Chapter 2

Weak Topologies and Reflexivity

2.1 Topological Vector Spaces and Locally Convex Spaces

Definition 2.1.1. [Topological Vector Spaces and Locally Convex Spaces]

Let $E$ be a vector space over $\mathbb{K}$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $\mathcal{T}$ be a topology on $E$. We call $(E, \mathcal{T})$ (or simply $E$, if there cannot be a confusion), a topological vector space, if the addition:

$$+ : E \times E \to E, \quad (x, y) \mapsto x + y,$$

and the multiplication by scalars

$$\cdot : \mathbb{K} \times E \to E, \quad (\lambda, x) \mapsto \lambda x,$$

are continuous functions. A topological vector space is called locally convex if $0$ (and thus any point $x \in E$) has a neighbourhood basis consisting of convex sets.

Remark. Topological vector spaces are in general not metrizable. Thus, continuity, closedness, and compactness etc, cannot be described by sequences. We will need nets.

Assume that $(I, \leq)$ is a directed set. This means

- (reflexivity) $i \leq i$, for all $i \in I$,
- (transitivity) if for $i, j, k \in I$ we have $i \leq j$ and $j \leq k$, then $i \leq k$, and
(existence of upper bound) for any $i, j \in I$ there is a $k \in I$, so that $i \leq k$ and $j \leq k$.

A net is a family $(x_i : i \in I)$ indexed over a directed set $(I, \leq)$.

A subnet of a net $(x_i : i \in I)$ is a net $(y_j : j \in J)$, together with a map $j \mapsto i_j$ from $J$ to $I$, so that $x_{i_j} = y_j$, for all $j \in J$, and for all $i_0 \in I$ there is a $j_0 \in J$, so that $i_j \geq i_0$ for all $j \geq j_0$.

**Note:** A subnet of a sequence is not necessarily a subsequence.

**Definition 2.1.2.** In a topological space $(T, \mathcal{T})$, we say that a net $(x_i : i \in I)$ converges to $x$, if for all open sets $U$ with $x \in U$ there is an $i_0 \in I$, so that $x_i \in U$ for all $i \geq i_0$. If $(T, \mathcal{T})$ is Hausdorff $x$ is unique and we denote it by $\lim_{i \in I} x_i$.

Using nets we can describe continuity, closedness, and compactness in arbitrary topological spaces:

a) A map between two topological spaces is continuous if and only if the image of converging nets are converging.

b) A subset $A$ of a topological space $S$ is closed if and only if the limit point of every converging net in $A$ is in $A$.

c) A topological space $S$ is compact if and only if every net has a convergent subnet.

In order to define a topology on a vector space $E$ which turns $E$ into a topological vector space we (only) need to define an appropriate neighborhood basis of $0$.

**Proposition 2.1.3.** Assume that $(E, \mathcal{T})$ is a topological vector space. And let

$$\mathcal{U}_0 = \{U \in \mathcal{T}, 0 \in U\}.$$  

Then

a) For all $x \in E$, $x + \mathcal{U}_0 = \{x + U : U \in \mathcal{U}_0\}$ is a neighborhood basis of $x$,

b) for all $U \in \mathcal{U}_0$ there is a $V \in \mathcal{U}_0$ so that $V + V \subset U$,

c) for all $U \in \mathcal{U}_0$ and all $R > 0$ there is a $V \in \mathcal{U}_0$, so that

$$\{\lambda \in \mathbb{K} : |\lambda| < R\} \cdot V \subset U,$$
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d) for all $U \in \mathcal{U}_0$ and $x \in E$ there is an $\varepsilon > 0$, so that $\lambda x \in U$, for all $\lambda \in \mathbb{K}$ with $|\lambda| < \varepsilon$,

e) if $(E, \mathcal{T})$ is Hausdorff, then for every $x \in E$, $x \neq 0$, there is a $U \in \mathcal{U}_0$ with $x \notin U$,

f) if $E$ is locally convex, then for all $U \in \mathcal{U}_0$ there is a convex $V \in \mathcal{T}$, with $V \subseteq U$.

Conversely, if $E$ is a vector space over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and

$$\mathcal{U}_0 \subset \{ U \in \mathcal{P}(E) : 0 \in U \}$$

is non empty and is downwards directed, i.e. if for any $U, V \in \mathcal{U}_0$, there is a $W \in \mathcal{U}_0$, with $W \subseteq U \cap V$ and satisfies (b), (c) and (d), then

$$\mathcal{T} = \{ V \subseteq E : \forall x \in V \exists U \in \mathcal{U} : x + U \subseteq V \},$$

defines a topological vector space for which $\mathcal{U}_0$ is a neighborhood basis of $0$. $(E, \mathcal{T})$ is Hausdorff if $\mathcal{U}$ also satisfies (e) and locally convex if it satisfies (f).

Proof. Assume $(E, \mathcal{T})$ is a topological vector space and $\mathcal{U}_0$ is defined as above.

We observe that for all $x \in E$ the linear operator $T_x : E \to E$, $z \mapsto z + x$ is continuous. Since also $T_x \circ T_{-x} = T_{-x} \circ T_x = Id$, it follows that $T_x$ is an homeomorphism, which implies (a). Property (b) follows from the continuity of addition at 0. Indeed, we first observe that $\mathcal{U}_{0,0} = \{ V \times V : V \in \mathcal{U}_0 \}$ is a neighborhood basis of $(0,0)$ in $E \times E$, and thus, if $U \in \mathcal{U}_0$, then there exists a $V \in \mathcal{U}_0$ so that

$$V \times V \subseteq (\cdot + \cdot)^{-1}(U) = \{ (x, y) \in E \times E : x + y \in U \},$$

and this translates to $V + V \subseteq U$.

The claims (c) and (d) follow similarly from the continuity of scalar multiplication at 0. If $E$ is Hausdorff then $\mathcal{U}_0$ clearly satisfies (e) and it clearly satisfies (f) if $E$ is locally convex.

Now assume that $\mathcal{U}_0 \subset \{ U \in \mathcal{P}(E) : 0 \in U \}$ is non empty and downwards directed, that for any $U, V \in \mathcal{U}_0$, there is a $W \in \mathcal{U}_0$, with $W \subseteq U \cap V$, and that $\mathcal{U}_0$ satisfies (b), (c) and (d). Then

$$\mathcal{T} = \{ V \subseteq E : \forall x \in V \exists U \in \mathcal{U} : x + U \subseteq V \},$$
is finitely intersection stable and stable by taking (arbitrary) unions. Also \( \varnothing, E \in \mathcal{T} \). Thus \( \mathcal{T} \) is a topology. Also note that for \( x \in E \),

\[
\mathcal{U}_x = \{ x + U : U \in \mathcal{U}_0 \}
\]

is a neighborhood basis of \( x \).

We need to show that addition and multiplication by scalars is continuous. Assume \( (x_i : i \in I) \) and \( (y_i : i \in I) \) converge in \( E \) to \( x \in E \) and \( y \in E \), respectively, and let \( U \in \mathcal{U}_0 \). By (b) there is a \( V \in \mathcal{U}_0 \) with \( V + V \subset U \). We can therefore choose \( i_0 \) so that \( x_i \in x + V \) and \( y_i \in x + V \), for \( i \geq i_0 \), and, thus, \( x_i + y_i \in x + y + V + V \subset x + y + U \), for \( i \geq i_0 \). This proves the continuity of the addition in \( E \).

Assume \( (x_i : i \in I) \) converges in \( E \) to \( x \), \( (\lambda_i : i \in I) \) converges in \( \mathbb{K} \) to \( \lambda \) and let \( U \in \mathcal{U}_0 \). Then choose first (using property (b)) \( V \in \mathcal{U}_0 \) so that \( V + V \subset U \). Then, by property (c) choose \( W \in \mathcal{U}_0 \), so that for all \( \rho \in \mathbb{K} \), \( |\rho| \leq R := |\lambda| + 1 \) it follows that \( \rho W \subset V \) and, using (d) choose \( \varepsilon \in (0,1) \) so that \( \rho x \in W \), for all \( \rho \in \mathbb{K} \), with \( |\rho| \leq \varepsilon \). Finally choose \( i_0 \in I \) so that \( x_i \in x + W \) and \( |\lambda - \lambda_i| < \varepsilon \) (and thus \( |\lambda_i| < R \) for \( i \geq i_0 \)), for all \( i \geq i_0 \) in \( I_0 \) (and thus also \( |\lambda_i| < R \) for \( i \geq i_0 \)).

\[
\lambda_i x_i = \lambda_i(x_i - x) + (\lambda_i - \lambda)x + \lambda x + \lambda_i W + V \subset \lambda x + V + V \subset \lambda x + U.
\]

If \( \mathcal{U}_0 \) satisfies (e) and if \( x \neq y \) are in \( E \), then we can choose \( U \in \mathcal{U}_0 \) so that \( y - x \not\subset U \) and then, using the already proven fact that addition and multiplication by scalars is continuous, there is \( V \) so that \( V - V \subset U \). It follows that \( x + V \) and \( y + V \) are disjoint. Indeed, if \( x + v_1 = y + v_2 \), for some \( v_1, v_2 \in V \) it would follows that \( y - x = v_2 - v_1 \in U \), which is a contradiction.

If (f) is satisfied then \( E \) is locally convex since we observed before that \( \mathcal{U}_x = \{ x + U : U \in \mathcal{U}_0 \} \) is a neighborhood basis of \( x \), for each \( x \in E \). \( \square \)

Let \( E \) be a vector space over \( \mathbb{K} \), \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{K} = \mathbb{C} \), and let \( F \) be a subspace of

\[
E^\# = \{ f : E \to \mathbb{K} \text{ linear} \}.
\]

Assume that for each \( x \in E \) there is an \( x^* \in F \) so that \( x^*(x) \neq 0 \), we say in that case that \( F \) is separating the elements of \( E \) from 0. Consider

\[
\mathcal{U}_0 = \bigcap_{j=1}^n \{ x \in E : |x_i^*(x)| < \varepsilon_i : n \in \mathbb{N}, x_i^*, \in F, \text{ and } \varepsilon_i > 0, i = 1, \ldots, n \}.
\]

\( \mathcal{U}_0 \) is finitely intersection stable and it is easily checked that \( \mathcal{U}_0 \) satisfies that for assumptions (b)-(f). It follows therefore that \( \mathcal{U}_0 \) is the neighborhood basis of a topology which turns \( E \) into locally convex Hausdorff space.
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Definition 2.1.4. If $E$ is a topological vector space over $\mathbb{K}$, we call

$$E^* = \{ f : E \to \mathbb{K} : f \text{ linear and continuous} \}.$$  

Definition 2.1.5. [The Topology $\sigma(E, F)$]

Let $E$ be a vector space and let $F$ be a separating subspace of $E^\#$. Then we denote the locally convex Hausdorff topology generated by

$$U_0 = \bigcap_{j=1}^n \{ x \in E : |x_i^*(x)| < \varepsilon_i \} : n \in \mathbb{N}, x_i^* \in F, \text{ and } \varepsilon_i > 0, i = 1, \ldots, n \},$$

by $\sigma(E, F)$.

