Math 220 Final Exam Practice Problems
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The following are just a few representative problems. They are not meant to include examples of all possible problems that may be on the exam. You should also be prepared to work any problems similar to homework, examples from class, and the first three exams.

1. Prove or find a counterexample:

(a) For all real numbers x, x is irrational if, and only if, $10x$ is irrational. True.

($\Rightarrow$) For all real numbers x, if x is irrational, then 10x is irrational.

Proof: By contrapositive. We'll prove that for all real numbers x, if 10x is rational, then x is rational.
Let $x \in \mathbb{R}$ for which 10x is rational, i.e. $10x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}, b \neq 0$.
So $x = \frac{a}{10b}$, which is rational since $a \in \mathbb{Z}$, $10b \in \mathbb{Z}$, $10b \neq 0$.

($\Leftarrow$) For all real numbers x, if 10x is irrational, then x is irrational.

Proof By contrapositive. We'll prove that for all real numbers x, if x is rational, then 10x is rational.
Let $x \in \mathbb{R}$ such that x is rational, i.e. $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$, $b \neq 0$.
Then $10x = \frac{10a}{b}$, which is rational since $10a \in \mathbb{Z}$, $b \in \mathbb{Z}$, $b \neq 0$.

(b) For all real numbers x, x is irrational if, and only if, $\sqrt{2}x$ is irrational. False.

Counterexample: Let $x = \sqrt{2}$.
Then x is irrational and $\sqrt{2}x = 2$, which is rational.

Another counterexample: Let $x = 1$.
Then x is rational and $\sqrt{2}x = \sqrt{2}$ is irrational.

2. Prove by induction that for each positive integer n,

$$P(n): \quad 1+5+9+\cdots+(4n-3)=n(2n-1).$$

equivalently, $$\sum_{i=1}^{n}(4i-3)=n(2n-1)$$

Proof

1. Check that it is true for $n=1$, i.e. check that $P(1)$ is true:

$$1 = 1(2\cdot 1-1) \checkmark$$

2. Assume that $P(m)$ is true for some positive integer m, i.e.

assume $$1+5+9+\cdots+(4m-3) = m(2m-1). \quad \circledast$$

Then by the induction hypothesis $\circledast$,

$$\underbrace{1+5+9+\cdots+(4m-3)}_{\text{use } \circledast}+(4(m+1)-3) = m(2m-1)+(4(m+1)-3)$$

$$= 2m^2-m+4m+4-3$$

$$= 2m^2+3m+1$$

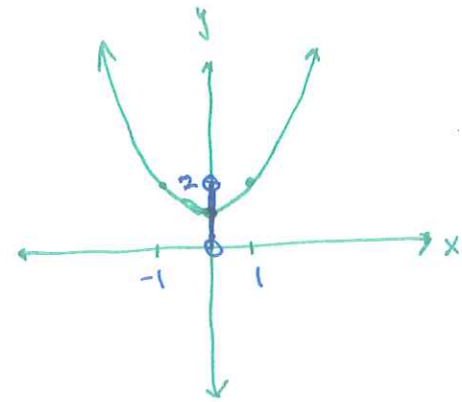
$$=(2m+1)(m+1)$$

$$=(m+1)(2m+1)$$

$$=(m+1)(2(m+1)-1),$$

So $P(m+1)$ is true. By the Principle of Mathematical Induction, $P(n)$ is true for all positive integers n. $\square$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2 + 1$.
(a) Find $f^{-1}((0,2))$ (where $(0,2)$ denotes an open interval).
of y-values



$$f^{-1}((0,2)) = \{x \in \mathbb{R} \mid f(x) \in (0,2)\}$$
$$= \{x \in \mathbb{R} \mid 0 < f(x) < 2\}$$
$$= \{x \in \mathbb{R} \mid 0 < x^2+1 < 2\}$$
$$= \{x \in \mathbb{R} \mid -1 < x^2 < 1\}$$
$$= \{x \in \mathbb{R} \mid -1 < x < 1\} = (-1,1)$$
interval of x-values

(b) Find the range of f.

$$\text{ran} f = \{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in \mathbb{R}\}$$
$$= \{y \in \mathbb{R} \mid y = x^2+1 \text{ for some } x \in \mathbb{R}\}$$
$$= [1, \infty)$$

(c) Is f one-to-one? Justify your answer.

No: $f(-1) = f(1)$

Also, f is not onto since $\text{ran} f = [1,\infty) \neq \mathbb{R}$

4. Let X and Y be sets, let A and B be subsets of X, and let $f : X \to Y$ be a function. Prove that if f is one-to-one and $f[A] \subseteq f[B]$, then $A \subseteq B$. *(This is a problem from Exam 3.)*

Proof Let $a \in A$. (We want to show that $a \in B$.)

Then $f(a) \in f[A]$, so $f(a) \in f[B]$ since $f[A] \subseteq f[B]$.

It follows that $f(a) = f(b)$ for some $b \in B$.

Since f is one-to-one, it follows that $a = b$.

Therefore $a \in B$.

So $A \subseteq B$.

