MATH 171H EXAM 1 2014 SOLUTIONS

1. (a) For every $\varepsilon>0$, there is a $\delta>0$ such that $|f(x)-L|<\varepsilon$ whenever $0<|x-a|<\delta$.

(b) Note that $|2x^2-x-15| = |(2x+5)(x-3)| = |2x+5||x-3|$. If $0<|x-3|<\delta$ and $\delta\le 1$, then $2<x<4$, which implies $9<2x+5<13$, and in particular, $|2x+5|<13$. So, given $\varepsilon>0$, we may let $\delta = \min\{1, \frac{\varepsilon}{13}\}$. Then, whenever $0<|x-3|<\delta$, we have $|2x^2-x-15| = |2x+5||x-3| < 13\delta \le 13\cdot\frac{\varepsilon}{13} = \varepsilon$.

2. (a) $\displaystyle\lim_{x\to 1} \frac{1-\frac{1}{x^2}}{\frac{1}{x}-1}\cdot\frac{x^2}{x^2} = \lim_{x\to 1}\frac{x^2-1}{x-x^2} = \lim_{x\to 1}\frac{(x+1)(x-1)}{x(1-x)} = \lim_{x\to 1}\frac{-(x+1)}{x} = -2$

(b) $\displaystyle\lim_{x\to 0}\left(\frac{1}{x\sqrt{1+x}} - \frac{1}{x}\cdot\frac{\sqrt{1+x}}{\sqrt{1+x}}\right) = \lim_{x\to 0}\left(\frac{1-\sqrt{1+x}}{x\sqrt{1+x}}\cdot\frac{1+\sqrt{1+x}}{1+\sqrt{1+x}}\right) = \lim_{x\to 0}\frac{1-(1+x)}{x\sqrt{1+x}\,(1+\sqrt{1+x})}$

$\displaystyle = \lim_{x\to 0}\frac{-1}{\sqrt{1+x}\,(1+\sqrt{1+x})} = -\frac{1}{2}$

(c) $\displaystyle\lim_{x\to\infty}\frac{1+x-5x^2}{\sqrt{4x^4+2}-1}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty}\frac{\frac{1}{x^2}+\frac{1}{x}-5}{\sqrt{4+\frac{2}{x^4}}-\frac{1}{x^2}} = \frac{-5}{\sqrt{4}} = -\frac{5}{2}$

3. (a) yes (by the Squeeze Theorem, $\lim_{x\to 0} x\sin(\frac{1}{x}) = 0 = f(0)$)

(b) no (another way to write the function is $f(x)=\begin{cases}-1 & \text{if } x<0\\ 0 & \text{if } x=0\\ 1 & \text{if } x>0\end{cases}$)

(c) yes ($\lim_{x\to 0} f(x) = 0 = f(0)$ since e.g. $-x\le f(x)\le x$ for all real numbers x)

4. (a) $\displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ or $\displaystyle\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

(b) $\displaystyle f'(x) = \lim_{h\to 0}\frac{(\sqrt{x+h}+1)-(\sqrt{x}+1)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$\displaystyle = \lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$

5. (a) T. By definition, $\mathrm{proj}_a b$ is the vector projection of b onto a, which is parallel to a. This can also be seen from the formula $\mathrm{proj}_a b = \frac{a\cdot b}{|a|^2}a$.

(b) F. Counterexample: $f(x)=\begin{cases}1 & \text{if } x\ge 0\\ -1 & \text{if } x<0\end{cases}$

(Many other counterexamples are possible.)

(c) F. Counterexample: $f(x)=\begin{cases}-2 & \text{if } x\le 0\\ 2 & \text{if } x>0\end{cases}$

(Many other counterexamples are possible.)

(d) T. Let $h=f-g$. Then h is continuous on the interval $[0,1]$ and $h(0)<0$, $h(1)>0$. By the Intermediate Value Theorem, there is a c in the interval $(0,1)$ such that $h(c)=0$, i.e. $f(c)=g(c)$.