If $E$ is a locally convex space we call $\sigma(E, E^*)$, as in the case of Banach spaces, the Weak Topology on $E$ and denote it also by $w$. If $E$, say $E = F^*$, for some locally convex space $F$, we call $\sigma(F^*, F)$ the weak* topology and denote it by $w^*$ (if no confusion can happen).

From the Hahn Banach Theorem for Banach spaces it follows that the weak topology turns a Banach space $X$ into a Hausdorff space, and we can see $(X, \sigma(X, X^*))$ as a locally convex space. Similarly $(X^*, \sigma(X^*, X))$ is a locally convex space which is Hausdorff.

Proposition 2.1.6. Assume that $X$ is a Banach space and that $X^*$ denotes its dual with respect to the norm. Then

$$(X, \sigma(X, X^*))^* = X^* \text{ and } (X^*, \sigma(X^*, X))^* = X.$$  

Proposition 2.1.6 follows from a more general principle.

Proposition 2.1.7. Let $E$ be a locally convex space and $E^*$ its dual space. Equip $E^*$ with the topology $\sigma(E^*, E)$. Then $(E^*, \sigma(E^*, E^*))$ and $(E, \sigma(E, E^*))$ are locally convex spaces whose duals are $E$ and $E^*$, respectively (where we identify $e \in E$ in the canonical way with a map defined on $E^*$).

Proof. It is clear that $E$ belongs to $(E^*, \sigma(E^*, E))^*$ in the following sense: If $e \in E$ and if $\chi(e)$ is the function on $E^*$ which assigns to $f \in E^*$ the scalar $\langle f, e \rangle$, then $\chi(e)$ is in $(E^*, \sigma(E^*, E))^*$. From now on we identify $e$ with $\chi(e)$ and simply write $e$ instead of $\chi(e)$.

Assume $\phi : E^* \to \mathbb{K}$ is linear and $\sigma(E^*, E)$-continuous.

$$U = \{ f \in E^* : |\langle \phi, f \rangle| < 1 \} = \phi^{-1}(-1, 1)$$
is then an $\sigma(E, E^*)$-open neighborhood and thus there are $e_1, e_2, \ldots, e_n \in E$ and $\varepsilon > 0$ so that

$$\bigcap_{j=1}^{n} \{ f \in E^* \mid |\langle e_j, f \rangle| < \varepsilon \} \subset U.$$ 

It follows from this that

$$\bigcap_{j=1}^{n} \ker(e_j) \subset \ker(\phi).$$

Indeed,

$$\bigcap_{j=1}^{n} \ker(e_j) = \bigcap_{\delta > 0} \bigcap_{j=1}^{n} \{ f \in E^* \mid |\langle e_j, f \rangle| < \delta \varepsilon \}$$

$$= \bigcap_{\delta > 0} \bigcap_{j=1}^{n} \{ f \in E^* \mid |\langle e_j, f \rangle| < \varepsilon \}$$

$$\subset \bigcap_{\delta > 0} \delta \cdot U$$

$$= \bigcap_{\delta > 0} \{ f \in E^* : |\langle \phi, f \rangle| < 1 \}$$

$$= \bigcap_{\delta > 0} \{ f \in E^* : |\langle \phi, f \rangle| < \delta \} = \ker(\phi).$$

Now an easy linear algebra argument implies that $\phi$ is a linear combination of $e_1, e_2, \ldots, e_n$ which yields that $\phi \in E$. □

**Proposition 2.1.8.** Let $E$ be a vector space and let $F$ be a separating subspace of $E^\#$.

For a net $(x_i)_{i \in I} \subset E$ and $x \in E$

$$\lim_{i \in I} x_i = x \text{ in } \sigma(E, F) \iff \forall x^* \in F \quad \lim_{i \in I} \langle x^*, x_i \rangle = \langle x^*, x \rangle.$$ 

**Exercises**

1. Show that in a topological space $(T, T)$ a set $A$ is closed if and only if for a net $(x_i)_{i \in I} \subset A$ which converges to some $x$ it follows that $x \in A$.  

2. Assume that $E$ is a locally convex vectors space. Show that $(E, \sigma(E, E^*))$ is also a local convex vector space and that $(E, \sigma(E, E^*))^* = E^*$.

3. Prove Proposition 2.1.8.
CHAPTER 2. WEAK TOPOLOGIES AND REFLEXIVITY

2.2 Geometric Version of the Hahn-Banach Theorem for locally convex spaces

We want to formulate a geometric version of the Hahn-Banach Theorem.

**Definition 2.2.1.** A subset $A$ of a vector space $V$ over $\mathbb{K}$ is called convex if for all $a, b \in A$ and all $\lambda \in [0, 1]$ also $\lambda a + (1 - \lambda)b \in A$.

If $A \subset V$ we define the convex hull of $A$ by

$$\text{conv}(A) = \bigcap \{ C : A \subset C \subset V, C \text{ convex} \}$$

$$= \left\{ \sum_{j=1}^{n} \lambda_j a_j : n \in \mathbb{N}, \lambda_j \in [0, 1], a_i \in A, \text{ for } i = 1, \ldots, n, \text{ and } \sum_{j=1}^{n} \lambda_j = 1 \right\}.$$

A subset $A \subset V$ is called absorbing if for all $x \in V$ there is an $0 < r < \infty$ so that $x/r \in A$. For an absorbing set $A$ we define the Minkowski functional by

$$\mu_A : V \to [0, \infty), x \mapsto \inf\{ \lambda > 0 : x/\lambda \in A \}.$$

$A$ is called symmetric if for all $\lambda \in \mathbb{K}$, $|\lambda| = 1$, and all $x \in A$, it follows that $\lambda x \in A$.

**Lemma 2.2.2.** Assume $C$ is a convex and absorbing subset of a vector space $V$. Then $\mu_C$ is a sublinear functional on $V$, and

$$\{ v \in V : \mu_C(v) < 1 \} \subset C \subset \{ v \in V : \mu_C(v) \leq 1 \}.$$

If $V$ is a locally convex space space and if $0$ is in the open kernel of $C$, then $\mu_C$ is continuous at 0.

**Proof.** Since $C$ is absorbing $0 \in C$ and $\mu_C(0) = 0$. If $u, v \in V$ and $\varepsilon > 0$ is arbitrary, we find $0 < \lambda_u < \mu_C(u) + \varepsilon$ and $0 < \lambda_v < \mu_C(v) + \varepsilon$, so that $u/\lambda_u \in C$ and $v/\lambda_v \in C$ and thus

$$\frac{u + v}{\lambda_u + \lambda_v} = \frac{\lambda_u}{\lambda_u + \lambda_v} \frac{u}{\lambda_u} + \frac{\lambda_v}{\lambda_u + \lambda_v} \frac{v}{\lambda_v} \in C,$$

which implies that $\mu_C(u + v) \leq \lambda_u + \lambda_v \leq \mu_C(u) + \mu_C(v) + 2\varepsilon$, and, since, $\varepsilon > 0$ is arbitrary, $\mu_C(u + v) \leq \mu_C(u) + \mu_C(v)$.

Finally for $\lambda > 0$ and $v \in V$

$$\mu_C(\lambda v) = \inf \left\{ r > 0 : \frac{\lambda v}{r} \in C \right\} = \lambda \inf \left\{ \frac{r}{\lambda} : \frac{\lambda v}{r} \in C \right\} = \lambda \mu_C(v).$$
2.2. GEOMETRIC VERSION OF THE HAHN-BANACH THEOREM

To show the first inclusion in (2.1) assume \( v \in V \) with \( \mu_C(v) < 1 \), there is a \( 0 < \lambda < 1 \) so that \( v/\lambda \in C \), and, thus,

\[
v = \frac{v}{\lambda} + (1 - \lambda)0 \in C.
\]

The second inclusion is clear since for \( v \in C \) it follows that \( v = \frac{v}{1} \in C \).

If \( V \) is a locally convex space and \( 0 \in C^0 \), then there is a an open convex neighborhood \( U \) of 0, so that \( 0 < \frac{1}{2} \in C \), and, thus, \( v = \frac{v}{2} \in C \).

**Theorem 2.2.3.** (The Geometric Hahn-Banach Theorem for locally convex spaces) Let \( C \) be a non empty, closed convex subset of a locally convex and Hausdorff space \( E \) and let \( x_0 \in E \setminus C \).

Then there is an \( x^* \in E^* \) so that

\[
\sup_{x \in C} \Re(\langle x^*, x \rangle) < \Re(\langle x^*, x_0 \rangle).
\]

**Proof.** We first assume that \( K = \mathbb{R} \) and we also assume w.l.o.g. that \( 0 \in C \) (otherwise pass to \( C - x \) and \( x_0 - x \) for some \( x \in C \)). Let \( U \) be convex open neighborhood of 0 so that \( C \cap (x_0 + U) = \emptyset \), then let \( V \) be an open neighborhood of 0 so that \( V - V \subset U \) and let \( D = C + V \).It follows that also \( (x_0 + V) \cap D = \emptyset \).

From Lemma 2.2.2 it follows that \( \mu_D \) is a sublinear functional on \( E \), which is continuous at 0.

On the one dimensional space \( Y = \text{span}(x_0) \) define

\[
f : Y \rightarrow \mathbb{R}, \quad ax_0 \mapsto a\mu_D(x_0).
\]

Then \( f(y) \leq \mu_D(y) \) for all \( y \in Y \) (if \( y = ax_0 \), with \( a > 0 \) this follows from the positive homogeneity of \( \mu_D \), and if \( a < 0 \) this is clear). By Theorem 1.4.2 we can extend \( f \) to a linear function \( F \), defined on all of \( E \), with \( F(x) \leq \mu_D(x) \) for all \( x \in E \). Since \( \mu_D \) is continuous at 0 it follows \( F \) is continuous at 0 and thus in \( E^* \).

Moreover, if \( x \in C \) it follows that \( F(x_0) > 1 \geq \sup_{x \in C} \mu_D(x) \geq 1 \). If \( K = \mathbb{C} \) we first choose \( F \), by considering \( E \) to be a real locally convex space, and then put \( f(x) = F(x) - iF(ix) \). It is then easily checked that \( F \) is a complex linear bounded functional on \( E \).
Corollary 2.2.4. Assume that $A$ and $B$ are two convex closed subsets of a locally convex space $E$, with for which there is an open neighborhood $U$ of 0 with $(A + U) \cap (B + U) = \emptyset$.

