Math 300 Exam 1
February 17, 2022
S. Witherspoon

Name Solutions

There are 6 questions, for a total of 100 points. Point values are written beside each question.

1. Consider the statement:
For all integers m and n, if m is even and n is even, then mn is a multiple of 4.

(a) [5 points] Write the converse of this statement.

For all integers m and n, if mn is a multiple of 4, then m is even and n is even.

(b) [5] Write the contrapositive of this statement.

For all integers m and n, if mn is not a multiple of 4, then m is odd or n is odd.

(c) [5] Write the negation of this statement.

There exist integers m and n such that m is even and n is even and mn is not a multiple of 4.

(d) [5] Which of the above four statements *(the proposition, its converse (a), its contrapositive (b), its negation (c))* are true? (You need not justify your answer.)

The proposition and its contrapositive.

(Counterexample to (a): $m=1$, $n=4$)

2. [15] Let P, Q, and R be statements. For each of parts (a) and (b), use a truth table to determine whether or not the two expressions are logically equivalent.

(a) $(P \to Q) \to R$ $\qquad$ $P \to (Q \to R)$

(b) $(P \to Q) \vee (P \to R)$ $\qquad$ $R \vee (\neg P \vee Q)$

P	Q	R	$(P \to Q)$	$(P \to Q) \to R$	$Q \to R$	$P \to (Q \to R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	F	F	T	F	F	T
F	T	T	T	T	T	T
F	F	F	T	F	T	T

Not logically equivalent.

P	Q	R	$P \to Q$	$P \to R$	$(P \to Q) \vee (P \to R)$	$\neg P$	$(\neg P) \vee Q$	$R \vee (\neg P \vee Q)$
T	T	T	T	T	T	F	T	T
T	T	F	T	F	T	F	T	T
T	F	T	F	T	T	F	F	T
T	F	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Logically equivalent

3. [15] Prove that for all integers n,

$$n^2 + 3n + 1 \equiv 1(\text{mod}4) \text{ or } n^2 + 3n + 1 \equiv 3(\text{mod}4).$$

Let n be an integer. Then $n \equiv r \pmod 4$ where $r=0$, $r=1$, $r=2$, or $r=3$

Case 1 $r=0$. Then $n^2+3n+1 \equiv 0^2+3\cdot 0+1 \equiv 1 \pmod 4$.

Case 2 $r=1$. Then $n^2+3n+1 \equiv 1^2+3\cdot 1+1 \equiv 5 \equiv 1 \pmod 4$.

Case 3 $r=2$. Then $n^2+3n+1 \equiv 2^2+3\cdot 2+1 \equiv 4+6+1 \equiv 11 \equiv 3 \pmod 4$.

Case 4 $r=3$. Then $n^2+3n+1 \equiv 3^2+3\cdot 3+1 \equiv 9+9+1 \equiv 19 \equiv 3 \pmod 4$.

Therefore, $n^2+3n+1 \equiv 1 \pmod 4$ or $n^2+3n+1 \equiv 3 \pmod 4$.

4. [15] Prove that for all integers n, $3|n^2$ if, and only if, $3|n$.

Rewrite this biconditional statement:
For all integers n, if $3|n^2$ then $3|n$, and if $3|n$ then $3|n^2$.

<u>Proof</u>: Let n be an integer.

We first prove that if $3|n^2$ then $3|n$ by proving the contrapositive. This is the statement: If $3 \nmid n$, then $3 \nmid n^2$.

Assume $3 \nmid n$. Then $n = 3k+1$ or $n = 3k+2$ for some integer k.

<u>Case 1</u>. $n = 3k+1$. Then $n^2 = (3k+1)^2 = 9k^2+6k+1 = 3(3k^2+2k)+1$, so $3 \nmid n^2$.

<u>Case 2</u> $n = 3k+2$. Then $n^2 = (3k+2)^2 = 9k^2+12k+4 = 3(3k^2+4k+1)+1$, so $3 \nmid n^2$.

We next prove that if $3|n$ then $3|n^2$.
Assume $3|n$ so that $n = 3k$ for some integer k.
Then $n^2 = (3k)^2 = 9k^2 = 3(3k^2)$, which is divisible by 3.

5. [15] Prove that $\sqrt{3}$ is irrational.

Proof by Contradiction:

Assume $\sqrt{3}$ is rational, so that $\sqrt{3} = \frac{p}{q}$ for integers p, q such that p, q have no common factors. Then, squaring both sides,

$$3 = \frac{p^2}{q^2}$$

So $3q^2 = p^2$.

Thus $3 \mid p^2$. It follows that $3 \mid p$ (e.g. by #4).

So $p = 3k$ for some integer k. Then

$$3q^2 = (3k)^2 = 9k^2.$$

Dividing both sides by 3, we have

$$q^2 = 3k^2$$

so that $3 \mid q^2$. It follows that $3 \mid q$ (e.g. by #4).

Thus $3 \mid p$ and $3 \mid q$, so p and q have a common factor of 3. This contradicts the assumption that p and q have no common factors. Therefore $\sqrt{3}$ is irrational.

6. [20] *(True/False/Counterexample.)* For each statement, determine whether it is true or false, and accordingly write "T" or "F" in the blank. If the statement is false, provide a counterexample. (No need to prove true statements.)

__F__ The set of positive integers is closed under division.

Counterexample : $\frac{1}{2}$ is not an integer

(there are many other counterexamples)

__T__ The set of positive rational numbers is closed under division.

__F__ The set of irrational numbers is closed under multiplication.

Counterexample: $(\sqrt{2})(\sqrt{2}) = 2$

__T__ For all rational numbers x, $x^2 \neq 2$.

__F__ For all real numbers x, $x^2 \neq 2$.

Counterexample: $x = \sqrt{2}$ (or $x = -\sqrt{2}$)