Section 3.9: Slopes and tangents of parametric curves

Theory:

As we have seen before the parametrized curve

\[ \vec{r}(t) = (2 \sin(t), \cos(t)), \]

describes an ellipse whose implicit equation is given by

\[ \frac{x^2}{2} + y^2 = 1. \]

Note, that the same implicit equation can be described by different parametrized vector functions. For example, the following parametrized curves describe all the curve in (**)..

1. \[ \vec{r}(t) = (2 \sin(2t), \cos(2t)), \]
2. \[ \vec{r}(t) = (2 \sin(t + \pi), \cos(t + \pi)), \]
3. \[ \vec{r}(t) = (2 \sin(t^2), \cos(t^2)), \]

Assume (\( \star \)) describes the movement of a particle on the plane. Then (1) describes the movement of the particle on the same curve, but with twice the speed, (2) describes the same curve, but with a different starting point, and (3) also describes the movement on the same curve, but with increasing speed.
Problems:

Problem 1. Find the equation of tangent line to the curve
\[ \vec{r}(t) = (t + 1, t^2 + 4) \]
at the point (2, 5).

Solution:

The point (2, 5) is on the curve, indeed:
\[ (t + 1, t^2 + 4) = (2, 5), \text{ for } t = 1 \text{ (and only that } t). \]
\[ \vec{r}'(t) = (1, 2t), \text{ thus } \vec{r}'(1) = (1, 2). \]
So slope of tangent through (2, 5) is \( \frac{y'(1)}{x'(1)} = 2 \).
Equation of tangent through (2, 5):
\[ \frac{y - 5}{x - 2} = 2 \text{ or } y = 2x - 2 \cdot 2 + 5 = 2x + 1. \]
Problem 2. Find the equation of the tangent line to the parametrized curve

\[ \vec{r}(t) = (\sin(t), \tan(x)) \]

at the point \( t = \frac{\pi}{4} \).

Solution:

\[ \vec{r}(\frac{\pi}{4}) = (\sin\left(\frac{\pi}{4}\right), \tan\left(\frac{\pi}{4}\right)) = \left(\sin\left(\frac{\pi}{4}\right), \frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)}\right) = \left(\frac{\sqrt{2}}{2}, 1\right). \]

\[ \vec{r}'(t) = \frac{d}{dt}(\sin(t) \tan(t)) = (\cos(t), \sec^2(t)), \]

Thus,

\[ \vec{r}'\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \left(\frac{\sqrt{2}}{2}\right)^2\right) = \left(\frac{\sqrt{2}}{2}, 2\right). \]

Slope of tangent:

\[ \frac{y'(\frac{\pi}{4})}{x'(\frac{\pi}{4})} = \frac{2}{\sqrt{2}} = \frac{4}{\sqrt{2}} \]

Tangent

\[ \frac{y - 1}{x - \frac{\sqrt{2}}{2}} = \frac{4}{\sqrt{2}}, \text{ or } y = \frac{4}{\sqrt{2}}x - 2 + 1 = \frac{4}{\sqrt{2}}x - 1. \]
Problem 3. Find the points, where the tangents are horizontal and vertical:

\[ \vec{r}(t) = (t + t^2, t^2 - t). \]

Solution:

\[ \vec{r}'(t) = (1 + 2t, 2t - 1). \]

\[ x'(t) = 0, \text{ for } t = -\frac{1}{2}, \text{ and } y'\left(-\frac{1}{2}\right) \neq 0, \text{ thus vertical tangent at } t = -\frac{1}{2}, \]

\[ y'(t) = 0, \text{ for } t = \frac{1}{2}, \text{ and } x'\left(\frac{1}{2}\right) \neq 0, \text{ thus horizontal tangent at } t = \frac{1}{2}. \]