product $\mathcal{V} \otimes^h \mathcal{W}$ may be realized as the set of those T in $\mathcal{V} \otimes^{eh} \mathcal{W}$ which admit a representation $T = \sum_{i\in I} v_i \otimes w_i$ where $\sum_{i\in I} v_i v_i^*$ and $\sum_{i\in I} w_i^* w_i$ converge in norm. It is easy to see that any element T of $\mathcal{V} \otimes^h \mathcal{W}$ may thus be written with a countable index set as $T = \sum_{i=1}^{\infty} v_i \otimes w_i$.

The space $\mathcal{V} \otimes^{eh} \mathcal{W}$ has two natural, though more extrinsic descriptions. First, if $\mathcal{V}$ and $\mathcal{W}$ are each weak* closed subspaces of $\mathcal{B}(\mathcal{H})$, they have respective preduals $\mathcal{V}_*$ and $\mathcal{W}_*$. For example,

$$\mathcal{V}_* = \mathcal{B}(\mathcal{H})_*/\{\omega \in \mathcal{B}(\mathcal{H})_* : \omega(v) = 0 \text{ for all } v \text{ in } \mathcal{V}\}$$

which is an operator space when endowed with the quotient structure from the predual operator space structure on $\mathcal{B}(\mathcal{H})_*$. Then $\mathcal{V} \otimes^{eh} \mathcal{W}$ is the dual space of $\mathcal{V}_* \otimes^h \mathcal{W}_*$ via the pairing

$$\left\langle \sum_{i\in I} v_i \otimes w_i, \omega \otimes \nu \right\rangle = \sum_{i\in I} \omega(v_i)\nu(w_i). \tag{2.1}$$

A proof of this can be found in [4] or [9]. In particular, $\mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H}) \cong \left(\mathcal{B}(\mathcal{H})_* \otimes^h \mathcal{B}(\mathcal{H})_*\right)^*$.

Let $\mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H}))$ denote the space of normal completely bounded operators on $\mathcal{B}(\mathcal{H})$. The map $\theta : \mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H}) \to \mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H}))$ given by

$$\theta\left(\sum_{i\in I} v_i \otimes w_i\right) x = \sum_{i\in I} v_i x w_i, \text{ for } x \text{ in } \mathcal{B}(\mathcal{H})$$

is a surjective isometry by [13] or [21]. Moreover, θ is still an isometry when restricted to the spaces $\mathcal{V} \otimes^{eh} \mathcal{W}$ or $\mathcal{V} \otimes^h \mathcal{W}$. For notational ease we will simply identify $\mathcal{V} \otimes^{eh} \mathcal{W}$ and $\mathcal{V} \otimes^h \mathcal{W}$ as subspaces of $\mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H}))$ in the sequel, and omit the map θ. In particular, we view $\mathcal{B}(\mathcal{H}) \otimes^h \mathcal{B}(\mathcal{H})$ as being the completion in the completely bounded operator norm of the space of elementary operators $x \mapsto \sum_{i=1}^n v_i x w_i$ on $\mathcal{B}(\mathcal{H})$. The composition of operators in $\mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H}))$ induces a product in $\mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$, making it a Banach algebra. This product is given on elementary tensors by

$$(a \otimes b)\circ(c \otimes d) = ac \otimes db.$$

The following is an extension of a theorem from [2], whose proof is much like the one offered there.

Proposition 2.1. *If $\mathcal{A}$ and $\mathcal{B}$ are norm closed subalgebras of $\mathcal{B}(\mathcal{H})$, then $\mathcal{A} \otimes^{eh} \mathcal{B}$ is a subalgebra of $\mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$. If $\mathcal{V}$ is a (left) $\mathcal{A}$-module and $\mathcal{W}$ is a (right) $\mathcal{B}$-module in $\mathcal{B}(\mathcal{H})$, then $\mathcal{V} \otimes^{eh} \mathcal{W}$ is a (left) $\mathcal{A} \otimes^{eh} \mathcal{B}$-module in $\mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$.*

If $\Omega \in \mathcal{B}(\mathcal{H})^*$ then the *left* and *right slice maps* $L_\Omega, R_\Omega : \mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ are given for $T = \sum_{i\in I} v_i \otimes w_i$ by

$$L_\Omega T = \sum_{i\in I} \Omega(v_i) w_i \quad \text{and} \quad R_\Omega T = \sum_{i\in I} \Omega(w_i) v_i. \tag{2.2}$$

These series each converge in norm as is shown in [22, Thm. 2.2]. Moreover, it is shown there that for any pair of closed subspaces $\mathcal{V}$ and $\mathcal{W}$ of $\mathcal{B}(\mathcal{H})$, $\mathcal{V} \otimes^{eh} \mathcal{W}$ consists exactly of those T in $\mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$ for which $L_\Omega T \in \mathcal{W}$ and $R_\Omega T \in \mathcal{V}$ for each Ω in $\mathcal{B}(\mathcal{H})^*$ (or for which $L_\omega T \in \mathcal{W}$ and $R_\omega T \in \mathcal{V}$ for each ω in $\mathcal{B}(\mathcal{H})_*$). These results extend [21, Thm. 4.5].

We will finish this section with a theorem on completely positive maps which will be useful in Section 4. We will first need some general preliminary results which are modeled on results from [21].

A closed subalgebra $\mathcal{B}$ of $\mathcal{B}(\mathcal{H})$ is called *locally cyclic* if for each finite dimensional subspace $\mathcal{L}$ of $\mathcal{H}$, there is a vector ξ in $\mathcal{H}$ such that $\overline{\mathcal{B}\xi} \supset \mathcal{L}$. We note, for example, that if $\mathcal{B}$ is a maximal abelian self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ then it is locally cyclic. Indeed if $\xi_1, \dots, \xi_n$ span $\mathcal{L}$, consider the orthogonal projections $p_1, p_2, \dots, p_n$ whose respective ranges are

$$\overline{\mathcal{B}\xi_1},\ \overline{\mathcal{B}\xi_2} \ominus \overline{\mathcal{B}\xi_1},\ \dots,\ \overline{\mathcal{B}\xi_n} \ominus \bigoplus_{i=1}^{n-1} \overline{\mathcal{B}\xi_i}.$$

Then each $p_i \in \mathcal{B}' = \mathcal{B}$, and $\xi = \xi_1 + p_2\xi_2 + \cdots + p_n\xi_n$ satisfies $\overline{\mathcal{B}\xi} \supset \mathcal{L}$.

The following is an adaptation of [21, Thm. 2.1].

Lemma 2.2. *If $\mathcal{B}$ is a locally cyclic C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ and $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a positive map which is also a $\mathcal{B}$-bimodule map, then T is completely positive.*

Proof. Let us fix n, a positive matrix $[x_{ij}]$ in $\mathrm{M}_n(\mathcal{B}(\mathcal{H}))$ and a column vector $\boldsymbol{\xi} = [\xi_1 \cdots \xi_n]^{\mathrm{t}}$ in $\mathcal{H}^n$ with $\|\boldsymbol{\xi}\| < 1$. Then, given $\varepsilon > 0$, there is vector ξ in $\mathcal{H}$ and elements $b_1, \dots, b_n$ in $\mathcal{B}$ such that the vector $\boldsymbol{\eta} = [b_1\xi \cdots b_n\xi]^{\mathrm{t}}$ satisfies $\|\boldsymbol{\xi} - \boldsymbol{\eta}\| < \varepsilon$ and $\|\boldsymbol{\eta}\| < 1$. Leting $T^{(n)} : \mathrm{M}_n(\mathcal{B}(\mathcal{H})) \to \mathrm{M}_n(\mathcal{B}(\mathcal{H}))$ be the amplification of T, we have

$$\left\langle T^{(n)}[x_{ij}]\boldsymbol{\eta}|\boldsymbol{\eta}\right\rangle = \left\langle [Tx_{ij}]\begin{bmatrix} b_1\xi \\ \vdots \\ b_n\xi \end{bmatrix} \mid \begin{bmatrix} b_1\xi \\ \vdots \\ b_n\xi \end{bmatrix}\right\rangle$$

$$= \sum_{i,j=1}^{n} \langle b_i^* T(x_{ij}) b_j \xi|\xi\rangle = \left\langle T\left(\sum_{i,j=1}^{n} b_i^* x_{ij} b_j\right)\xi|\xi\right\rangle \geq 0$$

and

$$\left|\left\langle T^{(n)}[x_{ij}]\boldsymbol{\eta}|\boldsymbol{\eta}\right\rangle - \left\langle T^{(n)}[x_{ij}]\boldsymbol{\xi}|\boldsymbol{\xi}\right\rangle\right| < \left(\left\|T^{(n)}\right\| + 1\right)\varepsilon.$$

Since ε can be chosen arbitrarily small, we conclude that $\langle T^{(n)}[x_{ij}]\boldsymbol{\xi}|\boldsymbol{\xi}\rangle \geq 0$. Hence T is completely positive. □

If a family of operators $\{b_i\}_{i\in I}$ from $\mathcal{B}(\mathcal{H})$ defines a bounded row matrix $B = [\cdots b_i \cdots]$, i.e. $\sum_{i\in I} b_i b_i^*$ converges weak* in $\mathcal{B}(\mathcal{H})$, then the product $B\cdot\boldsymbol{\lambda} = \sum_{I\in I} \lambda_i b_i$ converges in norm and thus defines an element of $\mathcal{B}(\mathcal{H})$ for each $\boldsymbol{\lambda} = [\cdots \lambda_i \cdots]^{\mathrm{t}}$ in $\ell^2(I)$. We say that the set $\{b_i\}_{i\in I}$ is *strongly independent* if $B\cdot\boldsymbol{\lambda} = 0$ only when $\boldsymbol{\lambda} = 0$. This is an obvious extension of the usual notion of linear independence, and can be easily adapted to column matrices. Elements of $\mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$ admit many different representations, and strong independence was introduced in [21] to handle the difficulties caused by this.

The following is an adaptation of [21, Thm. 3.1].