Then there is an $x^* \in E^*$ and $\alpha \in \mathbb{R}$ so that

$$\Re(\langle x^*, x \rangle) \leq \alpha \leq \Re(\langle x^*, y \rangle), \text{ for all } x \in A \text{ and } y \in B.$$ 

Proof. Consider

$$C = A - B = \{x - y : x \in A \text{ and } y \in B\}.$$ 

we note that $0 \not\in C$ is convex and that

Applying Theorem 2.2.3 we obtain an $x^* \in X^*$ so that

$$\sup_{x \in C} \Re(\langle x^*, x \rangle) < \Re(\langle x^*, 0 \rangle) = 0.$$ 

But this means that for all $x \in A$ and all $y \in B \Re(\langle x^*, x - y \rangle) < 0$ and thus

$$\Re(\langle x^*, x \rangle) < \Re(\langle x^*, y \rangle).$$

An easy consequence of the geometrical version of the Hahn-Banach Theorem 2.2.3 is the following two observation.

Proposition 2.2.5. If $A$ is a convex subset of a Banach space $X$ then

$$\overline{A}^{\omega} = \overline{A}^\| \|.$$ 

If a representation of the dual space of a Banach space $X$ is not known, it might be hard to verify weak convergence of a sequence directly. The following Corollary of Proposition 2.2.5 states an equivalent criterium for a sequence to be weakly null without using the dual space of $X$.

Corollary 2.2.6. For a bounded sequence $(x_n)$ in Banach space $X$ it follows that $(x_n)$ is weakly null if and only if for all subsequences $(z_n)$, all $\varepsilon > 0$ there is a convex combination $z = \sum_{j=1}^k \lambda_j z_j$ of $(z_j)$ (i.e. $\lambda_i \geq 0$, for $i = 1, 2, \ldots, k$, and $\sum_{j=1}^l \lambda_j = 1$) so that $\|z\| \leq \varepsilon$. 
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Exercises

1. Prove Proposition 2.2.5.

2. Let \( X \) be a Banach space with norm \( \| \cdot \| \). Show that \( \mu_{B_X} = \| \cdot \| \).

3. Prove Corollary 2.2.6.

4. Show that \( \ell_1 \) is not isomorphic to a subspace of \( c_0 \).

5. Show that there is an \( x^* \in \ell^*_\infty \) so that

   a) \( \| x^* \| = 1 \),
   
   b) \( \langle x^*, x \rangle = \lim_{i \to \infty} x_i \), for \( x = (x_i) \in c = \{ (\xi_i) : \lim_{i \to \infty} \xi_i \text{ exists} \} \)
   
   c) If \( x = (\xi_i) \in \ell_\infty \), and \( \xi_i \geq 0 \), for \( i \in \mathbb{N} \), then \( \langle x^*, x \rangle \geq 0 \), and

   d) If \( x = (\xi_i) \in \ell_\infty \) and \( x' = (\xi_2, \xi_3, \ldots) \) then \( \langle x^*, x' \rangle = \langle x^*, x \rangle \)
2.3 Reflexivity and Weak Topology

**Proposition 2.3.1.** If \( X \) is a Banach space and \( Y \) is a closed subspace of \( X \), then \( \sigma(Y, Y^*) = \sigma(X, X^*) \cap Y \), i.e. the weak topology on \( Y \) is the weak topology on \( X \) restricted to \( Y \).

**Theorem 2.3.2.** (Theorem of Alaoglu, c.f. [Fol, Theorem 5.18]) \( B_{X^*} \) is \( w^* \) compact for any Banach space \( X \).

**Sketch of a proof.** Consider the map

\[
\Phi : B_X^* \to \prod_{x \in X} \{ \lambda \in \mathbb{K} : |\lambda| \leq \|x\| \}, \quad x^* \mapsto (x^*(x) : x \in X).
\]

Then we check that \( \Phi \) is continuous with respect to \( w^* \) topology on \( B_X^* \) and the product topology on \( \prod_{x \in X} \{ \lambda \in \mathbb{K} : |\lambda| \leq \|x\| \} \), has a closed image, and is a homeomorphism from \( B_X^* \) onto its image.

Since by the Theorem of Tychanoff \( \prod_{x \in X} \{ \lambda \in \mathbb{K} : |\lambda| \leq \|x\| \} \) is compact, \( \Phi(B_{X^*}) \) is a compact subset, which yields (via the homeomorphism \( \Phi^{-1} \)) that \( B_{X^*} \) is compact in the \( w^* \) topology.

**Theorem 2.3.3.** (Theorem of Goldstein) \( B_X \) is (via the canonical embedding) \( w^* \) dense in \( B_{X^{**}} \).

The proof follows easily from the following Lemma (see Problem 1).

**Lemma 2.3.4.** Let \( X \) be a Banach space and let \( x^{**} \in X^{**} \), with \( x^{**} \leq 1 \), and \( x_1^*, x_2^*, \ldots, x_n^* \in X^* \). Then

\[
\inf_{\|x\| \leq 1} \sum_{i=1}^n |\langle x^{**}, x_i^* \rangle - \langle x_i^*, x \rangle|^2 = 0.
\]

**Proof.** For \( x \in X \) put \( \phi(x) = \sum_{i=1}^n |\langle x^{**}, x_i^* \rangle - \langle x_i^*, x \rangle|^2 \) and \( \beta = \inf_{x \in B_X} \phi(x) \), and choose a sequence \( (x_j) \subset B_X \) so that \( \phi(x_j) \searrow \beta \), if \( j \to \infty \).

W.l.o.g we can also assume that \( \xi_i = \lim_{k \to \infty} \langle x_i^*, x_k \rangle \) exists for all \( i = 1, 2, \ldots, n \).

For any \( t \in [0, 1] \) and any \( x \in B_X \) we note for \( k \in \mathbb{N} \)

\[
\phi((1-t)x_k + tx) = \sum_{i=1}^n |\langle x^{**}, x_i^* \rangle - (1-t)\langle x_i^*, x_k \rangle - t\langle x_i^*, x \rangle|^2
\]
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\[
= \sum_{i=1}^{n} |\langle x^*, x_i^* \rangle - \langle x_i^*, x_k \rangle + t \langle x_i^*, x_k - x \rangle|^2
\]

\[
= \sum_{i=1}^{n} |\langle x^*, x_i^* \rangle - \langle x_i^*, x_k \rangle|^2
\]

\[
+ 2t \Re \left( \sum_{i=1}^{n} \left( \langle x^*, x_i^* \rangle - \langle x_i^*, x_k \rangle \langle x_i^*, x_k - x \rangle \right) \right) + t^2 \sum_{i=1}^{n} |\langle x_i^*, x_k \rangle - \langle x_i^*, x \rangle|^2
\]

\[
\rightarrow_{k \to \infty} \beta + 2t \Re \left( \sum_{i=1}^{n} \left( \langle x^*, x_i^* \rangle \xi_i - \langle x_i^*, x \rangle \right) \right) + t^2 \sum_{i=1}^{n} |\xi_i - \langle x_i^*, x \rangle|^2.
\]

From the minimality of \( \beta \) it follows that for all \( x \in B_X \), that for all \( t > 0 \) the second term needs to be non negative, and, thus,

\[
\Re \left( \sum_{i=1}^{n} \lambda_i \xi_i \right) \geq \Re (\langle x^*, x \rangle) \text{ with } x^* := \sum_{i=1}^{n} \lambda_i x_i^*,
\]

and thus

\[
\|x^*\| \leq \Re \left( \sum_{i=1}^{n} \lambda_i \xi_i \right).
\]

Indeed, write \( \langle x^*, x \rangle = re^{ia} \), then

\[
|\langle x^*, x \rangle| = e^{-ia} \langle x^*, x \rangle = \langle x^*, e^{-ia} x \rangle \leq \Re \left( \sum_{i=1}^{n} \lambda_i \xi \right).
\]

On the other hand, since \( x \in B_X \) is arbitrary,

\[
\|x^*\| \geq \limsup_{k \to \infty} \Re (\langle x^*, x_k \rangle) = \Re \left( \sum_{i=1}^{n} \lambda_i \xi_i \right)
\]

and thus

\[
\|x^*\| \geq \left| \sum_{i=1}^{n} \lambda_i \xi_i \right| \geq \Re \left( \sum_{i=1}^{n} \lambda_i \xi_i \right),
\]

which implies that

\[
\|x^*\| = \Re \left( \sum_{i=1}^{n} \lambda_i \xi_i \right).
\]

So

\[
\beta = \lim_{k \to \infty} \phi(x_k)
\]
Thus \( \beta = 0 \) which proves our claim. \( \square \)

**Theorem 2.3.5.** Let \( X \) be a Banach space. Then \( X \) is reflexive if and only if \( B_X \) is compact in the weak topology.

**Proof.** Let \( \chi : X \hookrightarrow X^{**} \) be the canonical embedding.

"\( \Rightarrow \)" If \( X \) is reflexive and thus \( \chi \) is onto it follows that \( \chi \) is an homeomorphism between \((B_X, \sigma(X, X^*))\) and \((B_X^{**}, \sigma(X^{**}, X^*))\). But by the Theorem of Alaoglu 2.3.2 \((B_X^{**}, \sigma(X^{**}, X^*))\) is compact.

"\( \Leftarrow \)" Assume that \((B_X, \sigma(X, X^*))\) is compact, and assume that \( x^{**} \in B_X^{**} \) we need to show that there is an \( x \in B_X \) so that \( \chi(x) = x^{**} \), or equivalently that \( \langle x^*, x \rangle = \langle x^{**}, x^* \rangle \) for all \( x^* \in X^* \).

For any finite set \( A = \{x_1^*, \ldots, x_n^*\} \subset X^* \) and for any \( \varepsilon > 0 \) we can, according to Lemma 2.3.4, choose an \( x_{(A, \varepsilon)} \in B_X \) so that

\[
\sum_{i=1}^{n} |\langle x^{**}, x_i^* \rangle - \langle x_i^*, x_k \rangle|^2 \leq \varepsilon.
\]

The set

\[
I = \{(A, \varepsilon) : A \subset X^* \text{ finite and } \varepsilon > 0\},
\]
2.3. REFLEXIVITY AND WEAK TOPOLOGY

is directed via \((A, \varepsilon) \leq (A', \varepsilon') : \iff A \subset A' \text{ and } \varepsilon' \leq \varepsilon\). Thus, by compactness, the net \((x_{(A, \varepsilon)} : (A, \varepsilon) \in I)\) must have a subnet \((z_j : j \in J)\) which converges weakly to some element \(x \in B_X\).

We claim that \(\langle x^*, x \rangle = \langle x^{**}, x^* \rangle\), for all \(x^* \in X^*\). Indeed, let \(j \mapsto i_j\) be the map from \(J\) to \(I\), so that \(z_j = x_{i_j}\), for all \(j \in J\), and so that, for any \(i_0 \in I\) there is a \(j_0\) with \(i_j \geq i_0\), for \(j \geq j_0\). Let \(x^* \in X^*\) and \(\varepsilon > 0\).