Definitions $f[A] = \{ y \in Y \mid y = f(a) \text{ for some } a \in A \}$.

$f[B] = \{ y \in Y \mid y = f(b) \text{ for some } b \in B \}$.

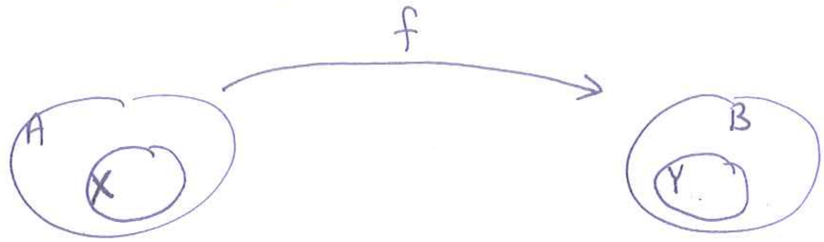
5. Let A and B be sets, and let $f : A \to B$ be a function. Let X be a subset of A, and let Y be a subset of B for which $f[X] \subseteq Y$.

(a) Prove that $X \subseteq f^{-1}[Y]$.

Let $x \in X$. (We want to show that $x \in f^{-1}(Y)$.)

Then $f(x) \in f[X]$. Since $f[X] \subseteq Y$, this implies $f(x) \in Y$.

Therefore $x \in f^{-1}[Y]$, and so $X \subseteq f^{-1}(Y)$. $\square$

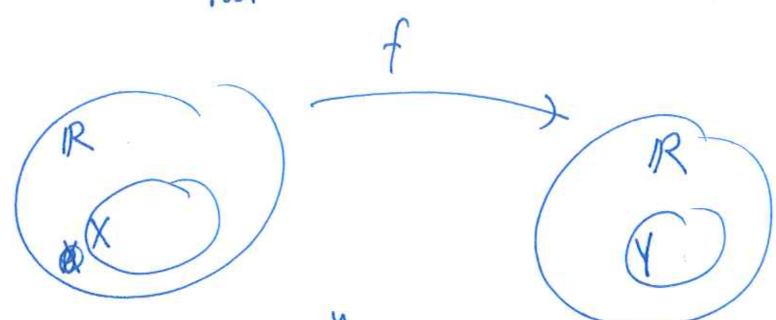


$f[X] = \{ b \in B \mid b = f(x) \text{ for some } x \in X \}$.

(Definitions) $f^{-1}[Y] = \{ a \in A \mid f(a) \in Y \}$ or $\{ a \in A \mid f(a) = y \text{ for some } y \in Y \}$.

(b) If $f[X] = Y$, is it necessarily true that $X = f^{-1}[Y]$? Justify your answer.

No.



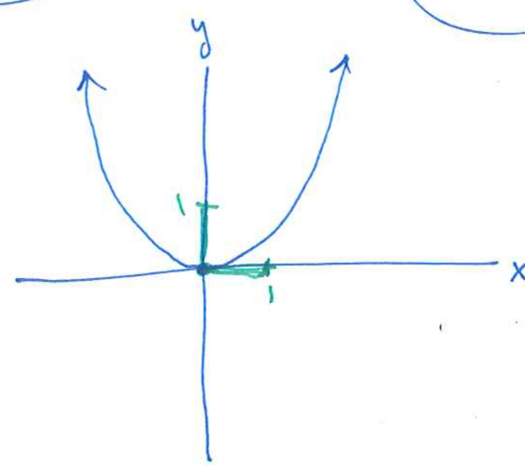
$f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

$X = [0,1]$

$f[X] = [0,1]$ so set $Y = [0,1]$

$f^{-1}[Y] = [-1,1] \neq X$



6. Find a solution to the equation $14x + 18y = 114$ in which x and y are integers.

Euclidean Algorithm:

$$18 = 14 \cdot 1 + 4$$
$$14 = 4 \cdot 3 + 2 \longrightarrow 2 = 14 - 4 \cdot 3$$
$$4 = 2 \cdot 2$$

$$= 14 - (18 - 14) \cdot 3$$
$$= 14 - 18 \cdot 3 + 14 \cdot 3$$

so $2 = 14 \cdot 4 + 18 \cdot (-3)$.

Multiply by 57:

$$57 \cdot 2 = 14 \cdot 4 \cdot 57 + 18 \cdot (-3) \cdot 57$$
$$= 14(57 \cdot 4) + 18((-3) \cdot 57)$$
$$= 14(228) + 18(-171)$$

Solution $x = 228$, $y = -171$

7. Consider the following two sets:

$$A = \{n \in \mathbb{Z} \mid n = 4x + 3y \text{ for some } x, y \in \mathbb{Z}\}$$
$$B = \{n \in \mathbb{Z} \mid n = 4x + 15y \text{ for some } x, y \in \mathbb{Z}\}$$

(a) List at least 5 elements of A and at least 5 elements of B.

A : 1 (since $\gcd(4,3)=1$) and so 2 is also an element since multiplying both sides of an equation $1 = 4x_0 + 3y_0$ by 2 yields an expression $2 = 4(2x_0) + 3(2y_0)$ so $2 \in A$. Similarly, $3, 4, 5 \in A$.