Lemma 2.3. *If $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ and $T \in \mathcal{A} \otimes^{eh} \mathcal{A}$, then T is completely positive if and only if there is a strongly independent family $\{a_i\}_{i\in I}$ from $\mathcal{A}$ for which $\sum_{i\in I} a_i a_i^*$ converges weak* in $\mathcal{B}(\mathcal{H})$ and $T = \sum_{i\in I} a_i \otimes a_i^*$.*

Proof. We need only to prove that the first condition implies the second.

If T is completely positive and normal on $\mathcal{B}(\mathcal{H})$, then its restriction to the algebra of compact operators $T|_{\mathcal{K}(\mathcal{H})}$ is a completely positve map which determines T. Using Stinespring's theorem and the representation theory for $\mathcal{K}(\mathcal{H})$, just as in [21, Thm. 3.1] or [13], we obtain a family $\{b_j\}_{j\in J}$ from $\mathcal{B}(\mathcal{H})$ for which $\sum_{j\in J} b_j b_j^*$ converges weak* in $\mathcal{B}(\mathcal{H})$ and $T=\sum_{j\in J} b_j\otimes b_j^*$. We see that J can be any index set whose cardinality coincides with the Hilbertian dimension of $\mathcal{H}$. Let $B=[\cdots b_j\cdots]$.

Now we let

$$\mathcal{L}=\{\boldsymbol{\lambda}\in\ell^2(J): B\cdot\boldsymbol{\lambda}=0\}$$

and partition $J=I'\cup I$ in such a way that there is an orthonormal basis $\{\boldsymbol{\lambda}_j\}_{j\in J}$ of $\ell^2(J)$ for which

$$\overline{\text{span}}\{\boldsymbol{\lambda}_i\}_{i\in I'}=\mathcal{L}\quad\text{ and }\quad\overline{\text{span}}\{\boldsymbol{\lambda}_i\}_{i\in I}=\mathcal{L}^\perp.$$

Let U denote the $J\times J$ unitary matrix whose columns are the vectors $\{\boldsymbol{\lambda}_j\}_{j\in J}$. Let $A=[\cdots a_j\cdots]=B\cdot U$. Note that $a_j=0$ for each j in I'. Then for any x in $\mathcal{B}(\mathcal{H})$, letting x^J denote the $J\times J$ diagonal matrix which is the amplification of x, we have

$$Tx=\sum_{j\in J} b_j x b_j^* = Bx^J B^* = B\cdot Ux^J U^*\cdot B^* = Ax^J A^* = \sum_{i\in I} a_i x a_i^*.$$

We have that $\{a_i\}_{i\in I}$ is strongly independent, for if $\boldsymbol{\alpha}=[\cdots\alpha_i\cdots]^{\rm t}$ in $\ell^2(I)$ is such that $A\cdot\boldsymbol{\alpha}=0$, then

$$0=A\cdot\boldsymbol{\alpha}=\sum_{i\in I}\alpha_i a_i=\sum_{i\in I}\alpha_i B\cdot\boldsymbol{\lambda}_i = B\cdot\left(\sum_{i\in I}\alpha_i\boldsymbol{\lambda}_i\right)$$

so $\sum_{i\in I}\alpha_i\boldsymbol{\lambda}_i\in\mathcal{L}\cap\mathcal{L}^\perp$, whence $\boldsymbol{\alpha}=0$. Hence

$$T=\sum_{i\in I} a_i\otimes a_i^*$$

where $\{a_i\}_{i\in I}$ is strongly independent. It remains to show that $\{a_i\}_{i\in I}\subset\mathcal{A}$.

Since $\{a_i\}_{i\in I}$ is strongly independent, so too is $\{a_i^*\}_{i\in I}$. Hence by [1, Lem. 2.2], the space

$$\left\{[\cdots\Omega(a_i^*)\cdots]^{\rm t}:\Omega\in\mathcal{B}(\mathcal{H})^*\right\}$$

is dense in $\ell^2(I)$. Thus, given a fixed index i_0 in I, there is a (not necessarily bounded) sequence $\{\Omega_n\}_{n=1}^{\infty}$ from $\mathcal{B}(\mathcal{H})^*$ such that

$$a_{i_0} = \lim_{n\to\infty} \sum_{i\in I} \Omega_n(a_i^*)a_i = \lim_{n\to\infty} R_{\Omega_n}T.$$

Since $R_\Omega T \in \mathcal{A}$ for each right slice map R_Ω, it follows that $a_{i_0} \in \mathcal{A}$. □

If E is any locally compact space we let

$$\begin{aligned} \mathrm{V}_0(E) &= \mathcal{C}_0(E) \otimes^h \mathcal{C}_0(E) \\ \mathrm{V}^0(E) &= \mathcal{C}_0(E) \otimes^{eh} \mathcal{C}_0(E) \\ \text{and } \mathrm{V}^b(E) &= \mathcal{C}_b(E) \otimes^{eh} \mathcal{C}_b(E). \end{aligned} \tag{2.3}$$

These spaces are discussed in [22]. These all may be regarded as Banach algebras of functions on $E\times E$ by Proposition 2.1. However, as pointed out in [20], an element u of $\mathrm{V}^b(E)$ may not be continuous on $E\times E$, even if E is compact. Hovever, if $\mathcal{C}$ is a closed subalgebra of $\mathcal{C}_b(E)$ (say $\mathcal{C} = \mathcal{C}_0(E)$), then for each $u \in \mathcal{C} \otimes^{eh} \mathcal{C} \subset \mathrm{V}^b(E)$, the pointwise slices, $u(\cdot, x)$ and $u(x, \cdot)$ for any fixed x in E, will always be elements of $\mathcal{C}$. In the case where E is a compact group, $\mathrm{V}_0(E)$ is discussed in [23], and in a profound way in [25]. We note that by Grothendieck's Inequality, $\mathrm{V}_0(E) = \mathcal{C}_0(E) \otimes^\gamma \mathcal{C}_0(E)$ (projective tensor product), up to equivalent norms.

If $u : E\times E \to \mathbb{C}$, we say that u is *positive definite* if for any finite collection of elements $x_1, \ldots, x_n$ from E, the matrix $[u(x_i, x_j)]$ is of positive type.

If $\mathcal{A}$ is any abelian C*-algebra for which there is a locally compact space E and an injective $*$-homomorphism $F : \mathcal{A} \to \mathcal{C}_b(E)$, then there is an isometric algebra homomorphism $F \otimes F : \mathcal{A} \otimes^{eh} \mathcal{A} \to \mathrm{V}^b(E)$, by [9] or [22, Cor. 2.3].

The following theorem generalises [24, Thm. 5.1].

Theorem 2.4. *Let $\mathcal{A}$ be an abelian C*-subalgebra of $\mathcal{B}(\mathcal{H})$ for which there is a locally compact space E and an injective $*$-homomorphism $F : \mathcal{A} \to \mathcal{C}_b(E)$. If $T \in \mathcal{A} \otimes^{eh} \mathcal{A}$ and $u = (F \otimes F)T$, so $u \in F(\mathcal{A}) \otimes^{eh} F(\mathcal{A}) \subset \mathrm{V}^b(E)$, then the following are equivalent:*

(i) *T is positive.*

(ii) *T is completely positive.*

(iii) *u is positive definite.*

Proof. (i)⇒(ii) If $\mathcal{B}$ is any maximal abelian subalgebra of $\mathcal{B}(\mathcal{H})$ which contains $\mathcal{A}$, then T is a $\mathcal{B}$-bimodule map. The result then follows from Lemma 2.2.

(ii)⇒(iii) By Lemma 2.3 we have that $T = \sum_{i\in I} a_i \otimes a_i^*$ for some family of elements from $\mathcal{A}$ for which $\sum_{i\in I} a_i a_i^*$ converges weak* in $\mathcal{B}(\mathcal{H})$. Let $\varphi_i = F(a_i)$ in $\mathcal{C}_b(E)$, so

$$u = \sum_{i\in I} \varphi_i \otimes \bar{\varphi}_i \quad \text{and} \quad \left\| \sum_{i\in I} |\varphi_i|^2 \right\|_\infty < \infty.$$

Let $\xi : E \to \ell^2(I)$ be given by $\xi(x) = \big(\varphi_i(x)\big)_{i\in I}$. Then for each (x, y) in $E\times E$, we have that

$$u(x, y) = \langle \xi(x)|\xi(y)\rangle \tag{2.4}$$

and hence u is positive definite.

(iii)⇒(i) Since u is positive definite function, then by [12, §3.1], there is a Hilbert space $\mathcal{L}$ and a bounded function $\xi : E \to \mathcal{L}$ such that (2.4) holds. Let p be the orthogonal projection on $\mathcal{L}$ whose range is $\overline{\text{span}}\{\xi(x)\}_{x\in E}$, and let $\{\xi_i\}_{i\in I}$ be an orthonormal basis for $p\mathcal{L}$. Then for each i the function

$$\varphi_i = \langle \xi(\cdot)|\xi_i\rangle$$

is in $F(\mathcal{A})$. Indeed, given $\varepsilon > 0$ we can find $\alpha_1, \dots, \alpha_n$ from $\mathbb{C}$ and $y_1, \dots, y_n$ from E, such that

$$\left\| \xi_i - \sum_{k=1}^n \alpha_k \xi(y_k) \right\| < \varepsilon$$

whence

$$\left\| \varphi_i - \sum_{k=1}^n \bar{\alpha}_k u(\cdot, y_k) \right\|_\infty = \left\| \langle \xi(\cdot)|\xi_i\rangle - \sum_{k=1}^n \bar{\alpha}_k \langle \xi(\cdot)|\xi(y_k)\rangle \right\|_\infty < \|\xi\|_\infty \varepsilon.$$

Hence φ_i can be uniformly approximated arbitrarily closely by elements of $F(\mathcal{A})$, and our conclusion holds. It then follows by Parseval's Identity that for any (x,y) in $E\times E$

$$u(x,y) = \langle p\xi(x)|p\xi(y)\rangle = \sum_{i\in I} \langle \xi(x)|\xi_i\rangle \langle \xi_i|\xi(y)\rangle = \sum_{i\in I} \varphi_i(x)\overline{\varphi_i(y)}.$$