Put \(i_0 = \{x^*\}, \varepsilon \in I\), choose \(j_0\), so that \(i_j \geq i_0\), for all \(j \geq j_0\), and choose \(j_1 \in J\), \(j_1 \geq j_0\), so that \(|\langle x - z_j, x^* \rangle| < \varepsilon\), for all \(j \geq j_1\). It follows therefore that (note that for \(i_{j_1} = (A, \varepsilon')\) it follows that \(x^* \in A\) and \(\varepsilon' \leq \varepsilon\))

\[
|\langle x^{**} - x, x^* \rangle| \leq |\langle x^{**} - x_{i_{j_1}}, x^* \rangle| + |\langle z_{j_1} - x, x^* \rangle| \leq \varepsilon + \varepsilon = 2\varepsilon.
\]

Since \(\varepsilon > 0\) and \(x^* \in X^*\) were arbitrary we deduce our claim.

\(\square\)

**Theorem 2.3.6.** For a Banach space \(X\) the following are equivalent.

\(a)\) \(X\) is reflexive,
\(b)\) \(X^*\) is reflexive,
\(c)\) every closed subspace of \(X\) is reflexive.

**Proof.** “(a)⇒(c)” Assume \(Y \subset X\) is a closed subspace. Proposition 2.2.5 yields that \(B_Y = B_X \cap Y\) is a \(\sigma(X, X^*)\)-closed and, thus, \(\sigma(X, X^*)\)-compact subset of \(B_X\). Since, by the Theorem of Hahn-Banach (Corollary 1.4.4), every \(y^* \in Y^*\) can be extended to an element in \(X^*\), it follows that \(\sigma(Y, Y^*)\) is the restriction of \(\sigma(X, X^*)\) to the subspace \(Y\). Thus, \(B_Y\) is \(\sigma(Y, Y^*)\)-compact, which implies, by Theorem 2.3.5 that \(Y\) is reflexive.

“(a)⇒(b)” If \(X\) is reflexive then \(\sigma(X^*, X^{**}) = \sigma(X^*, X)\). Since by the Theorem of Alaoglu 2.3.2 \(B_X^*\) is \(\sigma(X^*, X)\)-compact the claim follows from Theorem 2.3.5.

“(c)⇒(a)” clear.

“(b)⇒(a)” If \(X^*\) is reflexive, then, by “(a)⇒(b)” \(X^{**}\) is also reflexive and thus, the implication “(a)⇒(c)” yields that \(X\) is reflexive.

\(\square\)

Similar ideas as in the proof of Theorem 2.3.3 are used to show the following result which characterizes when a Banach space \(X\) is a dual space of another space.

**Theorem 2.3.7.** Assume that \(X\) is a Banach space and \(Z\) is a closed subspace of \(X^*\), so that \(B_X\) is compact in the topology \(\sigma(X, Z)\), and so that \(\|x\| = \sup_{z \in B_Z} |z(x)|\).
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Then $Z^*$ is isometrically isomorphic to $X$ and the map

$$T : X \to Z^*, \quad x \mapsto f_x, \text{ with } f_x(z) = \langle z, x \rangle, \text{ for } x \in X \text{ and } z \in Z,$$

is an isometrical isomorphism onto $Z^*$.

Proof. We first note that $T(B_X)$ is $\sigma(Z^*, Z)$ dense in $B_{Z^*}$. Indeed, if this is not true we can apply the Geometric Hahn Banach Theorem for locally convex spaces (Theorem 2.2.3) applied to the locally convex space $(Z^*, \sigma(Z^*, Z))$ whose dual is by Proposition 2.1.6 $(Z, \sigma(Z, Z^*))$, and obtain elements $z^* \in S_{Z^*}$ and $z \in S_Z$ so that

$$1 = \|z\| = \sup_{x \in B_X} \langle x, z \rangle < \langle z^*, z \rangle = 1,$$

which is a contradiction.

Secondly, our assumption says that $T(B_X)$ is $\sigma(Z^*, Z)$-compact. To see that note that if $(x_i)_{i \in I}$ is a net in $X$ and $z^* \in Z^*$, then

$$(f_{x_i})_{i \in I} \text{ converges to } z^* \text{ with respect to } \sigma(Z^*, Z) \quad \iff \lim_{i \in I} \langle x_i, z \rangle = \langle z^*, z \rangle \text{ for all } z \in Z \quad \iff z^* \in T(B_X) \text{ and } \sigma(X, Z^*) - \lim_{i \in I} \langle x_i, z \rangle = z^* \text{ (By assumption).}$$

Exercises

1. Show Theorem 2.3.3 using Lemma 2.3.4.

2. Prove Proposition 2.3.1.

3. Show that $B_{\ell_1^*}$ is not sequentially compact in the $w^*$-topology.
   Hint: Consider the unit vector basis of $\ell_1$ seen as subsequence of $B_{\ell_1^*}$.

4. Prove that for a Banach space $X$ every $w^*$-converging sequence in $X^*$ is bounded, but that if $X$ is infinite dimensional, $X^*$ contains nets $(x_i^* : i \in I)$ which converge to 0, but so that for every $c > 0$ and all $i \in I$ there is a $j_0 \geq i$, with $\|x_j\| \geq c$, whenever $j \geq j_0$.

5. Show that in each infinite dimensional Banach space $X$ there is a weakly null net in $S_X$. 
6.* Prove that every weakly null sequence in $\ell_1$ is norm null.

Hint: Assume that $(x_n) \subset S_{\ell_1}$ is weakly null. Then there is a subsequence $x_{n_k}$ and a block sequence $(z_k)$ so that $\lim_{k \to \infty} \|x_{n_k} - z_k\| = 0$.

Here we mean by a block sequence a sequence $(z_n)$ in $\ell_1$ of the form

$$z_n = (0, 0, 0 \ldots, 0, \zeta(n_{k-1} + 1), \zeta(n, n_{k-1} + 2)), \ldots, \zeta(n, n_k), 0, 0, \ldots),$$

where $1 = n_1 < n_2 < n_3 < \ldots$. 
2.4 Annihilators, Complemented Subspaces

Definition 2.4.1. (Annihilators, Pre-Annihilators)
Assume $X$ is a Banach space. Let $M \subseteq X$ and $N \subseteq X^*$. We call
\[ M^\perp = \{ x^* \in X^* : \forall x \in M \langle x^*, x \rangle = 0 \} \subset X^* , \]
the annihilator of $M$ and
\[ N_\perp = \{ x \in X : \forall x^* \in N \langle x^*, x \rangle = 0 \} \subset X , \]
the pre-annihilator of $N$.

Proposition 2.4.2. Let $X$ be a Banach space, and assume $M \subseteq X$ and $N \subseteq X^*$.

a) $M^\perp$ is a closed subspace of $X^*$, $M^\perp = (\text{span}(M))^\perp$, and $(M^\perp)_\perp = \text{span}(M)$,
b) $N_\perp$ is a closed subspace of $X$, $N_\perp = (\text{span}(N))_\perp$, and $\text{span}(N) \subset (N_\perp)^\perp$.
c) $\text{span}(M) = X \iff M^\perp = \{0\}$.

Proposition 2.4.3. If $X$ is Banach space and $Y \subseteq X$ is a closed subspace then $(X/Y)^*$ is isometrically isomorphic to $Y^\perp$ via the operator
\[ \Phi : (X/Y)^* \to Y^\perp , \text{ with } \Phi(z^*)(x) = z^*(\overline{x}). \]
(recall $\overline{x} := x + Y \subseteq X/Y$ for $x \in X$).

Proof. Let $Q : X \to X/Y$ be the quotient map.

For $z^* \in (X/Y)^*$, $\Phi(z^*)$, as defined above, can be written as $\Phi(z^*) = z^* \circ Q$. Thus $\Phi(z^*) \in X^*$. Since $Q(Y) = \{0\}$ it follows that $\Phi(z^*) \in Y^\perp$.

For $z^* \in (X/Y)^*$ we have
\[ \| \Phi(z^*) \| = \sup_{x \in B_X} \langle z^*, Q(x) \rangle = \sup_{\overline{x} \in B_{X/Y}} \langle z^*, \overline{x} \rangle = \| z^* \|_{(X/Y)^*} , \]
where the second equality follows on the one hand from the fact that $\| Q(x) \| \leq \| x \|$, for $x \in X$, and on the other hand, from the fact that for any $\overline{x} \in X/Y$ there is a sequence $(y_n) \subseteq Y$ so that $\limsup_{n \to \infty} \| x + y_n \| = \| \overline{x} \|$. Thus $\Phi$ is an isometric embedding. If $x^* \in Y^\perp \subseteq X^*$, we define
\[ z^* : X/Y \to \mathbb{K} , \quad x + Y \mapsto \langle x^*, x \rangle . \]
First note that this map is well defined (since \( \langle x^*, x + y_1 \rangle = \langle x^*, x + y_2 \rangle \) for \( y_1, y_2 \in Y \)). Since \( x^* \) is linear, \( z^* \) is also linear, and \( |\langle z^*, \overline{x} \rangle| = |\langle x^*, x \rangle| \), for all \( x \in X \), and thus \( \|z^*\|(X/Y)^* = \|x^*\| \). Finally, since
\[
\langle \Phi(z^*), x \rangle = \langle z^*, Q(x) \rangle = \langle x^*, x \rangle,
\]
it follows that \( \Phi(z^*) = x^* \), and thus that \( \Phi \) is surjective. \( \square \)

**Proposition 2.4.4.** Assume \( X \) and \( Y \) are Banach spaces and \( T \in L(X, Y) \). Then
\[
(2.2) \quad T(X)^\perp = \mathcal{N}(T^*) \quad \text{and} \quad T^*(Y^*) \subset \mathcal{N}(T)^\perp
\]
\[
(2.3) \quad \overline{T(X)} = \mathcal{N}(T^*)_\perp \quad \text{and} \quad T^*(Y^*)_\perp = \mathcal{N}(T).
\]

**Proof.** We only prove (2.2). The verification of (2.3) is similar. For \( y^* \in Y^* \)
\[
y^* \in T(X)^\perp \iff \forall x \in X \quad \langle y^*, T(x) \rangle = 0
\]
\[
\iff \forall x \in X \quad \langle T^*(y^*), x \rangle = 0
\]
\[
\iff T^*(y^*) = 0 \iff y^* \in \mathcal{N}(T^*),
\]
which proves the first part of (2.2), and for \( y^* \in Y^* \) and all \( x \in \mathcal{N}(T) \),
it follows that \( \langle T^*(y^*), x \rangle = \langle y^*, T(x) \rangle = 0 \), which implies that \( T^*(Y^*) \subset \mathcal{N}(T)^\perp \), and, thus, \( T^*(X^*) \subset \mathcal{N}(T)^\perp \) since \( \mathcal{N}(T)^\perp \) is closed. \( \square \)

**Definition 2.4.5.** Let \( X \) be a Banach space and let \( U \) and \( V \) be two closed subspaces of \( X \). We say that \( X \) is the complemented sum of \( U \) and \( V \) and we write \( X = U \oplus V \), if for every \( x \in X \) there are \( u \in U \) and \( v \in V \), so that \( x = u + v \) and so that this representation of \( x \) as sum of an element of \( U \) and an element of \( V \) is unique.

