MATH 220 - Exam 2 - Partial Solutions

1. For each integer $n \geq 1$, let $P(n)$ be the statement that $\sum_{k=0}^{n-1} 3^k = \frac{1}{2}(3^n - 1)$.

Proof that $P(n)$ is true for all $n \geq 1$:

(1) $P(1)$ is true: $\sum_{k=0}^{1-1} 3^k = 3^0 = 1$, and $\frac{1}{2}(3^1 - 1) = \frac{1}{2}(2) = 1$.

(2) Assume that for some integer $m \geq 1$, $P(m)$ is true, that is,

$$\sum_{k=0}^{m-1} 3^k = \frac{1}{2}(3^m - 1).$$

Then

$$\sum_{k=0}^{(m+1)-1} 3^k = \sum_{k=0}^{m} 3^k = \left(\sum_{k=0}^{m-1} 3^k\right) + 3^m$$

$$= \frac{1}{2}(3^m - 1) + 3^m \quad \text{(by the induction hypothesis)}$$

$$= \frac{3^m - 1 + 2\cdot 3^m}{2}$$

$$= \frac{3\cdot 3^m - 1}{2}$$

$$= \frac{3^{m+1} - 1}{2} = \frac{1}{2}(3^{m+1} - 1),$$

that is, $P(m+1)$ is true.

By mathematical induction, $P(n)$ is true for all integers $n \geq 1$.

2. For each integer $n \geq 1$, let $P(n)$ be the statement that $a_n = 4^n - 3^n$.
Proof that $P(n)$ is true for all $n \geq 1$:

(1) $P(1)$ is true: $a_1 = 1$, $4^1 - 3^1 = 1$
$P(2)$ is true: $a_2 = 7$, $4^2 - 3^2 = 16 - 9 = 7$

(2) Assume that for some positive integer m, $m \geq 2$,
$$a_k = 4^k - 3^k$$
for all integers k for which $1 \leq k \leq m$.

Then
$$\begin{aligned} a_{m+1} &= 7a_m - 12a_{m-1} && \text{(by definition of the sequence)} \\ &= 7(4^m - 3^m) - 12(4^{m-1} - 3^{m-1}) && \text{(by the induction hypothesis)} \\ &= 7 \cdot 4^m - 7 \cdot 3^m - 12 \cdot 4^{m-1} + 12 \cdot 3^{m-1} \\ &= 7 \cdot 4^m - 7 \cdot 3^m - 3 \cdot 4^m + 4 \cdot 3^m \\ &= 4 \cdot 4^m - 3 \cdot 3^m \\ &= 4^{m+1} - 3^{m+1}, \end{aligned}$$

so $P(m+1)$ is true.

By strong mathematical induction, $P(n)$ is true for all positive integers n.

3. (a) A: $-5, -2, 1, 4, 7$ $\quad B$: $14, -5, 4, 13, 22$ (answers will vary)

(b) No: $1 \in A$ but $1 \notin B$ (answers will vary)

(c) Yes: Let $n \in B$, i.e. $n = 4 - 9s$ for some $s \in \mathbb{Z}$. Then
$$\begin{aligned} n = 4 - 9s &= 3 - 9s + 1 \\ &= 3(1 - 3s) + 1 \\ &= 3r + 1 \quad \text{for } r = 1 - 3s. \end{aligned}$$
Since r is an integer, $n \in A$. Therefore $B \subseteq A$.

4. $B - A = B \cap \bar{A}$, and $(A \cup B) \cap \bar{A} = (A \cap \bar{A}) \cup (B \cap \bar{A})$ (by distributivity)
$= \phi \cup (B \cap \bar{A})$
$= B \cap \bar{A}$.

Therefore, $B - A = (A \cup B) \cap \bar{A}$.
This can also be proven by arguing containment in both directions.

5. (a) $A_1 \cap A_2 = [0, \frac{1}{2}]$, $\quad A_1 \cup A_2 = [-1, 1]$
(b) $\bigcap_{i=1}^{\infty} A_i = \{0\}$, $\quad \bigcup_{i=1}^{\infty} A_i = (-\infty, 1]$

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F Counterexample: $U = \mathbb{Z}$, $A = \{1\}$, $B = \{1, 2\}$, so $A \subseteq B$
$\bar{A} = \mathbb{Z} - \{1\}$, $\bar{B} = \mathbb{Z} - \{1, 2\}$, so $\bar{A} \not\subseteq \bar{B}$ since $2 \in \bar{A}$, $2 \notin \bar{B}$

F Counterexample: $A = \{2\}$, $B = \{1, 2, 3\}$, $C = \{1\}$
$A \cup (B \cap C) = \{2\} \cup \{1\} = \{1, 2\}$, $(A \cup B) \cap C = \{1, 2, 3\} \cap \{1\} = \{1\}$

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F Counterexample: $n = -1$
$(-\infty, -1] \cup [1, \infty) \neq \mathbb{R}$ since $0 \in \mathbb{R}$, $0 \notin (-\infty, -1] \cup [1, \infty)$