Hence we may write

$$u = \sum_{i\in I} \varphi_i \otimes \bar{\varphi}_i \quad \text{with} \quad \left\| \sum_{i\in I} |\varphi_i|^2 \right\|_\infty = \|\xi\|_\infty^2 < \infty.$$

Letting $a_i = F^{-1}(\varphi_i)$ in $\mathcal{A}$, we get that $T = (F\otimes F)^{-1}u = \sum_{i\in I} a_i \otimes a_i^*$ and is thus positive. $\square$

3. Representations of Groups in Completely Bounded Maps

Let G be a locally compact group, let $\mathcal{A}$ be a unital Banach algebra which is also a dual space with predual $\mathcal{A}_*$, and let $\alpha : G \to \mathcal{A}_{\rm inv}$ be a weak* continuous bounded homomorphism where $\mathcal{A}_{\rm inv}$ denotes the group of invertible elements in $\mathcal{A}$. In particular we assume $\alpha(e)$ is the unit of $\mathcal{A}$ Denote the space of bounded complex Borel measures on G by $\mathrm{M}(G)$. Recall that $\mathrm{M}(G)$ is the dual space to the space $\mathcal{C}_0(G)$ of continuous functions vanishing at infinity. Recall too that $\mathrm{M}(G)$ is a Banach algebra via *convolution*: for each μ, ν in $\mathrm{M}(G)$ we define $\mu{*}\nu$ by

$$\int_G \varphi d\mu{*}\nu = \int_G \int_G \varphi(st) d\mu(s) d\nu(t) \tag{3.1}$$

for each φ in $\mathcal{C}_0(G)$. We note that since each of μ and ν can be approximated in norm by compactly supported bounded measures, (3.1) holds for any choice of φ in $\mathcal{C}_b(G)$ too. If $\mu \in \mathrm{M}(G)$, let

$$\alpha_1(\mu) = \text{weak*-}\int_G \alpha(s) d\mu(s)$$

i.e. if $\omega \in \mathcal{A}_*$, then $\langle \alpha_1(\mu), \omega\rangle = \int_G \langle \alpha(s), \omega\rangle \, d\mu(s)$. Then $\alpha_1 : \mathrm{M}(G) \to \mathcal{A}$ is a bounded linear map for if $\|\alpha\|_\infty = \sup_{s\in G} \|\alpha(s)\|$, then

$$\|\alpha_1(\mu)\| = \sup_{\omega \in \mathrm{b}_1(\mathcal{A}_*)} \left| \int_G \langle \alpha(s), \omega\rangle \, d\mu(s) \right| \leq \int_G \|\alpha\|_\infty \, d|\mu|(s) = \|\alpha\|_\infty \, \|\mu\|_1 . \tag{3.2}$$

Recall that the dual $\mathcal{A}^*$ is a contractive $\mathcal{A}$-bimodule where for b in $\mathcal{A}$ and F in $\mathcal{A}^*$ we define $b{\cdot}F$ and $F{\cdot}b$ in $\mathcal{A}^*$ by $\langle a, b{\cdot}F\rangle = \langle ab, F\rangle$ and $\langle a, F{\cdot}b\rangle = \langle ba, F\rangle$, for each a in $\mathcal{A}$. We say that a subspace Ω of $\mathcal{A}^*$ is a right $\alpha(G)$-submodule if $\omega{\cdot}\alpha(s) \in \Omega$, for each ω in Ω and s in G.

Proposition 3.1. *Let G, $\mathcal{A}$ and α be as above. Moreover, suppose that $\mathcal{A}_*$ is both a left $\mathcal{A}$-submodule of $\mathcal{A}^*$ and a right $\alpha(G)$-submodule. Then $\alpha_1 : \mathrm{M}(G) \to \mathcal{A}$ is a unital algebra homomorphism.*

Proof. If $\mu, \nu \in \mathrm{M}(G)$ and $\omega \in \mathcal{A}^*$ then

$$\begin{aligned}\langle \alpha_1(\mu)\alpha_1(\nu), \omega\rangle &= \langle \alpha_1(\mu), \alpha_1(\nu)\cdot\omega\rangle \\ &= \int_G \langle \alpha(s), \alpha_1(\nu)\cdot\omega\rangle \, d\mu(s) \\ &= \int_G \langle \alpha_1(\nu), \omega\cdot\alpha(s)\rangle \, d\mu(s) \\ &= \int_G \int_G \langle \alpha(t), \omega\cdot\alpha(s)\rangle \, d\nu(t) d\mu(s) \\ &= \int_G \int_G \langle \alpha(st), \omega\rangle \, d\nu(t) d\mu(s).\end{aligned}$$

where the hypotheses guarantee that $\alpha_1(\nu)\cdot\omega \in \mathcal{A}_*$ and that $\omega\cdot\alpha(s) \in \mathcal{A}_*$, for each s. By Fubini's Theorem we have that

$$\int_G \int_G \langle \alpha(st), \omega\rangle \, d\nu(t) d\mu(s) = \int_G \int_G \langle \alpha(st), \omega\rangle \, d\mu(s) d\nu(t) = \langle \alpha_1(\mu * \nu), \omega\rangle$$

where we note that $(s,t) \mapsto \langle \alpha(st), \omega\rangle$ is continuous and bounded, hence $\mu\times\nu$-integrable.

That $\alpha_1(\delta_e) = \alpha(e)$ follows from that $\mathcal{A}_*$ is a separating for $\mathcal{A}$. Hence α_1 is a unital map. □

By a symmetric argument, the above proposition also holds if $\mathcal{A}_*$ is assumed to be both a right $\mathcal{A}$-submodule of $\mathcal{A}^*$ and a left $\alpha(G)$-submodule.

Example 3.2. **(i)** Let $\mathcal{X}$ be a Banach space admitting a predual $\mathcal{X}_*$. Then we have that $\mathcal{A} = \mathcal{B}(\mathcal{X})$ is a dual unital Banach algebra admitting a predual $\mathcal{A}_* = \mathcal{X} \otimes^\gamma \mathcal{X}_*$, via the dual pairing

$$\langle T, x \otimes \omega\rangle = \langle Tx, \omega\rangle \text{ for } T \text{ in } \mathcal{A},\ x \text{ in } \mathcal{X} \text{ and } \omega \text{ in } \mathcal{X}_*.$$

Here $\otimes^\gamma$ denotes the *projective tensor product*. We have then that $\mathcal{A}_*$ is a left $\mathcal{A}$ submodule of $\mathcal{A}^*$. Indeed, for any S, T in $\mathcal{A}$ and elementary tensor $x \otimes \omega$ in $\mathcal{A}_*$ we have that,

$$\langle ST, x \otimes \omega\rangle = \langle STx, \omega\rangle = \langle S, (Tx) \otimes \omega\rangle$$

so $T\cdot(x \otimes \omega) = (Tx) \otimes \omega$.

If $\mathcal{B}^\sigma(\mathcal{X})$ denotes the weak*-weak* continuous bounded linear maps on $\mathcal{X}$ then $\mathcal{A}_*$ is a right $\mathcal{B}^\sigma(\mathcal{X})$-submodule of $\mathcal{A}^*$. Thus we obtain the situation of Proposition 3.1 whenever $\alpha : G \to \mathcal{A}_{\mathrm{inv}}$ is a weak* continuous bounded homomorphism whose range is in $\mathcal{B}^\sigma(\mathcal{X})$. In particular, this happens when $\mathcal{X}$ is reflexive and α is a non-degenerate strong operator continuous representation on $\mathcal{X}$.

(ii) The example above can be easily modified for the case where $\mathcal{V}$ is a dual operator space and $\mathcal{A} = \mathcal{CB}(\mathcal{V})$.

(iii) There is a standard identification $\mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H})) \cong \mathcal{CB}(\mathcal{K}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$, and thus an identification of $\mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H}) \cong \mathcal{CB}(\mathcal{K}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$. In fact, as shown in [4], this latter identification is a weak* homeomorphism. Indeed, using standard identifications with row and column Hilbert spaces and the *operator projective tensor product*, $\hat{\otimes}$ (see [3] or [8, II.9.3]), we have

$$\begin{aligned}\mathcal{B}(\mathcal{H})_* \otimes^h \mathcal{B}(\mathcal{H})_* &\cong \left(\overline{\mathcal{H}}_r \otimes^h \mathcal{H}_c\right) \otimes^h \left(\overline{\mathcal{H}}_r \otimes^h \mathcal{H}_c\right)\\ &\cong \overline{\mathcal{H}}_r \otimes^h \left(\mathcal{H}_c \otimes^h \overline{\mathcal{H}}_r\right) \otimes^h \mathcal{H}_c \cong \overline{\mathcal{H}}_r \hat{\otimes} \mathcal{K}(\mathcal{H}) \hat{\otimes} \mathcal{H}_c\\ &\cong \mathcal{K}(\mathcal{H}) \hat{\otimes} \overline{\mathcal{H}}_r \hat{\otimes} \mathcal{H}_c \cong \mathcal{K}(\mathcal{H}) \hat{\otimes} \mathcal{B}(\mathcal{H})_*.\end{aligned}$$

On elementary tensors this identification is given by

$$(\xi^* \otimes \eta) \otimes (\zeta^* \otimes \vartheta) \mapsto (\eta \otimes \zeta^*) \otimes (\xi^* \otimes \vartheta)$$

where for vectors ξ, η in $\mathcal{H}$ we let $\xi \otimes \eta^*$ denote the usual rank 1 operator and $\xi^* \otimes \eta$ the usual vector functional. Now if $T = \sum_{i\in I} a_i \otimes b_i$ in $\mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$ then, in the dual pairing (2.1), we have that

$$\langle T, (\xi^* \otimes \eta) \otimes (\zeta^* \otimes \vartheta)\rangle = \sum_{i\in I} \langle a_i\eta|\xi\rangle \langle b_i\vartheta|\zeta\rangle .$$

Meanwhile, in the $\mathcal{CB}(\mathcal{K}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$–$\mathcal{K}(\mathcal{H}) \hat{\otimes} \mathcal{B}(\mathcal{H})_*$ duality we have that

$$\begin{aligned}\langle T, (\eta \otimes \zeta^*) \otimes (\xi^* \otimes \vartheta)\rangle &= \left\langle \sum_{i\in I} a_i\eta \otimes (b_i^*\zeta)^*, \xi^* \otimes \vartheta \right\rangle\\ &= \sum_{i\in I} \langle a_i\eta|\xi\rangle \langle b_i\vartheta|\zeta\rangle .\end{aligned}$$