We say that a closed subspace \( Y \) of \( X \) is complemented in \( X \) if there is a closed subspace \( Z \) of \( X \) so that \( X = Y \oplus Z \).

**Remark.** Assume that the Banach space \( X \) is the complemented sum of the two closed subspaces \( U \) and \( V \). We note that this implies that \( U \cap V = \{0\} \).

We can define two maps
\[
P : X \to U \text{ and } Q : X \to V
\]
where we define \( P(x) \in U \) and \( Q(x) \in V \) by the equation \( x = P(x) + Q(y) \), with \( P(x) \in U \) and \( Q(x) \in V \) (which, by assumption, has a unique solution).
Note that \( P \) and \( Q \) are linear. Indeed if \( P(x_1) = u_1, P(x_2) = u_2, Q(x_1) = v_1 \),
\( Q(x_2) = v_2 \), then for \( \lambda, \mu \in \mathbb{K} \) we have \( \lambda x_1 + \mu x_2 = \lambda u_1 + \mu u_2 + \lambda v_1 + \mu v_2 \), and thus, by uniqueness \( P(\lambda x_1 + \mu x_2) = \lambda u_1 + \mu u_2 \), and \( Q(\lambda x_1 + \mu x_2) = \lambda v_1 + \mu v_2 \).

Secondly it follows that \( P \circ P = P \), and \( Q \circ Q = Q \). Indeed, for any \( x \in X \) we write \( P(x) = P(x) + 0 \in U + V \), and since this representation of \( P(x) \) is unique it follows that \( P(P(x)) = P(x) \). The argument for \( Q \) is the same.

Finally it follows that, again using the uniqueness argument, that \( P \) is the identity on \( U \) and \( Q \) is the identity on \( V \).

We therefore proved that

a) \( P \) is linear,

b) the image of \( P \) is \( U \)

c) \( P \) is idempotent, i.e. \( P^2 = P \)

We say in that case that \( P \) is a linear projection onto \( U \). Similarly \( Q \) is a linear projection onto \( V \), and \( P \) and \( Q \) are complementary to each other, meaning that \( P(X) \cap Q(X) = \{0\} \) and \( P+Q = 1 \). A linear map \( P : X \to X \) with the properties (a) and (c) is called projection.

The next Proposition will show that \( P \) and \( Q \) as defined in above remark are actually bounded.

**Lemma 2.4.6.** Assume that \( X \) is the complemented sum of two closed subspaces \( U \) and \( V \). Then the projections \( P \) and \( Q \) as defined in above remark are bounded.

**Proof.** Consider the norm \( \| \cdot \| \) on \( X \) defined by

\[
\| x \| = \| P(x) \| + \| Q(x) \|, \quad \text{for } x \in X.
\]

We claim that \( (X, \| \cdot \|) \) is also a Banach space. Indeed if \( (x_n) \subset X \) with

\[
\sum_{n=1}^{\infty} \| x_n \| = \sum_{n=1}^{\infty} \| P(x_n) \| + \sum_{n=1}^{\infty} \| Q(x_n) \| < \infty.
\]

Then \( u = \sum_{n=1}^{\infty} P(x_n) \in U \), \( v = \sum_{n=1}^{\infty} Q(x_n) \in V \) (\( U \) and \( V \) are assumed to be closed) converge in \( U \) and \( V \) with respect to \( \| \cdot \| \), respectively. Since \( \| \cdot \| \leq \| \cdot \| \) also \( x = \sum_{n=1}^{\infty} x_n \) converges with respect to \( \| \cdot \| \) and

\[
x = \sum_{n=1}^{\infty} x_n = \lim_{n \to \infty} \sum_{j=1}^{n} (P(x_j) + Q(x_j)) = \lim_{n \to \infty} \sum_{j=1}^{n} P(x_j) + \lim_{n \to \infty} \sum_{j=1}^{n} Q(x_j) = u + v,
\]
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and

\[ \left\| x - \sum_{j=1}^{n} x_n \right\| = \left\| u - \sum_{j=1}^{n} P(x_n) + v - \sum_{j=1}^{n} Q(x_n) \right\| \]

\[ = \left\| u - \sum_{j=1}^{n} P(x_n) \right\| + \left\| v - \sum_{j=1}^{n} Q(x_n) \right\| \rightarrow_{n \to \infty} 0, \]

(here all series are meant to converge with respect to \( \| \cdot \| \)) which proves that 

\( (X, \| \cdot \|) \) is complete.

Since the identity is a bijective linear bounded operator from \( (X, \| \cdot \|) \) to \( (X, \| \cdot \|) \) it has by Corollary 1.3.6 of the Closed Graph Theorem a continuous inverse and is thus an isomorphy. Since \( \| P(x) \| \leq \| x \| \) and \( \| Q(x) \| \leq \| x \| \)
we deduce our claim. \( \square \)

**Proposition 2.4.7.** Assume that \( X \) is a Banach space and that \( P : X \rightarrow X \), is a bounded projection onto a closed subspace of \( X \).

Then \( X = P(X) \oplus N(P) \).

**Theorem 2.4.8.** There is no linear bounded operator \( T : \ell_{\infty} \rightarrow \ell_{\infty} \) so that the kernel of \( T \) equals to \( c_0 \).

**Corollary 2.4.9.** \( c_0 \) is not complemented in \( \ell_{\infty} \).

**Proof of Theorem 2.4.8.** For \( n \in \mathbb{N} \) we let \( e_n^* \) be the \( n \)-th coordinate functional on \( \ell_{\infty} \), i.e.

\[ e_n^* : \ell_{\infty} \rightarrow \mathbb{K}, \quad x = (x_j) \mapsto x_n. \]

Step 1. If \( T : \ell_{\infty} \rightarrow \ell_{\infty} \) is bounded and linear, then

\[ \mathcal{N}(T) = \bigcap_{n=1}^{\infty} \mathcal{N}(e_n^* \circ T). \]

Indeed, note that

\[ x \in \mathcal{N}(T) \iff \forall n \in \mathbb{N} \quad e_n^*(T(x)) = \langle e_n^*, T(x) \rangle = 0. \]

In order to prove our claim we will show that \( c_0 \) cannot be the intersection of the kernel of countably many functionals in \( \ell_{\infty}^* \).

Step 2. There is an uncountable family \( (N_{\alpha} : \alpha \in I) \) of infinite subsets of \( \mathbb{N} \) for which \( N_{\alpha} \cap N_{\beta} \) is finite whenever \( \alpha \neq \beta \) are in \( I \).
Write the rational numbers \( \mathbb{Q} \) as a sequence \( (q_j : j \in \mathbb{N}) \), and choose for each \( r \in \mathbb{R} \) a sequence \( (n_k(r) : k \in \mathbb{N}) \), so that \( (q_{n_k(r)} : k \in \mathbb{N}) \) converges to \( r \). Then, for \( r \in \mathbb{R} \) let \( N_r = \{ n_k(r) : k \in \mathbb{N} \} \). The family \( (N_r : r \in \mathbb{R}) \) then satisfies the claim in Step 2.

For \( i \in I \), put \( x_\alpha = 1_{N_\alpha} \in \ell_\infty \), i.e.

\[
x_\alpha = (\xi_k^{(\alpha)} : k \in \mathbb{N}) \quad \text{with} \quad \xi_k^{(\alpha)} = \begin{cases} 1 & \text{if } k \in N_\alpha \\ 0 & \text{if } k \notin N_\alpha. \end{cases}
\]

Step 3. If \( f \in \ell_\infty^* \) and \( c_0 \subset \mathcal{N}(f) \) then \( \{ \alpha \in I : f(x_\alpha) \neq 0 \} \) is countable.

In order to verify Step 3 let \( A_n = \{ \alpha : |f(x_\alpha)| \geq 1/n \} \), for \( n \in \mathbb{N} \). It is enough to show that for \( n \in \mathbb{N} \) the set \( A_n \) is finite. To do so, let \( \alpha_1, \alpha_2, \ldots, \alpha_k \) be distinct elements of \( A_n \) and put \( x = \sum_{j=1}^k \text{sign}(f(x_{\alpha_j})) x_{\alpha_j} \) (for \( a \in \mathbb{C} \) we put \( \text{sign}(a) = a/|a| \)) and deduce that \( f(x) \geq k/n \). Now consider \( M_j = N_{\alpha_j} \setminus \bigcup_{i \neq j} N_{\alpha_i} \). Then \( N_{\alpha_j} \setminus M_j \) is infinite, and thus it follows for

\[
\tilde{x} = \sum_{j=1}^k \text{sign}(f(x_{\alpha_j})) 1_{M_j}
\]

that \( f(x) = f(\tilde{x}) \) (since \( x - \tilde{x} \in c_0 \)). Since the \( M_j \), \( j = 1, 2, \ldots, k \) are pairwise disjoint, it follows that \( \|\tilde{x}\|_\infty = 1 \), and thus

\[
\frac{k}{n} \leq f(x) = f(\tilde{x}) \leq \|f\|.
\]

Which implies that that \( A_n \) can have at most \( n\|f\| \) elements.

Step 4. If \( c_0 \subset \bigcap_{n=1}^\infty \mathcal{N}(f_n) \), for a sequence \( (f_n) \subset \ell_\infty^* \), then there is an \( \alpha \in I \) so that \( x_\alpha \in \bigcap_{n=1}^\infty \mathcal{N}(f_n) \). In particular this implies that \( c_0 \neq \bigcap_{n \in \mathbb{N}} \mathcal{N}(f_n) \).

Indeed, Step 3 yields that

\[
C = \{ \alpha \in I : f_n(x_\alpha) \neq 0 \text{ for some } n \in \mathbb{N} \} = \bigcup_{n \in \mathbb{N}} \{ \alpha \in I : f_n(x_\alpha) \neq 0 \},
\]

is countable, and thus \( I \setminus C \) is not empty.

\[\square\]

**Remark.** Assume that \( Z \) is any subspace of \( \ell_\infty \) which is isomorphic to \( c_0 \), then \( Z \) is not complemented. The proof of that statement is a bit harder.

**Theorem 2.4.10.** [So] Assume \( Y \) is a subspace of a separable Banach space \( X \) and \( T : Y \to c_0 \) is linear and bounded. Then \( T \) can be extended to a linear and bounded operator \( \bar{T} : X \to c_0 \). Moreover, \( \bar{T} \) can be chosen so that \( \|\bar{T}\| \leq 2\|T\| \).
Corollary 2.4.11. Assume that \( X \) is a separable Banach space which contains a subspace \( Y \) which is isomorphic to \( c_0 \). Then \( Y \) is complemented in \( X \).