B : 1 (since $\gcd(4,15)=1$)
Similarly, $2, 3, 4, 5 \in B$.

(b) Is $A = B$? Prove or disprove.

Yes. $A = \mathbb{Z}$ $B = \mathbb{Z}$

(By the above explanation for part (a).)

Proof that $A = \mathbb{Z}$: By its definition, $A \subseteq \mathbb{Z}$. We must prove that $\mathbb{Z} \subseteq A$. Let $z \in \mathbb{Z}$.

Since $1 = \gcd(4,3)$, $1 = 4x_0 + 3y_0$ for some $x_0, y_0 \in \mathbb{Z}$. Multiply by z to obtain: $z = 4x_0 z + 3y_0 z$. Let $x = x_0 z$, $y = y_0 z$, to have $z = 4x + 3y$. So $z \in A$ (since $x, y \in \mathbb{Z}$). Therefore $\mathbb{Z} \subseteq A$. It follows that $A = \mathbb{Z}$.

A similar proof shows that $B = \mathbb{Z}$.

Therefore $A = B$.

8. Let $A = \{1, 2, 3\}$ and let X be the set of all bijective functions $f : A \to A$. Define a relation $\sim$ on X by $f \sim g$ if and only if $f(1) = g(1)$.
(a) Prove that $\sim$ is an equivalence relation.

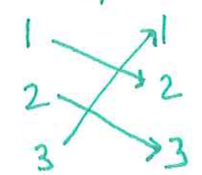
reflexive Let $f \in X$. Then $f \sim f$ since $f(1) = f(1)$.
So $\sim$ is reflexive.

symmetric Let $f, g \in X$ such that $f \sim g$. (We want to show $g \sim f$.)
Since $f \sim g$, $f(1) = g(1)$.
So $g(1) = f(1)$, which implies $g \sim f$.
So $\sim$ is symmetric.

transitive Let $f, g, h \in X$ such that $f \sim g$ and $g \sim h$. (We want to show $f \sim h$.)
Since $f \sim g$ and $g \sim h$, we have $f(1) = g(1)$ and $g(1) = h(1)$.
So $f(1) = h(1)$, and it follows that $f \sim h$.
Therefore $\sim$ is transitive.

(b) Find all elements in the equivalence class of the function f defined by $f(1) = 2$, $f(2) = 3$, and $f(3) = 1$.

f

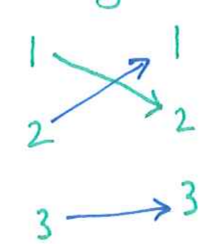


Let $g \in X$ such that $f \sim g$, i.e. $f(1) = g(1)$.

$[f] = \{f, g\}$

where $g(1) = 2$, $g(2) = 1$, $g(3) = 3$

g



9. Prove that if n is an integer for which $5 \nmid n$, then $n^2 \equiv 1 \mod 5$ or $n^2 \equiv 4 \mod 5$.

Since $5 \nmid n$, there are 4 cases to consider:

Case 1 $n \equiv 1 \mod 5$.

Then $n^2 \equiv 1^2 \mod 5$

$\equiv 1 \mod 5$.

Case 2 $n \equiv 2 \mod 5$.

Then $n^2 \equiv 2^2 \mod 5$

$\equiv 4 \mod 5$.

Case 3 $n \equiv 3 \mod 5$.

Then $n^2 \equiv 3^2 \mod 5$

$\equiv 9 \mod 5$

$\equiv 4 \mod 5$.

Case 4 $n \equiv 4 \mod 5$.

Then $n^2 \equiv 4^2 \mod 5$

$\equiv 16 \mod 5$

$\equiv 1 \mod 5$.

In each case, $n^2 \equiv 1 \mod 5$ or $n^2 \equiv 4 \mod 5$.

10. Let $\mathbb{Z}_9$ be the set of congruence classes of integers modulo 9. Find the subset of $\mathbb{Z}_9 - \{[0]_9\}$ consisting of all elements $[a]_9$ for which there exists $[x]_9 \in \cancel{\mathbb{Z}_9}$ $\mathbb{Z}_9 - \{[0]_9\}$ such that $[a]_9 \cdot_9 [x]_9 = [0]_9$.

i.e. $ax \equiv 0 \bmod 9$ $\left(\begin{matrix} 0 \leq a, x \leq 8, \\ a \neq 0, \ x \neq 0 \end{matrix} \quad a, x \in \mathbb{Z} \right)$

i.e. $9 \mid ax$

i.e. we want integers between 1 and 8 whose product is a multiple of 9

$a = 3$ $\left(\text{since } \underset{a=3}{3} \cdot \underset{x=3}{3} = 9\right)$

$a = 6$ $\left(\text{since } \underset{a=6}{6} \cdot \underset{x=3}{3} = 18\right)$ $\left(\text{or } 6 \cdot 6 = 36\right)$ $a=6, x=6$

Answer: 3, 6

(these are the integers between 1 and 8 that are not relatively prime to 9)