Now for every elementary tensor $k\otimes\omega$ in $\mathcal{K}(\mathcal{H})\hat{\otimes}\mathcal{B}(\mathcal{H})_*$ and T in $\mathcal{B}(\mathcal{H})\otimes^{eh}\mathcal{B}(\mathcal{H})$, we have that $(k\otimes\omega)\cdot T = k\otimes(\omega\cdot T) \in \mathcal{K}(\mathcal{H})\hat{\otimes}\mathcal{B}(\mathcal{H})_*$. Hence $\mathcal{B}(\mathcal{H})_*\otimes^h\mathcal{B}(\mathcal{H})_* \cong \mathcal{K}(\mathcal{H})\hat{\otimes}\mathcal{B}(\mathcal{H})_*$ is a right module for $\mathcal{B}(\mathcal{H})\otimes^{eh}\mathcal{B}(\mathcal{H})$. We note that $\mathcal{B}(\mathcal{H})_*\otimes^h\mathcal{B}(\mathcal{H})_*$ is a left $\mathcal{B}(\mathcal{H})\otimes^h\mathcal{B}(\mathcal{H})$-module. □

We will identify the group algebra $\mathrm{L}^1(G)$ with the closed ideal in $\mathrm{M}(G)$ of measures which are absolutely continuous with respect to the left Haar measure m (whose integral we will denote $\int_G\cdots ds$). We will identify the discrete group algebra $\ell^1(G)$ with the closed subspace of $\mathrm{M}(G)$ generated by all of the Dirac measures $\{\delta_s : s\in G\}$. We let

$$\mathrm{M}_\alpha = \overline{\alpha_1(\mathrm{M}(G))},\quad \mathrm{C}_\alpha = \overline{\alpha_1(\mathrm{L}^1(G))}\quad\text{and}\quad \mathrm{D}_\alpha = \overline{\alpha_1(\ell^1(G))}$$

where each of the closures is in the norm topology of $\mathcal{A}$.

The following proposition is surely well-known, though we have been unable to find it in the literature.

Proposition 3.3. *Given G, $\mathcal{A}$ and α satifying the hypotheses of Proposition 3.1, α is norm continuous if and only if C_α is unital.*

Proof. Let (e_U) be the bounded approximate identity for $\mathrm{L}^1(G)$ given by $e_U = \frac{1}{m(U)}1_U$ (normalised indicator function), indexed over the family of all relatively compact neighbourhoods of the identity e in G, partially ordered by reverse inclusion.

"$\Rightarrow$" Let $\varepsilon>0$. Let V be any relatively compact neighbourhood of e for which $\|\alpha(s)-\alpha(e)\|<\varepsilon$ for each s in V. Then for any relatively compact neighbourhood U of e which is contained in V we have

$$\begin{aligned}\|\alpha_1(e_U)-\alpha(e)\| &= \left\|\frac{1}{m(U)}\int_U \alpha(s)ds-\alpha(e)\right\|\\ &\le \frac{1}{m(U)}\int_U \|\alpha(s)-\alpha(e)\|\,ds<\varepsilon\end{aligned}$$

where the second from last inequality is proved just as in (3.2). Thus $\alpha(e) = \lim_U \alpha_1(e_U)$ in norm, so $\alpha(e)\in\mathrm{C}_\alpha$. Now $\alpha(e)$ is the unit for $\mathcal{A}$, and hence the unit for C_α.

" $\Leftarrow$ " It is a standard fact that $\lim_U \alpha_1(e_U) = \alpha(e)$ in the weak* topology of $\mathcal{A}$. Indeed, $\lim_U \int_G e_U(s)\varphi(s)ds = \varphi(e)$ for any continuous function φ; set $\varphi = \langle\alpha(\cdot),\omega\rangle$ for any ω in $\mathcal{A}_*$. Now let E be the unit for C_α. We will establish that $E = \alpha(e)$, the unit of $\mathcal{A}$. First, the map $s \mapsto \alpha(s)E$ is norm continuous. Indeed $E \in \mathrm{C}_\alpha$ and can thus be norm approximated by $\{\alpha_1(f) : f \in \mathrm{L}^1(G)\}$. Moreover, if $f \in \mathrm{L}^1(G)$ then we have that

$$\|\alpha(s)\alpha_1(f) - \alpha_1(f)\| = \|\alpha_1(\delta_s * f - f)\| \leq \|\alpha\|_\infty \|\delta_s * f - f\|_1 \xrightarrow{s\to e} 0$$

where the inequality follows from (3.2) and limit follows from [15, 20.4]. Next, for any compact neighbourhood U of e we have that

$$\alpha_1(e_U) = \alpha_1(e_U)E = \frac{1}{m(U)}\int_U \alpha(s)ds\cdot E = \frac{1}{m(U)}\int_U \alpha(s)Eds$$

where we note that right multiplication is weak*-continuous in $\mathcal{A}$, by hypothesis. Now, let $\varepsilon > 0$ be given, and find a neighbourhood V of e in G such that $\|\alpha(s)E - E\| < \varepsilon$ for each s in V. Then for any relatively compact neighbourhood U of E, contained in V, we have that

$$\|\alpha_1(e_U) - E\| = \left\|\frac{1}{m(U)}\int_U \alpha(s)Eds - E\right\| \leq \frac{1}{m(U)}\int_U \|\alpha(s)E - E\|\,ds < \varepsilon$$

where the second from last inequality is proved just as in (3.2). Hence we have that $\lim_U \alpha_1(e_U) = E$ in norm, so, *a fortiori*, weak*-$\lim_U \alpha_1(e_U) = E$. It then follows from above that $E = \alpha(e)$, so $\alpha(e) \in \mathrm{C}_\alpha$. Thus

$$\|\alpha(s) - \alpha(e)\| = \|\alpha(s)E - E\| \xrightarrow{s\to e} 0.$$

Hence α is norm continuous at e, and thus norm continuous on all of G. □

Corollary 3.4. *For G, $\mathcal{A}$ and α as above, the following are equivalent:*

(i) *α is norm continuous* **(ii)** $\mathrm{C}_\alpha = \mathrm{M}_\alpha$ **(iii)** $\mathrm{C}_\alpha = \mathrm{D}_\alpha$.

Proof. (i)$\Leftrightarrow$(ii) If α is norm continuous, then C_α contains the unit $\alpha(e)$ by Proposition 3.3. Hence, C_α is an ideal in M_α, containing the unit. Conversely, if $\mathrm{C}_\alpha = \mathrm{M}_\alpha$ then C_α is unital, and norm continuity of α follows from Proposition 3.3.

(i)⇒(iii) Since (ii) holds, the inclusion $\mathrm{C}_\alpha \supset \mathrm{D}_\alpha$ is clear. To obtain the opposite inclusion, note that for any continuous function of compact support φ – the family of which is dense in $\mathrm{L}^1(G)$ – the function $s \mapsto \varphi(s)\alpha(s)$, from G to D_α, can be uniformly approximated by Borel simple functions. Hence $\alpha_1(\varphi) = \int_G \varphi(s)\alpha(s)ds$ may be regarded as a Bochner integral, and is thus in D_α, since each $\alpha(s) \in \mathrm{D}_\alpha$. It then follows that $\alpha_1(\mathrm{L}^1(G)) \subset \mathrm{D}_\alpha$ and hence $\mathrm{C}_\alpha \subset \mathrm{D}_\alpha$.

(iii)⇒(i) Since $\mathrm{C}_\alpha \supset \mathrm{D}_\alpha$, C_α is unital, and the result follows from Proposition 3.3. □

Now suppose that $\pi : G \to \mathcal{U}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})_{\mathrm{inv}}$ is a strongly continuous unitary representation (which is equivalent to it being weak* continuous). We will define $\pi_1 : \mathrm{M}(G) \to \mathcal{B}(\mathcal{H})$ as above, but will use the notation

$$\mathrm{M}^*_\pi = \overline{\pi_1(\mathrm{M}(G))}, \quad \mathrm{C}^*_\pi = \overline{\pi_1(\mathrm{L}^1(G))} \quad \text{and} \quad \mathrm{D}^*_\pi = \overline{\pi_1(\ell^1(G))}$$

to indicate that these are C*-algebras. Using von Neumann's double commutant theorem, we have that C^*_π and D^*_π each generate the same von Neumann algebra, VN_π. We note that $\mathrm{M}^*_\pi \subset \mathrm{VN}_\pi$ but $\mathrm{M}^*_\pi \neq \mathrm{VN}_\pi$ in general. Thus, in particular, there is no reason to suspect that M^*_π is a dual space.

Proposition 3.5. *If $\mu \in \mathrm{M}(G)$, then*

$$\Gamma_\pi(\mu) = \int_G \pi(s) \otimes \pi(s)^* d\mu(s) \tag{3.3}$$

defines an element of $\mathcal{B}(\mathcal{H}) \otimes^{w^*h} \mathcal{B}(\mathcal{H})$, and the integral converges in the weak topology, i.e. for each x in $\mathcal{B}(\mathcal{H})_* \otimes^h \mathcal{B}(\mathcal{H})_*$,*

$$\langle \Gamma_\pi(\mu), x\rangle = \int_G \langle \pi(s) \otimes \pi(s)^*, x\rangle \, d\mu(s).$$

Moreover,

(i) *$\Gamma_\pi : \mathrm{M}(G) \to \mathcal{B}(\mathcal{H}) \otimes^{w^*h} \mathcal{B}(\mathcal{H})$ is a contractive homomorphism whose range is contained in the algebra $\mathrm{M}^*_\pi \otimes^{eh} \mathrm{M}^*_\pi$.*

(ii) *$\Gamma_\pi(\mathrm{L}^1(G)) \subset \mathrm{C}^*_\pi \otimes^{eh} \mathrm{C}^*_\pi$.*

(iii) *$\Gamma_\pi(\ell^1(G)) \subset \mathrm{D}^*_\pi \otimes^h \mathrm{D}^*_\pi$.*

(iv) *If π is norm continuous, then* $\Gamma_\pi(\mathrm{M}(G)) \subset \mathrm{D}_\pi^* \otimes^h \mathrm{D}_\pi^*$.