Proof. Let \( T : Y \to c_0 \) be an isomorphism. Then extend \( T \) to \( \tilde{T} : X \to c_0 \) and put \( P = T^{-1} \circ \tilde{T} \).

Proof of Theorem 2.4.10. Note that an operator \( T : Y \to c_0 \) is defined by a \( \sigma(Y^*, Y) \) null sequence \( (y_n^*) \subset Y^* \), i.e.

\[
T : Y \to c_0, \quad y \mapsto \langle y_n^*, y \rangle : n \in \mathbb{N}.
\]

We would like to use the Hahn Banach theorem and extend each \( y_n^* \) to an element \( x_n^* \in X_n^* \), with \( \|y_n^*\| = \|x_n^*\| \), and define

\[
\tilde{T}(x) := (\langle x_n^*, x \rangle : n \in \mathbb{N}), \quad x \in X.
\]

But the problem is that \( (x_n^*) \) might not be \( \sigma(X^*, X) \) convergent to \( 0 \), and thus we can only say that \( (\langle x_n^*, x \rangle : n \in \mathbb{N}) \in \ell_\infty \), but not necessarily in \( c_0 \). Thus we will need to change the \( x_n^* \) somehow so that they are still extensions of the \( y_n^* \) but also \( \sigma(X^*, X) \) null.

Let \( B = \|T\|B_{X^*} \). \( B \) is \( \sigma(X^*, X) \)-compact and metrizable (since \( X \) is separable). Denote the metric which generates the \( \sigma(X^*, X) \)-topology by \( d(\cdot, \cdot) \). Put \( K = B \cap Y^\perp \). Since \( Y^\perp \subset X^* \) is \( \sigma(X^*, X) \)-closed, \( K \) is \( \sigma(X^*, X) \)-compact. Also note that every \( \sigma(X^*, X) \)-accumulation point of \( (x_n^*) \) lies in \( K \). Indeed, this follows from the fact that \( x_n^*(y) = y_n^*(y) \to_{n \to \infty} 0 \), for all \( y \in Y \). This implies that \( \lim_{n \to \infty} d(x_n^*, K) = 0 \), thus we can choose \( (z_n^*) \subset K \) so that \( \lim_{n \to \infty} d(x_n^*, z_n^*) = 0 \), and thus \( (x_n^* - z_n^*) \) is \( \sigma(X^*, X) \)-null and for \( y \in Y \) it follows that \( \langle x_n^* - z_n^*, y \rangle = \langle x_n^*, y \rangle, n \in \mathbb{N} \). Choosing therefore

\[
\tilde{T} : X \to c_0, \quad x \mapsto (\langle x_n^* - z_n^*, x \rangle : n \in \mathbb{N}),
\]

yields our claim.

Remark. Zippin [Zi] proved the converse of Theorem: if \( Z \) is an infinite-dimensional separable Banach space admitting a projection from any separable Banach space \( X \) containing it, then \( Z \) is isomorphic to \( c_0 \).

Exercises
1. Prove Proposition 2.4.2.
2. a) Assume that \( \ell_\infty \) isomorphic to a subspace \( Y \) of some Banach space \( X \), then \( Y \) is complemented in \( X \).
b) Assume $Z$ is a closed subspace of a Banach space $X$, and $T : Z \to \ell_\infty$ is linear and bounded. Then $T$ can be extended to a linear and bounded operator $\tilde{T} : X \to \ell_\infty$, with $\|\tilde{T}\| = \|T\|$.

3. Show that for a Banach space $X$, the dual space $X^*$ is isometrically isomorphic to a complemented subspace of $X^{***}$, via the canonical embedding.

4. Prove Proposition 2.4.7.

5. Prove (2.3) in Proposition 2.4.4.
2.5 The Theorem of Eberlein Smulian

For infinite dimensional Banach spaces the weak topology is not metrizable (see Exercise 1). Nevertheless compactness in the weak topology can be characterized by sequences.

**Theorem 2.5.1.** (The Theorem of Eberlein- Smulian)
Let $X$ be a Banach space. For subset $K$ the following are equivalent.

a) $K$ is relatively $\sigma(X,X^*)$ compact, i.e. $K_{\sigma(X,X^*)}$ is compact.

b) Every sequence in $K$ contains a $\sigma(X,X^*)$-convergent subsequence.

c) Every sequence in $K$ has a $\sigma(X,X^*)$-accumulation point.

We will need the following Lemma.

**Lemma 2.5.2.** Let $X$ be a Banach space and assume that there is a countable set $C = \{x_n^* : n \in \mathbb{N}\} \subset B_{X^*}$, so that $C_{\perp} = \{0\}$. In that case we say that $C$ is total for $X$.

Consider for $x, y$

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n}|\langle x_n^*, x - y \rangle|.$$ 

Then $d$ is a metric on $X$, and for any $\sigma(X,X^*)$-compact set $K$, $\sigma(X,X^*)$ coincides on $K$ with the metric generated by $d$.

The proof of Lemma 2.5.2 goes along the lines of Exercise 1 in this section.

**Lemma 2.5.3.** Assume that $X$ is separable. Then there is a countable total set $C \subset X^*$.

**Proof.** Let $D \subset X$ be dense, and choose by the Corollary 1.4.6 of the Theorem of Hahn Banach for each element $x \in D$, an element $y_x^* \in S_{X^*}$ so that $\langle y_x^*, x \rangle = \|x\|$. Put $C = \{y_x^* : x \in D\}$. If $x \in X$, $x \neq 0$, is arbitrary then there is a sequence $(x_k) \subset D$, so that $\lim_{k \to \infty} x_k = x$, and thus $\lim_{k \to \infty} \langle y_{x_k}^*, x \rangle = \|x\| > 0$. Thus there is an $x^* \in C$ so that $\langle x^*, x \rangle \neq 0$, which implies that $C$ is total.

$\square$
Proof of Theorem 2.5.1. 

“(a)⇒(b)” Assume that $K$ is $\sigma(X,X^*)$-compact (if necessary, pass to the closure) and let $(x_n) \subset K$ be a sequence, and put $X_0 = \operatorname{span}(x_n : n \in \mathbb{N})$. $X_0$ is a separable Banach space. By Proposition 2.2.5 the topology $\sigma(X_0,X_0^*)$ coincides with the restriction of $\sigma(X,X^*)$ to $X_0$. Thus, $K_0 = K \cap X_0$ is $\sigma(X_0,X_0^*)$-compact. Since $X_0$ is separable, by Lemma 2.5.3 there exists a countable set $C \subset B_{X_0^*}$, so that $C_\perp = \{0\}$.

It follows therefore from Lemma 2.5.2 that $(K_0,\sigma(X_0,X_0^*) \cap K_0)$ is metrizable and thus $(x_n)$ has a convergent subsequence in $K_0$. Again, using the fact that on $X_0$ the weak topology coincides with the weak topology on $X$, we deduce our claim.

“(b)⇒(c)” clear.

“(c)⇒(a)” Assume $K \subset X$ satisfies (c). We first observe that $K$ is (norm) bounded. Indeed, for $x^* \in X^*$, the set $A_{x^*} = \{ \langle x^*, x \rangle : x \in K \} \subset \mathbb{K}$ is the continuous image of $A$ (under $x^*$) and thus has the property that every sequence has an accumulation point in $\mathbb{K}$. This implies that $A_{x^*}$ is bounded in $\mathbb{K}$ for all $x^* \in X^*$, but this implies by the Banach Steinhaus Theorem 1.3.8 that $A \subset X$ must be bounded.

Let $\chi : X \hookrightarrow X^{**}$ be the canonical embedding. By the Theorem of Alaoglu 2.3.2, it follows that $\overline{\chi(K)}^{\sigma(X^{**},X^*)}$ is $\sigma(X^{**},X^*)$-compact. Therefore it will be enough to show that $\overline{\chi(K)}^{\sigma(X^{**},X^*)} \subset \chi(X)$ (because this would imply that every net $(\chi(x_i) : i \in I) \subset \chi(K)$ has a subnet which $\sigma(\chi(X),X^*)$ converges to some element $\chi(x) \in \chi(X)$).

So, fix $x_{0}^{**} \in \overline{\chi(K)}^{(X^{**},X^*)}$. Recursively we will choose for each $k \in \mathbb{N}$, $x_k \in K$, and for each $k \in \mathbb{N}$ a finite set $A_k^* \subset S_{X^*}$, so that

\begin{equation}
(2.4) \quad \left|\langle x_{0}^{**} - \chi(x_k), x^* \rangle\right| < \frac{1}{k} \quad \text{for all } x^* \in \bigcup_{0 \leq j < k} A_j^*, \text{ if } k \geq 1,
\end{equation}

\begin{equation}
(2.5) \quad \forall x^{**} \in \operatorname{span}(x_{0}^{**}, \chi(x_j), 0 \leq j \leq k) \|x^{**}\| \geq \max_{x^* \in A_k^*} |\langle x^{**}, x^* \rangle| \geq \frac{\|x^{**}\|}{2}.
\end{equation}

For $k = 0$ choose $A_0^* = \{x^*\}$, $x^* \in S_{X^*}$, with $|\langle x^*(x_{0}^{**}) \rangle| \geq \|x_{0}^{**}\|/2$, then condition (2.5) is satisfied, while condition (2.4) is vacuous.

Assuming that $x_1, x_2, \ldots, x_{k-1}$ and $A_0^*, A_1^*, \ldots, A_{k-1}^*$ have been chosen for some $k > 1$, we can first choose $x_k \in K$ so that (2.4) is satisfied (since $A_j^*$ is finite for $j = 1, 2, \ldots, k - 1$), and then, since $\operatorname{span}(x_{0}^{**}, \chi(x_j), j \leq k)$ is a finite dimensional space we can choose $A_k^* \subset S_{X^*}$ so that (2.5) holds.

By our assumption (c) the sequence $(x_k)$ has an $\sigma(X,X^*)$-accumulation point $x_0$. By Proposition 2.3.1 it follows that

\[ x_0 \in Y = \overline{\operatorname{span}(x_k : k \in \mathbb{N})} = \overline{\operatorname{span}(x_k : k \in \mathbb{N})^{\sigma(X,X^*)}}. \]
We will show that \(x^*_0 = \chi(x_0)\) (which will finish the proof). First note that for any \(x^* \in \bigcup_{j \in \mathbb{N}} A_j^*\)

\[
\left| \langle x^*_0 - \chi(x_0), x^* \rangle \right| \leq \liminf_{k \to \infty} \left( \left| \langle x^*_0 - \chi(x_k), x^* \rangle \right| + \left| \langle x^*, x_k - x_0 \rangle \right| \right) = 0.
\]

Secondly consider the space \(Z = \text{span}(x^*_0, \chi(x_k), k \in \mathbb{N})^\perp \subset X^{**}\) it follows from (2.5) that the set of restrictions of elements of \(\bigcup_{k=1}^\infty A_k^*\) to \(Y\) is total in \(Z\) and thus that

\[
x^*_0 - \chi(x_0) \in Z \cap \left( \bigcup_{k=1}^\infty A_k^* \right) = \{0\},
\]

which implies our claim. \(\square\)

Exercises

1. Prove that if \(X\) is a separable Banach space \((B_{X^*}, \sigma(X^*, X))\) is metrizable.

2. For an infinite dimensional Banach space prove that \((X, \sigma(X, X^*))\) is not metrizable.
   Hint: Exercise 4 in Section 2.3.