Proof. (i) First, let us see that, for each μ in $\mathrm{M}(G)$, the integral in (3.3) converges as claimed. This amounts to verifying that $s \mapsto \pi(s) \otimes \pi(s)^*$ is a weak* continuous representation from G into $(\mathcal{B}(\mathcal{H}) \otimes^{w^*h} \mathcal{B}(\mathcal{H}))_{\mathrm{inv}}$, i.e. that $s \mapsto \langle \pi(s) \otimes \pi(s)^*, x\rangle$ is continuous for each x in $\mathcal{B}(\mathcal{H})_* \otimes^h \mathcal{B}(\mathcal{H})_*$, by (2.1). If $x \in \mathcal{B}(\mathcal{H})_* \otimes^h \mathcal{B}(\mathcal{H})_*$ and $\varepsilon > 0$, then there is x_ε in $\mathcal{B}(\mathcal{H})_* \otimes \mathcal{B}(\mathcal{H})_*$ such that $\|x - x_\varepsilon\|_h < \varepsilon$. The function $x_{\varepsilon,\pi}$, given by $s \mapsto \langle \pi(s) \otimes \pi(s)^*, x_\varepsilon\rangle$, is clearly continuous on G, and $\|x_\pi - x_{\varepsilon,\pi}\|_\infty \leq \|x - x_\varepsilon\|_h < \varepsilon$. Thus, taking choices of ε tending to 0, we see that x_π is a continuous function on G.

Since $\|\pi(s) \otimes \pi(s)^*\|_{w^*h} = 1$ for each s in G, the contractivity of Γ_π follows from (3.2). That Γ_π is a homomorphism follows from Proposition 3.1 and Example 3.2 (iii).

To see that $\Gamma_\pi(\mu) \in \mathrm{M}_\pi^* \otimes^{eh} \mathrm{M}_\pi^*$, for any given μ in $\mathrm{M}(G)$, we will inspect the image of a typical weak*-weak* continuous left slice map on $\Gamma_\pi(\mu)$ and use [22, Thm. 2.2]. If $\omega \in \mathcal{B}(\mathcal{H})_*$, then

$$(3.4) \qquad L_\omega\left(\Gamma_\pi(\mu)\right) = \int_G \langle \pi(s), \omega\rangle\, \pi(s)^* d\mu(s) = \int_G \pi(s) d(\omega_\pi \mu)^\vee(s) \in \mathrm{M}_\pi^*$$

where $\omega_\pi \mu$ is the measure with Radon derivative $d(\omega_\pi\mu)/d\mu = \omega_\pi$ (here $\omega_\pi(s) = \langle \pi(s), \omega\rangle$), and $\nu^\vee(E) = \nu(E^{-1}) = \overline{\nu^*(E)}$ for any Borel measure ν. The computation for any right slice map is similar.

(ii) This follows from a computation similar to (3.4).

(iii) If $\mu = \sum_{s\in G} \alpha(s)\delta_s$, where $\sum_{s\in G} |\alpha(s)| < \infty$, then since $\pi(s) \otimes \pi(s)^* \in \mathrm{D}_\pi^* \otimes^h \mathrm{D}_\pi^*$ for each s in G, it follows too that

$$\Gamma_\pi(\mu) = \sum_{s\in G} \alpha(s)\pi(s) \otimes \pi(s)^* \in \mathrm{D}_\pi^* \otimes^h \mathrm{D}_\pi^*.$$

(iv) If we let $\alpha : G \to (\mathcal{B}(\mathcal{H}) \otimes^{w^*h} \mathcal{B}(\mathcal{H}))_{\mathrm{inv}}$ be given by $\alpha(s) = \pi(s) \otimes \pi(s)^*$, then α is norm continuous. Hence

$$\Gamma_\pi(\mathrm{M}(G)) \subset \mathrm{M}_\alpha = \mathrm{D}_\alpha \subset \mathrm{D}_\pi^* \otimes^h \mathrm{D}_\pi^*$$

by (iii) above and Corollary 3.4. □

Remark 3.6. We note that (3.3) also converges in the $\mathcal{CB}(\mathcal{B}(\mathcal{H}))$–$(\mathcal{B}(\mathcal{H})\widehat{\otimes}\mathcal{B}(\mathcal{H})_*)$ topology. Indeed, if $a \in \mathcal{B}(\mathcal{H})$ and $\eta^* \otimes \xi$ is a vector functional in $\mathcal{B}(\mathcal{H})_*$, then for any s in G we have that

$$\langle \pi(s) \otimes \pi(s)^*, a \otimes (\eta^* \otimes \xi)\rangle = \langle \pi(s)a\pi(s)^*, \eta^* \otimes \xi\rangle = \langle a, (\pi(s)^*\eta)^* \otimes \pi(s)^*\xi\rangle$$

where $s \mapsto (\pi(s)^*\eta)^* \otimes \pi(s)^*\xi$ is continuous in the norm topology of $\mathcal{B}(\mathcal{H})_*$. Hence $s \mapsto \langle \pi(s) \otimes \pi(s)^*, a \otimes (\eta^* \otimes \xi)\rangle$ is continuous. In particular, for each a in $\mathcal{B}(\mathcal{H})$ and μ in $\mathrm{M}(G)$ we have that

$$\Gamma_\pi(\mu)a = \int_G \pi(s)a\pi(s)^* d\mu(s)$$

where the integral converges in the weak* topology of $\mathcal{B}(\mathcal{H})$

We observe that it is possible, for each μ in $\mathrm{M}(G)$, to see $\Gamma_\pi(\mu)|_{\mathcal{K}(\mathcal{H})}$ as an integral converging in the point-norm topology. However, our approach for obtaining (3.3) better lends itself to (4.4). □

We let the *augmentation ideal* in $\mathrm{L}^1(G)$ be given by

$$\mathrm{I}_0(G) = \left\{ f \in \mathrm{L}^1(G) : \int_G f(s)ds = 0 \right\}.$$

Theorem 3.7. *For any strongly continuous representation $\pi : G \to \mathcal{U}(\mathcal{H})$, the following are equivalent:*

(i) *π is norm continuous.*

(ii) $\Gamma_\pi(\mathrm{L}^1(G)) \subset \mathrm{C}^*_\pi \otimes^h \mathrm{C}^*_\pi$.

(iii) *there is an f in $\mathrm{L}^1(G) \setminus \mathrm{I}_0(G)$ such that $\Gamma_\pi(f) \in \mathrm{C}^*_\pi \otimes^h \mathrm{C}^*_\pi$.*

Proof. That (i) implies (ii) follows from Proposition 3.5 (iv) and the fact that $\mathrm{C}^*_\pi = \mathrm{D}^*_\pi$. That (ii) implies (iii) is trivial. Suppose now that f satisfies statement (iii). Without loss of generality, we may suppose that $\int_G f(s)ds = 1$. Then by [4], there exist sequences $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}}$ from C^*_π such that $\sum_{i=1}^\infty a_i a_i^*$ and $\sum_{i=1}^\infty b_i^* b_i$ converge in norm, and

$$\Gamma_\pi(f)x = \sum_{i=1}^\infty a_i x b_i$$

for each $x \in \mathcal{B}(\mathcal{H})$. But it then follows from Remark 3.6 that

$$I = \int_G f(s)\pi(s)I\pi(s)^* ds = \Gamma_\pi(f)I = \sum_{i=1}^{\infty} a_i b_i \in \mathrm{C}^*_\pi.$$

Hence π is norm continuous by Proposition 3.3. □

In the next section, we will address the necessity of the assumption that $f \in \mathrm{L}^1(G) \setminus \mathrm{I}_0(G)$ in (iii) above.

It is interesting to note that the kernel of Γ_π is related to the kernel of a more familiar representation. Below, we will let $\overline{\mathcal{H}}$ denote the conjugate Hilbert space and $\bar{\pi} : G \to \mathcal{U}(\overline{\mathcal{H}})$ denote the conjugate representation. We will also let $\pi \otimes \bar{\pi} : G \to \mathcal{U}(\mathcal{H} \otimes_2 \overline{\mathcal{H}})$ be the usual tensor product of representations on the Hilbert space $\mathcal{H} \otimes_2 \overline{\mathcal{H}}$.

Proposition 3.8. $\ker \Gamma_\pi = \ker(\pi \otimes \bar{\pi})_1$.

Proof. We have that $\mu \in \ker \Gamma_\pi$ if and only if

$$0 = \langle \Gamma_\pi(\mu), \omega_{\xi,\eta} \otimes \omega_{\zeta,\vartheta} \rangle$$

for every elementary tensor of vector functionals $\omega_{\xi,\eta} \otimes \omega_{\zeta,\vartheta}$ in $\mathcal{B}(\mathcal{H})_* \otimes^h \mathcal{B}(\mathcal{H})_*$. (Note that we earlier had used the notation $\omega_{\xi,\eta} = \eta^* \otimes \xi$.) We may compute

$$\begin{aligned}
\langle \Gamma_\pi(\mu), \omega_{\xi,\eta} \otimes \omega_{\zeta,\vartheta} \rangle &= \int_G \langle \pi(s)\xi|\eta\rangle \, \langle \pi(s)^*\zeta|\vartheta\rangle \, d\mu(s) \\
&= \int_G \langle \pi(s)\xi|\eta\rangle \, \overline{\langle \pi(s)\vartheta|\zeta\rangle} d\mu(s) \\
&= \int_G \left\langle \pi \otimes \bar{\pi}(s) \, \xi \otimes \bar{\vartheta} | \eta \otimes \bar{\zeta} \right\rangle d\mu(s) \\
&= \left\langle (\pi \otimes \bar{\pi})_1(\mu) \, \xi \otimes \bar{\vartheta} | \eta \otimes \bar{\zeta} \right\rangle .
\end{aligned}$$

Thus it follows that $\mu \in \ker \Gamma_\pi$ if and only if $\mu \in \ker(\pi \otimes \bar{\pi})_1$. □

In particular, if we let $\mathrm{F}_{\pi\otimes\bar{\pi}}$ be the linear space generated by all of the coefficient functions, $s \mapsto \left\langle \pi \otimes \bar{\pi}(s) \, \xi \otimes \bar{\vartheta} | \eta \otimes \bar{\zeta} \right\rangle$, we see that $\mu \in \ker \Gamma_\pi$ exactly when μ, as a functional on $\mathcal{C}_b(G)$, annihilates $\mathrm{F}_{\pi\otimes\bar{\pi}}$.