3. Prove that for two Banach spaces \(X\) and \(Y\), the adjoint of a linear bounded operator \(T : X \to Y\) is \(w^*\)-continuous (i.e \(\sigma(Y^*, Y) = \sigma(X^*, X)\)-continuous).

4. Show that \(\ell_1\) isometric to a subspace of \(C[0, 1]\).

5.* Show that \(\ell_1\) is not complemented in \(C[0, 1]\).

You might need to prove the following two things:

a) There is no separable complemented subspace \(X\) of \(\ell_\infty\) which contains \(c_0\) (this follows from the fact that \(c_0\) is not complemented in \(\ell_\infty\) but complemented in each separable super space)

b) Every separable sub space \(X\) of \(M[0, 1]\) sits inside another separable subspace \(Y\) of \(M[0, 1]\) which is complemented in \(M[0, 1]\).

Idea for b) take dense sequence \((\mu_n)\)in \(S_X\) take \(\nu = \sum 2^{-n}|\mu_n|\) then prove that \(L_1(\nu)\) is complemented in \(X\).
2.6 Characterizations of Reflexivity by Ptáčk

We present several characterization of the reflexivity of a Banach space, due to Pták [Pták]. We assume in this section that our Banach spaces are defined over the real field \( \mathbb{R} \).

**Theorem 2.6.1.** The following conditions for a Banach space \( X \) are equivalent

1. \( X \) is not reflexive.

2. For each \( \theta \in (0, 1) \) there are sequences \( (x_i)_{i=1}^{\infty} \subset B_X \) and \( (x_i^*)_{i=1}^{\infty} \subset B_{X^*} \), so that

\[
x_j^*(x_i) = \begin{cases} 
\theta & \text{if } j \leq i, \text{ and } \\
0 & \text{if } j > i.
\end{cases}
\]

3. For some \( \theta > 0 \) there are sequences \( (x_i)_{i=1}^{\infty} \subset B_X \) and \( (x_i^*)_{i=1}^{\infty} \subset B_{X^*} \), for which (2.6) holds.

4. For each \( \theta \in (0, 1) \) there is a sequence \( (x_i)_{i=1}^{\infty} \subset B_X \), so that

\[
\text{dist}(\text{conv}(x_1, \ldots, x_n), \text{conv}(x_{n+1}, x_{n+2}, \ldots)) \geq \theta.
\]

5. For some \( \theta > 0 \) there is a sequence \( (x_i)_{i=1}^{\infty} \subset B_X \), so that (2.7) holds.

For the proof we will need Helly’s Lemma.

**Lemma 2.6.2.** Let \( Y \) be an infinite-dimensional normed linear space \( y_1^*, y_2^*, \ldots, y_n^* \in Y^* \), \( M > 0 \) and let \( c_1, c_2, \ldots, c_n \) be scalars.

The following are equivalent

1. **The Moment Condition**
   
   For all \( \varepsilon > 0 \) there exists \( y \in Y \) with
   
   \[
   \|y\| = M + \varepsilon \text{ and } y_k^*(y) = c_k \text{ for } k = 1, 2, \ldots, n.
   \]

2. **Helly’s Condition**
   
   \[
   \left| \sum_{j=1}^{n} a_j c_j \right| \leq M \left\| \sum_{j=1}^{n} a_j y_j^* \right\| \text{ for any sequence } (a_j)_{j=1}^{n} \text{ of scalars.}
   \]
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Proof. “(M) ⇒ (H)”. Let ϵ > 0 and assume y ∈ Y satisfies the condition in (M). Then

\[ \left| \sum_{j=1}^{n} a_j c_j \right| = \left| \sum_{j=1}^{n} a_j y_j^* (y) \right| \leq \| y \| \cdot \left( \sum_{j=1}^{n} a_j y_j^* \right) = (M + \epsilon) \left( \sum_{j=1}^{n} a_j y_j^* \right), \]

which implies (H), since ϵ > 0 was arbitrary.

“(H) ⇒ (M)” Assume (H) and let ϵ > 0. We can assume that not all the y_k are vanishing (otherwise also all the c_k have to be equal to 0, and any y ∈ {y_1^*, y_2^*, ..., y_n^*} \perp \perp \text{span}(y_j^*: j = 1, 2, ..., k). This implies that if we have a y ∈ Y, with \| y \| = M + \epsilon and y_j^*(y) = c_j, for j = 1, 2, ..., k, then it also follows that y_j^*(y) = c_j, for j = k + 1, k + 2, ..., n. Indeed, for j = k + 1, k + 2, ..., n, choose scalars (a_i^(j): i = 1, 2, ..., k) so that y_j^* = \sum_{i=1}^{k} a_i^(j) y_i^*, for j = k + 1, k + 2, ..., n. Now, the inequality in (H) implies that c_j = \sum_{i=1}^{k} a_i^(j) c_i, for j = k + 1, k + 2, ..., n. Indeed, choose a_j = -1, a_i = 0, if i ∈ {k + 1, k + 2, ..., n} \{ j \} and a_i = a_i^(j), if i ∈ {1, 2, ..., k}, which implies that the right hand of the equation in (M) vanishes. This yields that y_j^*(y) = c_j, for j = k + 1, k + 2, ..., n.

We can therefore restrict ourselves to satisfy the second condition in (H) for all j = 1, 2, ..., k. Define for j = 1, 2, ..., k the affine subspace \( H_j = \{ y \in Y : y_j^*(y) = c_j \} \). Then G = \bigcap_{j=1}^{k} H_j is an affine subspace of codimension k. Note, that if we pick y ∈ G, then G = y + G_0, where G_0 is the closed subspace

\[ G_0 = \bigcap_{j=1}^{k} \{ y \in Y : y_j^*(y) = 0 \}. \]

We need to show that

\[ N := \inf \{ \| y \| : y \in G \} \leq M. \]

Then our claim would follow, since the Intermediate Value implies that there must be some y in G for which N < M + ϵ ≤ \| y \| < ∞. Without loss of generality we can assume that N > 0. We choose a functional g* in the dual of the span of G so that g*(y) = N, for all y ∈ G. This can be done by
picking a point \( y \in G \), and choosing by Hahn Banach \( g^* \in \text{span}(G) \), with \( g^*(y) = N \) and which vanishes on the linear closed subspace \( G_0 \).

Note that

\[
\text{span}(G) = \{ ry : y \in G \text{ and } r \in \mathbb{R} \}.
\]

We note that \( \|g^*\| \geq 1 \). Indeed, otherwise choose a sequence \( (y_n) \subset G \), with \( \lim_{n \to \infty} \|y_n\| = N \), and note that

\[
N = g^*(y_n) \leq \|g^*\| : \|y_n\| \to_{n \to \infty} \|g^*\|N < N
\]

which is a contradiction.

Secondly, we note that \( \|g^*\| \leq 1 \), we use (2.8) and find \( r \in \mathbb{R} \) and \( y \in G \) so that \( g^*(ry) > \|ry\| \geq |r|N \), which is a contradiction since \( g^*(ry) = rN \).

We let \( y^* \) be a Hahn Banach extension of \( g^* \) to a functional defined on all of \( Y \).

For all \( y \in Y \), we have that if \( y^*_j(y) = c_j \), for \( j = 1, 2, \ldots, k \) it follows that \( y^*(y) = N \). Thus, we have for all \( y \in Y \) if \( y^*_j(y) = 0 \), for \( j = 1, 2, \ldots, k \), then \( y^*(y) = 0 \), in other words, the intersection of the null spaces of the \( y^*_j \), \( j = 1, 2, \ldots, k \), is a subset of the null space of \( y^* \). This means that \( y^* \) is a linear combination of the \( y^*_j \), \( j = 1, 2, \ldots, k \), say \( y^* = \sum_{j=1}^{k} a_j y^*_j \). This also implies that \( N = y^*(y) = \sum_{j=1}^{k} a_j y^*_j(y) = \sum_{j=1}^{k} a_j c_j \), for \( y \in G \).

Thus, by our assumption \((H)\)

\[
N = \frac{N}{\|y^*\|} = \frac{\sum_{j=1}^{k} a_j c_j}{\left| \sum_{j=1}^{k} a_j y^*_j \right|} \leq M,
\]

which proves our claim (2.8) and finishes the proof of the Lemma.

**Proof of Theorem 2.6.1.** “(i) \( \Rightarrow \) (ii)” Assume that \( X \) is not reflexive and that \( \theta \in (0, 1) \). Since \( X \) is a closed subspace of \( X^{**} \) (as usual we are identifying \( X \) with its image under the canonical embedding into \( X^{**} \)), there is a functional \( x^{***} \in X^{***} \) that \( \|x^{***}\| = 1 \), \( x^{***}|_X \equiv 0 \) and \( x^{***}(x^{**}) > \theta \) for some \( x^{**} \in X^{**} \), with \( \|x^{**}\| < 1 \) (see Exercise 1).

Now we will choose inductively \( x_n \in B_X \) and \( x^*_n \in B_{X^*}, n \in \mathbb{N} \), at each step assuming that the condition (2.6) holds up to \( n \), and additionally, that \( x^{**}(x^*_n) = \theta \).

For \( n = 1 \) we simply choose \( x^*_1 \in S_{X^*} \) so that \( x^{**}(x^*_1) = \theta \) and then we choose \( x_1 \in B_X \) so that \( x^*_1(x_1) = \theta \). Assuming we have chosen \( x_1, x_2, \ldots, x_n \)
and \( x_1^*, x_2^*, \ldots, x_n^* \) so that

\[
x_j^*(x_i) = \begin{cases} 
\theta & \text{if } j \leq i \leq n, \\
0 & \text{if } i < j \leq n.
\end{cases}
\]  

Since \( x^{**}(x_j) = 0 \) for \( j = 1, 2, \ldots, n \) and \( x^{**}(x^*) > \theta \), the elements \( x_1, x_2, \ldots, x_n, x^* \), seen as functionals on \( X^* \), together with the numbers \( 0, 0, \ldots, 0, \theta \) and \( M = \frac{\theta}{\lambda} < 1 \) satisfy Helly’s condition (II). Indeed, for scalars \( a_1, \ldots, a_{n+1} \) we have

\[
|a_{n+1}|\theta = M |a_{n+1}x^{**}(x^*)|
\]

\[
= M \left| x^{**} \left( \sum_{j=1}^{n} a_j x_j + a_{n+1}x^* \right) \right| \leq M \left\| \sum_{j=1}^{n} a_j x^{**}(x_j) + a_{n+1}x^* \right\|.
\]

We can therefore choose an \( x_{n+1}^* \in X^* \), \( \|x_{n+1}^*\| \leq 1 \) so that \( x_{n+1}^*(x_j) = 0 \) for all \( j = 1, 2, \ldots, n \) and \( x^{**}(x_{n+1}^*) = \theta \).