4. Abelian Groups

For this section we let G be a locally compact *abelian* group, and we let $\widehat{G}$ denote its topological dual group. For each s in G, we will let $\hat{s}$ denote the associated unitary character on $\widehat{G}$, defined by $\hat{s}(\sigma) = \sigma(s)$ for each σ in $\widehat{G}$.

As above, we will let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a strongly continuous unitary representation. We let E_π denote the spectrum of C^*_π. Since C^*_π is a quotient of the enveloping C*-algebra $\mathrm{C}^*(G)$, and $\mathrm{C}^*(G) \cong \mathcal{C}_0(\widehat{G})$, we may consider E_π to be a closed subset of $\widehat{G}$. Moreover, the natural isomorphism $F_\pi : \mathrm{C}^*_\pi \to \mathcal{C}_0(E_\pi)$ satisfies

$$F_\pi\big(\pi_1(f)\big) = \hat{f}|_{E_\pi}$$

for each f in $\mathrm{L}^1(G)$, where $\hat{f}(\sigma) = \sigma_1(f) = \int_G f(s)\sigma(s)ds$ for each σ in $\widehat{G}$. We note that our notation $f \mapsto \hat{f}$, for the Fourier transform, differs from that of our main reference, [15].

We would like to be able to extend F_π to some suitable map $\bar{F}_\pi$ on VN_π. It is not clear that this can be done in general, but it can be done in many cases.

Lemma 4.1. *Consider the following conditions for π or G below:*

(a) *$\mathcal{H}$ admits a maximal countable family of mutually orthogonal cyclic subspaces for π.*

(b) *There is a family $\{U_i\}_{i\in I}$ of separable open subsets of E_π such that $E_\pi = \dot{\bigcup}_{i\in I} U_i$.*

(c) *$\widehat{G}$ has a separable open subgroup.*

(d) *G is compactly generated.*

Then, under any one of these conditions there exists a regular Borel measure ν on E_π, bounded on compacta, such that there is a normal $*$-homomorphism $\bar{F}_\pi : \mathrm{VN}_\pi \to \mathrm{L}^\infty(E_\pi, \nu)$ which extends F_π.

Proof. (a) By standard arguments (see [5, §7], for example), VN_π admits a faithful normal state ω. Then the measure ν given by

$$\int_{E_\pi} \varphi(\sigma)d\nu(\sigma) = \omega(F_\pi^{-1}\varphi) \tag{4.1}$$

for each φ in $\mathcal{C}_0(E_\pi)$, gives rise to the desired map $\bar{F}_\pi$.

(b) Since $E_\pi = \dot{\bigcup}_{i\in I} U_i$, we have that $\mathcal{C}_0(E_\pi) = c_0\text{-}\bigoplus_{i\in I}\mathcal{C}_0(U_i)$. If we let $\mathcal{C}_i = F_\pi^{-1}\big(\mathcal{C}_0(U_i)\big)$, then $\mathcal{M}_i = \overline{\mathcal{C}_i}^{w^*}$ is an ideal in VN_π. The ideals $\mathcal{M}_i$ are mutually orthogonal, and hence if $\{p_i\}_{i\in I}$ is the family of projections for which $\mathcal{M}_i = p_i\mathrm{VN}_\pi$ for each i, then $\sum_{i\in I} p_i = I$. Since each $\mathcal{C}_i$ is separable, each $\mathcal{M}_i$ is countably generated, and hence there is a normal state ω_i on VN_π with support projection p_i. Let ν_i be the measure on E_π associated with ω_i as in (4.1). Then $\mathrm{supp}(\nu_i) = U_i$ for each i, and $\nu = \bigoplus_{i\in I}\nu_i$ is the desired measure.

(c) If $\widehat{G}$ has a separable open subgroup X, let T be any transversal for X in $\widehat{G}$, and we have that $E_\pi = \dot{\bigcup}_{\tau\in T}(E_\pi \cap \tau X)$, and again we obtain (b).

(d) If G is compactly generated, then by [15, 9.8] there is a topological isomorphism $G \cong \mathbb{Z}^n\times\mathbb{R}^m\times K$, where K is compact. Then $\widehat{G} \cong \mathbb{T}^n\times\mathbb{R}^m\times\widehat{K}$, and the subgroup X corresponding to $\mathbb{T}^n\times\mathbb{R}^m$ is open and separable, and hence (c) holds. □

We will need to use an extension of F_π of a different nature than in the lemma above. Since C^*_π is an essential ideal in M^*_π, the map $F_\pi : \mathrm{C}^*_\pi \to \mathcal{C}_0(E_\pi)$ extends to an injective $*$-homomorphism $\tilde{F}_\pi : \mathrm{M}^*_\pi \to \mathcal{C}_b(E_\pi)$, such that $F_\pi(na) = \tilde{F}_\pi(n)F_\pi(a)$ for each n in M^*_π and a in C^*_π, by [18, 3.12.8]. We note that for each μ in $\mathrm{M}(G)$,

$$\tilde{F}_\pi(\pi_1(\mu)) = \hat{\mu}|_{E_\pi} \tag{4.2}$$

where for each μ in $\mathrm{M}(G)$, $\hat{\mu}(\sigma) = \sigma_1(\mu) = \int_G \sigma(s)d\mu(s)$. Thus $\mu \mapsto \hat{\mu}$ is the Fourier-Stieltjes transform. To see the validity of (4.2), observe that for each f in $\mathrm{L}^1(G)$ we have

$$\hat{\mu}\hat{f}|_{E_\pi} = \widehat{\mu{*}f}|_{E_\pi} = F_\pi(\pi_1(\mu{*}f)) = \tilde{F}_\pi(\pi_1(\mu))F_\pi(f) = \tilde{F}_\pi(\pi_1(\mu))\hat{f}|_{E_\pi}.$$

Thus it follows that $\tilde{F}_\pi(\pi_1(\mu))\varphi = \hat{\mu}\varphi$ for each φ in $\mathcal{C}_0(E_\pi)$.

If any of the conditions of Lemma 4.1 hold, then there exists a measure ν for which there is a normal extension $\bar{F}_\pi : \mathrm{VN}_\pi \to \mathrm{L}^\infty(E_\pi, \nu)$ of F_π. Then for any μ in $\mathrm{M}(G)$,

$$\bar{F}_\pi(\pi_1(\mu)) = \hat{\mu}|_{E_\pi} \tag{4.3}$$

where we identify $\mathcal{C}_b(E_\pi)$ as a closed subspace of $\mathrm{L}^\infty(E_\pi, \nu)$. To see (4.3), we note that if (a_β) is any bounded approximate identity in C^*_π, then weak*-$\lim_\beta a_\beta = I$ in VN_π, thus weak*-$\lim_\beta F_\pi(a_\beta) = 1_{E_\pi}$. Hence

$$\begin{aligned}\bar{F}_\pi(\pi_1(\mu)) &= \text{weak*-}\lim_\beta \bar{F}_\pi(\pi_1(\mu)a_\beta) = \text{weak*-}\lim_\beta F_\pi(\pi_1(\mu)a_\beta) \\ &= \text{weak*-}\lim_\beta \tilde{F}_\pi(\pi_1(\mu))F_\pi(a_\beta) = \tilde{F}_\pi(\pi_1(\mu)) = \hat{\mu}|_{E_\pi}.\end{aligned}$$

We will make use of the spaces $\mathrm{V}^b(E)$, $\mathrm{V}^0(E)$ and $\mathrm{V}_0(E)$, which were defined in (2.3). If ν is any non-negative measure on E, we let

$$\mathrm{V}^\infty(E,\nu) = \mathrm{L}^\infty(E,\nu) \otimes^{eh} \mathrm{L}^\infty(E,\nu).$$

Spaces of this type are discussed in [22].

Theorem 4.2. *If G is a locally compact abelian group and $\pi : G \to \mathcal{U}(\mathcal{H})$ is a strongly continuous unitary representation, then for any μ in $\mathrm{M}(G)$ and (σ,τ) in $E_\pi \times E_\pi$ we have that*

$$(\tilde{F}_\pi \otimes \tilde{F}_\pi)\Gamma_\pi(\mu)(\sigma,\tau) = \hat{\mu}(\sigma\tau^{-1}).$$

In particular, if E is any closed subset of $\widehat{G}$ and $\mu \in \mathrm{M}(G)$, then $u(\sigma,\tau) = \hat{\mu}(\sigma\tau^{-1})$ is an element of $\mathrm{V}^b(E)$, and $u \in \mathrm{V}^0(E)$ if $\mu \in \mathrm{L}^1(G)$. Moreover, if E is compact then $u \in \mathrm{V}_0(E)$.

Proof. The result will be established in three stages. The first two of these require additional hypotheses and are preparatory for the general case.

I. Suppose that any one of the conditions of Lemma 4.1 is satisfied. Let ν be the measure on E_π and let $\bar{F}_\pi : \mathrm{VN}_\pi \to \mathrm{L}^\infty(E_\pi, \nu)$ be the map given there.

If $\mu \in \mathrm{M}(G)$, we have that

$$(\tilde{F}_\pi \otimes \tilde{F}_\pi)\Gamma_\pi(\mu) = (\bar{F}_\pi \otimes \bar{F}_\pi)\int_G \pi(s)\otimes\pi(s)^* d\mu(s)$$

$$= \int_G \hat{s}|_{E_\pi} \otimes \bar{\hat{s}}|_{E_\pi} d\mu(s) \tag{4.4}$$

where the latter integral converges in the weak* topology of $\mathrm{V}^\infty(E_\pi,\nu)$.

For (σ,τ) in $E_\pi \times E_\pi$ let

$$u(\sigma,\tau) = \hat{\mu}(\sigma\tau^{-1}).$$

Then $u \in \mathrm{V}^b(E_\pi)$. Indeed, we have that $\hat{\mu} \in \mathrm{B}(\widehat{G})$, the *Fourier-Stieltjes algebra* which is defined in [10]. Thus there is a Hilbert space $\mathcal{L}$, a continuous unitary representation $\rho : G \to \mathcal{U}(\mathcal{L})$, and vectors ξ, η in $\mathcal{L}$ with $\|\mu\| = \|\xi\|\,\|\eta\|$, such that $\hat{\mu}(\sigma) = \langle \rho(\sigma)\xi|\eta\rangle$ for each σ in $\widehat{G}$. If $\{\xi_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{L}$, then we have, using Parseval's formula, that

$$\hat{\mu}(\sigma\tau^{-1}) = \left\langle \rho(\sigma\tau^{-1})\xi|\eta\right\rangle = \sum_{i\in I} \langle \rho(\sigma)\xi|\xi_i\rangle \langle \xi_i|\rho(\tau)\eta\rangle$$

for any (σ,τ) in $E_\pi \times E_\pi$. Hence

$$u = \sum_{i\in I} \langle \rho(\cdot)\xi|\xi_i\rangle \otimes \overline{\langle \rho(\cdot)\eta|\xi_i\rangle} \in \mathrm{V}^b(E_\pi)$$

with $\|u\|_{eh} \leq \|\xi\|\,\|\eta\| = \|\mu\|$. (This is similar to the proof of [23, Prop. 5.1].) We note that if $\mu \in \mathrm{L}^1(G)$, then ρ can be taken to be the left regular representation and hence each $\langle \rho(\cdot)\xi|\xi_i\rangle$ and $\langle \rho(\cdot)\eta|\xi_i\rangle$ is in $\mathcal{C}_0(E_\pi)$. Hence, in this case we would have that $u \in \mathrm{V}^0(E_\pi)$.

We wish to establish that

$$u = (\tilde{F}_\pi \otimes \tilde{F}_\pi)\Gamma_\pi(\mu). \tag{4.5}$$

We will do this by using the dual pairing (2.1). If $g \otimes h$ is an elementary tensor in $\mathrm{L}^1(E_\pi, \nu) \otimes^h \mathrm{L}^1(E_\pi, \nu)$, then

$$\begin{aligned}\langle u, g \otimes h \rangle &= \int_{E_\pi} \int_{E_\pi} g(\sigma) h(\tau) \hat{\mu}(\sigma\tau^{-1}) d\nu(\sigma) d\nu(\tau) \\ &= \int_{E_\pi} \int_{E_\pi} g(\sigma) h(\tau) \left(\int_G \sigma(s) \overline{\tau(s)} d\mu(s) \right) d\nu(\sigma) d\nu(\tau) \\ &= \int_G \left(\int_{E_\pi} g(\sigma) \hat{s}(\sigma) d\nu(\sigma) \right) \left(\int_{E_\pi} h(\tau) \overline{\hat{s}(\tau)} d\nu(\tau) \right) d\mu(s)\end{aligned}$$

where the version of Fubini's Theorem required is [15, 13.10], noting that g and h each have ν-σ-finite supports. On the other hand, by (4.4),

$$\begin{aligned}\langle (\tilde{F}_\pi \otimes \tilde{F}_\pi) \Gamma_\pi(\mu), g \otimes h \rangle &= \left\langle \int_G \hat{s}|_{E_\pi} \otimes \bar{\hat{s}}|_{E_\pi} d\mu(s), g \otimes h \right\rangle \\ &= \int_G \left(\int_{E_\pi} g(\sigma) \hat{s}(\sigma) d\nu(\sigma) \right) \left(\int_{E_\pi} h(\tau) \overline{\hat{s}(\tau)} d\nu(\tau) \right) d\mu(s)\end{aligned}$$

and this shows that (4.5) holds.

II. Suppose that μ is supported on a compactly generated open subgroup H of G.

Let us first compute the spectrum $E_{\pi|_H}$ of $\mathrm{C}^*_{\pi|_H}$. We note that $\widehat{H} = \widehat{G}|_H$ and that the restriction map $r : \widehat{G} \to \widehat{G}|_H$ is a homomorphic topological quotient map by [15, 24.5]. Moreover, $\ker r$ is compact, by [15, 23.29(a)]. Then $E_{\pi|_H} = r(E_\pi)$. To see this, observe that the map $\iota : \mathrm{L}^1(H) \to \mathrm{L}^1(G)$, which we define to be the inverse of $f \mapsto f|_H$, extends to an injective $*$-homomorphism $\iota_\pi : \mathrm{C}^*_{\pi|_H} \to \mathrm{C}^*_\pi$. In particular, then, each multiplicative linear functional on $\mathrm{C}^*_{\pi|_H}$ is necessarily the restriction of such a functional on C^*_π. Let $r_\pi = r|_{E_\pi}$. Then, the map $r_\pi : E_\pi \to r(E_\pi)$ induces an injective $*$-homomorphism $j_{r_\pi} : \mathcal{C}_0(r(E_\pi)) \to \mathcal{C}_0(E_\pi)$, whose image is the subalgebra of all functions which are constant on relative cosets of $\ker r$ in E_π. Now, if $g \in \mathrm{L}^1(H)$, and $\sigma \in E_\pi$ then

$$\widehat{\iota g}(\sigma) = \int_G \iota g(s) \sigma(s) ds = \int_H g(s) r(\sigma)(s) ds = \hat{g}(r_\pi(\sigma)) = j_{r_\pi} \hat{g}(\sigma) \tag{4.6}$$

from which it follows that every character on $\mathrm{C}^*_{\pi|_H}$ is from $r_\pi(E_\pi)$. Moreover, it follows from (4.6) that

$$F_\pi \circ \iota_\pi = j_{r_\pi} \circ F_{\pi|_H}.$$

Now, we let $\tilde{\iota} : \mathrm{M}(H) \to \mathrm{M}(G)$ be the homomorphism whose inverse is $\kappa \mapsto \kappa_H$, where for any Borel subset B of G, $\kappa_H(B) = \kappa(B \cap H)$. Then ι induces an injective $*$-homomorphism $\tilde{\iota}_\pi : \mathrm{M}^*_{\pi|_H} \to \mathrm{M}^*_\pi$. It follows from the discussion above that $\tilde{F}_{\pi|_H} : \mathrm{M}^*_{\pi|_H} \to \mathcal{C}_b(r(E_\pi))$. Then

$$\tilde{\jmath}_{r_\pi} \circ \tilde{F}_{\pi|_H} = \tilde{F}_\pi \circ \tilde{\iota}_\pi \tag{4.7}$$

where $\tilde{\jmath}_{r_\pi} : \mathcal{C}_b(E_{\pi|_H}) \to \mathcal{C}_b(E_\pi)$ is the map induced by $r_\pi : E_\pi \to E_{\pi|_H}$. Indeed, if $\kappa \in \mathrm{M}(H)$, then for each σ in E_π we have that $\widehat{\tilde{\iota}\kappa}(\sigma) = \tilde{\jmath}_{r_\pi}\hat{\kappa}(\sigma)$, by a computation analagous to (4.6), above. Next, we wish to establish that

$$\Gamma_\pi \circ \tilde{\iota} = (\tilde{\iota}_\pi \otimes \tilde{\iota}_\pi) \circ \Gamma_{\pi|_H}. \tag{4.8}$$

If $\kappa \in \mathrm{M}(H)$ and $x \in \mathcal{B}(\mathcal{H}) \otimes^h \mathcal{B}(\mathcal{H})$, then

$$\begin{aligned}\langle \Gamma_\pi(\tilde{\iota}\kappa), x \rangle &= \int_G \langle \pi(s) \otimes \pi(s)^*, x \rangle \, d\tilde{\iota}\kappa(s) \\ &= \int_H \langle \pi(s) \otimes \pi(s)^*, x \rangle \, d\kappa(s) = \left\langle \Gamma_{\pi|_H}(\kappa), x \right\rangle\end{aligned}$$

whence, as elements of $\mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$, $\Gamma_\pi(\tilde{\iota}\kappa) = \Gamma_{\pi|_H}(\kappa)$. However, in $\mathcal{B}(\mathcal{H})$, the inclusion map $\mathrm{M}^*_{\pi|_H} \hookrightarrow \mathrm{M}^*_\pi$ is the map $\tilde{\iota}_\pi$, and thus (4.8) holds.

Now, since μ is supported on H, we have that $\mu = \tilde{\iota}\kappa$ for some $\kappa \in \mathrm{M}(H)$. Then for each (σ, τ) in $E_\pi \times E_\pi$ we have that

$$\begin{aligned}\hat{\mu}(\sigma\tau^{-1}) &= \widehat{\tilde{\iota}\kappa}(\sigma\tau^{-1}) = \hat{\kappa}(r(\sigma\tau^{-1}) = \hat{\kappa}(r_\pi(\sigma) r_\pi(\tau^{-1})) \\ &= (\tilde{F}_{\pi|_H} \otimes \tilde{F}_{\pi|_H}) \Gamma_{\pi|_H}(\kappa)(r_\pi(\sigma), r_\pi(\tau)), \quad \text{by part I} \\ &= (\tilde{\jmath}_{r_\pi} \otimes \tilde{\jmath}_{r_\pi})(\tilde{F}_{\pi|_H} \otimes \tilde{F}_{\pi|_H}) \Gamma_{\pi|_H}(\kappa)(\sigma, \tau) \\ &= (\tilde{F}_\pi \otimes \tilde{F}_\pi)(\tilde{\iota}_\pi \otimes \tilde{\iota}_\pi) \Gamma_{\pi|_H}(\kappa)(\sigma, \tau), \quad \text{by (4.7)} \\ &= (\tilde{F}_\pi \otimes \tilde{F}_\pi) \Gamma_\pi(\tilde{\iota}\kappa)(\sigma, \tau), \quad \text{by (4.8)} \\ &= (\tilde{F}_\pi \otimes \tilde{F}_\pi) \Gamma_\pi(\mu)(\sigma, \tau).\end{aligned}$$

III. We now cover the case of a general μ in $\mathrm{M}(G)$.

Let U be a relatively compact symmetric open neighbourhood of the identity in G. Then $H = \bigcup_{n=1}^{\infty} U^n$ is a compactly generated open subgroup of G. We note that if T is a transversal for H in G then

$$\mu = \sum_{t\in T} \mu_{tH}$$

which is an absolutely summable series. For each t in T let

$$\mu_t = \delta_{t^{-1}} * (\mu_{tH})$$

so $\operatorname{supp}(\mu_t) \subset H$ and $\mu = \sum_{t\in T} \delta_t * \mu_t$. We then have that

$$\begin{aligned}(\tilde{F}_\pi \otimes \tilde{F}_\pi)\Gamma_\pi(\mu) &= \sum_{t\in T} (\tilde{F}_\pi \otimes \tilde{F}_\pi)\Gamma_\pi(\delta_t * \mu_t) \\ &= \sum_{t\in T} (\tilde{F}_\pi \otimes \tilde{F}_\pi)\left[(\pi(t)\otimes \pi(t)^*)\Gamma_\pi(\mu_t)\right] \\ &= \sum_{t\in T} (\hat{t}|_{E_\pi} \otimes \bar{\hat{t}}|_{E_\pi})(\tilde{F}_\pi \otimes \tilde{F}_\pi)\Gamma_\pi(\mu_t).\end{aligned}$$

Hence if $(\sigma,\tau) \in E_\pi \times E_\pi$, we obtain

$$\begin{aligned}(\tilde{F}_\pi \otimes \tilde{F}_\pi)\Gamma_\pi(\mu)(\sigma,\tau) &= \sum_{t\in T} \hat{t}(\sigma)\overline{\hat{t}(\tau)}(\tilde{F}_\pi \otimes \tilde{F}_\pi)\Gamma_\pi(\mu_t)(\sigma,\tau) \\ &= \sum_{t\in T} \hat{t}(\sigma)\overline{\hat{t}(\tau)}\hat{\mu}_t(\sigma\tau^{-1}), \qquad \text{by part II} \\ &= \sum_{t\in T} \hat{\delta}_t(\sigma\tau^{-1})\hat{\mu}_t(\sigma\tau^{-1}) \\ &= \sum_{t\in T} \widehat{\delta_t * \mu_t}(\sigma\tau^{-1}) = \hat{\mu}(\sigma\tau^{-1}).\end{aligned}$$

Thus our first claim is established in general.

If E is any closed subset of $\widehat{G}$ then by [14, 33.7] there is a representation $\pi : G \to \mathcal{U}(\mathcal{H})$ for which $E_\pi = E$. Hence

$$u(\sigma,\tau) = \hat{\mu}(\sigma\tau^{-1}) = (\tilde{F}_\pi \otimes \tilde{F}_\pi)\Gamma_\pi(\mu)(\sigma,\tau)$$

defines an element of $\mathrm{V}^b(E)$, and of $\mathrm{V}^0(E)$ if $\mu \in \mathrm{L}^1(G)$. If E is compact, we note that any representation π for which $E_\pi = E$, is norm continuous by Proposition 3.3, since $\mathrm{C}^*_\pi \cong \mathcal{C}_0(E)$, which is unital. Thus $\Gamma_\pi(\mathrm{M}(G)) = \Gamma_\pi(\mathrm{L}^1(G)) \subset \mathrm{C}^*_\pi \otimes^h \mathrm{C}^*_\pi$. Hence u, as above, is in $\mathrm{V}_0(E)$. □

We can now obtain a generalisation of [24, Prop. 5.7]. This is a straightforward corollary of Theorems 2.4 and 4.2.

Corollary 4.3. *If G is a locally compact abelian group, $\pi : G \to \mathcal{U}(\mathcal{H})$ is a strongly continuous representation and $\mu \in \mathrm{M}(G)$, then the following are equivalent:*

(i) $\Gamma_\pi(\mu)$ *is positive.*

(ii) $\Gamma_\pi(\mu)$ *is completely positive.*

(iii) $(\sigma, \tau) \mapsto \hat{\mu}(\sigma\tau^{-1})$ *is positive definite on* $E_\pi \times E_\pi$.

The next result follows directly from Theorem 4.2, but can also be deduced from Proposition 3.8.

Corollary 4.4. *If G is a locally compact abelian group and $\pi : G \to \mathcal{U}(\mathcal{H})$ is a strongly continuous representation then*

$$\ker \Gamma_\pi = \{\mu \in \mathrm{M}(G) : \hat{\mu}|_{E_\pi E_\pi^{-1}} = 0\}.$$

Let us now address the assumption that $f \in \mathrm{L}^1(G) \setminus \mathrm{I}_0(G)$ in Theorem 3.7 (iii). We want to show that having an f in $\mathrm{I}_0(G)$ for which $\Gamma_\pi(f) \subset \mathrm{C}^*_\pi \otimes^h \mathrm{C}^*_\pi$ does not imply that π is norm continuous. First, following Corollary 4.4 we see that that $\Gamma_\pi(f) = 0$ if the support of $\hat{f}$ misses the difference set $E_\pi E_\pi^{-1}$. Thus it is possible that $\Gamma_\pi(f) = 0 \in \mathrm{C}^*_\pi \otimes^h \mathrm{C}^*_\pi$, though π need not be norm continuous, i.e. E_π need not be compact. Thus we may ask if $\Gamma_\pi(\mathrm{L}^1(G)) \cap \left(\mathrm{C}^*_\pi \otimes^h \mathrm{C}^*_\pi\right) = \{0\}$ when π is not norm continuous. However, this may not happen, as the next example shows.

Example 4.5. Let $G = \mathbb{T}$, and identify $\widehat{\mathbb{T}} = \mathbb{Z}$. Define $\pi : \mathbb{T} \to \mathcal{U}(\ell^2(\mathbb{N}))$ for each z in $\mathbb{T}$ by

$$\pi(z)\big(\xi_n\big)_{n\in\mathbb{N}} = \left(z^{n^2}\xi_n\right)_{n\in\mathbb{N}}.$$

Then $E_\pi = \{n^2 : n \in \mathbb{N}\}$, which is not compact in $\mathbb{Z}$. Hence $\mathrm{C}^*_\pi \cong c_0(E_\pi)$, which is not unital, so π is not norm continuous on $\mathbb{T}$, by Proposition 3.3. Fix k in $\mathbb{Z} \setminus \{0\}$ and let $\hat{k}(z) = z^k$. Then for each pair n, m in $\mathbb{N}$, using normalized Haar measure on $\mathbb{T}$ and Theorem 4.2, we have that

$$(F_\pi \otimes F_\pi)\Gamma_\pi(\hat{k})(\hat{n}, \hat{m}) = \int_{\mathbb{T}} z^k z^{n^2} \bar{z}^{m^2} dz = \begin{cases} 1 & \text{if } m^2 - n^2 = k \\ 0 & \text{otherwise} \end{cases}.$$

The set of solutions to $m^2 - n^2 = (m-n)(m+n) = k$ is clearly finite; we shall write them $\{(n_1, m_1), \ldots, (n_{l(k)}, m_{l(k)})\}$. We then see that

$$(F_\pi \otimes F_\pi)\Gamma_\pi(\hat{k}) = \sum_{i=1}^{l(k)} 1_{(n_i, m_i)} = \sum_{i=1}^{l(k)} 1_{n_i} \otimes 1_{m_i} \in \mathrm{V}_0(E_\pi).$$

Hence $\Gamma_\pi(\hat{k}) \in \mathrm{C}^*_\pi \otimes^h \mathrm{C}^*_\pi$. In fact, since $\mathrm{I}_0(\mathbb{T}) = \overline{\mathrm{span}} \left\{ \hat{k} : k \in \mathbb{Z} \setminus \{0\} \right\}$, we have that $\Gamma_\pi(\mathrm{I}_0(\mathbb{T})) \subset \mathrm{C}^*_\pi \otimes^h \mathrm{C}^*_\pi$.

We remark that for a general locally compact abelian group G, and representation $\pi : G \to \mathcal{U}(\mathcal{H})$, $\mathrm{V}_0(E_\pi) \subset \mathcal{C}_0(E_\pi \times E_\pi)$. Thus if f in $\mathrm{L}^1(G)$ is such that $\hat{f}(\sigma) \neq 0$ for some σ in $\widehat{G}$ such that $E_\pi \cap \sigma E_\pi$ is not compact, then $\Gamma_\pi(f) \notin \mathrm{C}^*_\pi \otimes^h \mathrm{C}^*_\pi$ by Theorem 4.2. Thus if $E_\pi \cap \sigma E_\pi$ is compact for no σ in $\widehat{G}$ then we have that

$$\Gamma_\pi(\mathrm{L}^1(G)) \cap \left(\mathrm{C}^*_\pi \otimes^h \mathrm{C}^*_\pi \right) = \{0\}.$$

Note that $E_\pi \cap \sigma E_\pi$ is never compact if $E_\pi = \widehat{G}$, which occurs, for example when π is the left regular representation λ. It is shown in [24, Cor. 4.7] that Γ_λ is an isometry. This was extended to non-abelian groups in [11] and expanded upon in [16], while [17] contains a proof that Γ_λ is a complete isometry. An analogue for the Fourier algebra of an amenable group is shown in [23, Cor. 5.4].

Question 4.6. If $G = \mathbb{R}$, then $[0, \infty) \cap \big(s + [0, \infty)\big) = [\min\{s, 0\}, \infty)$ is never compact. Thus if π is a representation of $\mathbb{R}$ such that $E_\pi = [0, \infty)$, then $\Gamma_\pi : \mathrm{M}(\mathbb{R}) \to \mathrm{M}^*_\pi \otimes^{eh} \mathrm{M}^*_\pi$ is injective by Corollary 4.4. Is Γ_π isometric?

How about $\Gamma_\pi|_{L^1(\mathbb{R})}$? More generally, under what conditions for an arbitrary abelian group G and representation π is Γ_π, or $\Gamma_\pi|_{L^1(G)}$, a quotient map?

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