Secondly, we note that the functionals \( x_1^*, x_2^*, \ldots, x_{n+1}^* \), the numbers \( \theta, \theta, \ldots, \theta \), and the number \( M = \|x^{**}\| < 1 \) satisfy Helly’s condition. Indeed, for scalars \( a_1, \ldots, a_{n+1} \) we have

\[
\left| \sum_{j=1}^{n+1} a_j \theta \right| = \left| \sum_{j=1}^{n+1} a_j x^{**}(x_j^*) \right| \leq M \left\| \sum_{j=1}^{n+1} a_j x_j^* \right\|.
\]

We can therefore find \( x_{n+1} \in B_X \), so that \( x_j^*(x_{n+1}) = \theta \), for all \( j = 1, 2, \ldots, n \).

“(ii)⇒(iv)” and “(iii)⇒(v)” Fix a \( \theta \in (0,1) \) for which there are sequences \( (x_j) \subset B_X \) and \( (x_j^*) \subset B_{X^*} \) for which (2.6) holds. Let \( x = \sum_{j=1}^{n} a_j x_j \in \text{conv}(x_1, x_2, \ldots, x_n) \) and \( z = \sum_{j=n+1}^{\infty} b_j x_j \in \text{conv}(x_{n+1}, x_{n+2}, \ldots) \) then

\[
\|z - x\| \geq x_{n+1}^*(z - x) = x_{n+1}^*(y) = \theta,
\]

which implies our claim.

“(iv)⇒(v)” obvious.

“(v)⇒(i)” Assume that for \( \theta > 0 \) and the sequence \( (x_j) \subset B_X \) satisfies (2.7). Now assume that our claim is false and \( X \) is reflexive.

Define \( C_n = \text{conv}(x_j : j \geq n + 1) \), for \( n \in \mathbb{N} \), then the sets \( C_n \), \( n \in \mathbb{N} \), are weakly compact, \( C_1 \supset C_2 \supset \ldots \). Thus there is an element \( v \in \bigcap_{n \in \mathbb{N}} C_n \). We can approximate \( v \) by some \( u \in \text{conv}(x_j : j \in \mathbb{N}) \), with \( \|u - v\| < \theta/2 \). There is some \( n \) so that \( v \in \text{conv}(x_1, \ldots, x_n) \). But now it follows, since \( u \in C_{n+1} \), that \( \text{dist}(\text{conv}(x_1, \ldots, x_n), \text{conv}(x_{n+1}, x_{n+2}, \ldots)) \leq \|v - u\| < \theta/2 \), which is a contradiction and finishes the proof.

\( \square \)
Exercises

1. Assume that $X$ is not reflexive and $0 < \Theta < 1$. Then there exists an $x^{***} \in X^{***}$, $\|x^{***}\| = 1$, and $x^{**} \in X^{**}$, $\|x^{**}\| < 1$, so that $x^{***}(x^*) > \Theta$, and $X^{***}|_X \equiv 0$.

2.7 The Principle of Local Reflexivity

In this section we prove a result by J. Lindenstrauss and H. Rosenthal [LR] which states that for a Banach space $X$ the finite dimensional subspaces of the bidual $X^{**}$ are in a certain sense have “similar” finite dimensional subspaces of $X$.

Theorem 2.7.1. [LR] [The Principle of Local Reflexivity]

Let $X$ be a Banach space and let $F \subset X^{**}$ and $G \subset X^*$ be finite dimensional subspaces of $X^{**}$ and $X^*$ respectively.

Then, given $\varepsilon > 0$, there is a subspace $E$ of $X$ containing $F \cap X$ (we identify $X$ with its image under the canonical embedding) with $\dim E = \dim F$ and an isomorphism $T : F \to E$ with $\|T\| \cdot \|T^{-1}\| \leq 1 + \varepsilon$ such that

\begin{align}
T(x) &= x \text{ if } x \in F \cap X \text{ and} \\
\langle x^*, T(x^{**}) \rangle &= \langle x^{**}, x^* \rangle \text{ if } x^* \in G, x^{**} \in F.
\end{align}

We need several Lemmas before we can prove Theorem 2.7.1. The first one is a corollary of the Geometric Hahn-Banach Theorem

Proposition 2.7.2. (Variation of the Geometric Version of the Theorem of Hahn Banach)

Assume that $X$ is a Banach space and $C \subset X$ is convex with $C^0 \neq \emptyset$ and let $x \in X \setminus C$ (so $x$ could be in the boundary of $C$). Then there exists an $x^* \in X^*$ so that

$$\Re \langle x^*, z \rangle < \langle x^*, x \rangle \text{ for all } z \in C^0,$$

and, if moreover $C$ is absolutely convex (i.e. if $\rho x \in C$ for all $x \in C$ and $\rho \in \mathbb{K}$, with $|\rho| \leq 1$), then

$$|\langle x^*, z \rangle| < 1 = \langle x^*, x \rangle \text{ for all } z \in C^0.$$

Lemma 2.7.3. Assume $T : X \to Y$ is a bounded linear operator between the Banach spaces $X$ and $Y$ and assume that $T(X)$ is closed.

Suppose that for some $y \in Y$ there is an $x^{**} \in X^{**}$ with $\|x^{**}\| < 1$, so that $T^{**}(x^{**}) = y$. Then there is an $x \in X$, with $\|x\| < 1$ so that $T(x) = y$. 
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Proof. We first show that there is an \( x \in X \) so that \( T(x) = y \). Assume this where not true, then we could find by the Hahn-Banach Theorem (Corollary 1.4.5) an element \( y^* \in Y^* \), so that \( y^*(z) = 0 \), for all \( z \in T(X) \) and \( \langle y^*, y \rangle = 1 \) \((T(X) \text{ is closed})\). But this yields \( \langle T^*(y^*), x \rangle = \langle y^*, T(x) \rangle = 0 \), for all \( x \in X \), and, thus, \( T^*(y^*) = 0 \). Thus

\[
0 = \langle x^{**}, T^*(y^*) \rangle = \langle T^*(x^{**}), y^* \rangle = \langle y, y^* \rangle = 1,
\]

which is a contradiction.

Secondly, assume that \( y \in T(X) \setminus T(B_X^0) \). Since \( T \) is surjective onto its (closed) image \( Z = T(X) \) it follows from the Open Mapping Theorem that \( T(B_X^0) \) is open in \( Z \), and we can use variation of the geometric version of the Hahn-Banach Theorem, Proposition (2.7.2), and chose \( z^* \in Z^* \), so that \( \langle z^*, T(x) \rangle < 1 = \langle z^*, y \rangle \) for all \( x \in B_X^0 \). Again by the Theorem of Hahn-Banach (Corollary 1.4.4) we can extend \( z^* \) to an element \( y^* \) in \( Y^* \). It follows that

\[
\|T^*(y^*)\| = \sup_{x \in B_X^0} \langle T^*(y^*), x \rangle = \sup_{x \in B_X^0} \langle z^*, T(x) \rangle \leq 1,
\]

and thus, since \( \|x^{**}\| < 1 \), it follows that

\[
|\langle y^*, y \rangle| = |\langle y^*, T^**(x^{**}) \rangle| = |\langle x^{**}, T^*(y^*) \rangle| < 1,
\]

which is a contradiction. \( \Box \)

Lemma 2.7.4. Let \( T : X \to Y \) be a bounded linear operator between two Banach spaces \( X \) and \( Y \) with closed range, and assume that \( F : X \to Y \) has finite rank.

Then \( T + F \) also has closed range.

Proof. Assume the claim is not true. Put \( S = T + K \) and consider the map

\[
\overline{S} : X/\mathcal{N}(S) \to Y, \quad x + \mathcal{N}(S) \to S(x)
\]

which is a well defined linear bounded Operator, and which by Proposition 1.3.11 cannot be an isomorphism onto its image.

Therefore we can choose sequence \((\overline{\pi_n})\) in \( X/\mathcal{N}(S) \), with \( \|\pi_n\| = 1 \) and \( x_n \in \pi_n \), with \( 1 \leq \|x_n\| \leq 2 \), for \( n \in \mathbb{N} \), so that

\[
\lim_{n \to \infty} \overline{S}(\pi_n) = \lim_{n \to \infty} S(x_n) = 0 \text{ and } \text{dist}(x_n, \mathcal{N}(S)) \geq 1.
\]
Since the sequence \( (F(x_n) : n \in \mathbb{N}) \) is a bounded sequence in a finite dimensional space, we can, after passing to a subsequence, assume that \( (F(x_n) : n \in \mathbb{N}) \) converges to some \( y \in Y \) and, hence,

\[
\lim_{n \to \infty} T(x_n) = -y.
\]

Since \( T \) has closed range there is an \( x \in X \), so that \( T(x) = -y \). Using again the equivalences in Proposition 1.3.11 and the fact that \( T(x_n) \to -y = T(x) \), if \( n \not\to \infty \), it follows for some constant \( C > 0 \) that

\[
\lim_{n \to \infty} \text{dist}(x - x_n, \mathcal{N}(T)) \leq \lim_{n \to \infty} C \| T(x - x_n) \| = 0,
\]

and, thus,

\[
y - F(x) = \lim_{n \to \infty} F(x_n) - F(x) \in F(\mathcal{N}(T)),
\]

so we can write \( y - F(x) \) as

\[
y - F(x) = F(u), \text{ where } u \in \mathcal{N}(T).
\]

Thus

\[
\lim_{n \to \infty} \text{dist}(x_n - x - u, \mathcal{N}(T)) = 0 \text{ and } \lim_{n \to \infty} \| F(x_n) - F(x) - F(u) \| = 0.
\]

\( F|_{\mathcal{N}(T)} \) has also closed range, Proposition 1.3.11 yields (\( C \) being some positive constant)

\[
\lim_{n \to \infty} \text{dist}(x_n - x - u, \mathcal{N}(F) \cap \mathcal{N}(T)) \leq \lim_{n \to \infty} C \| F(x_n) - F(x) - F(u) \| = 0.
\]

Since \( T(x + u) = -y = -F(x + u) \) (by choice of \( u \)), and thus \( (T + F)(x + u) = 0 \) which means that \( x + u \in \mathcal{N}(T + F) \). Therefore

\[
\lim_{n \to \infty} \text{dist}(x_n, \mathcal{N}(T + F)) = \lim_{n \to \infty} \text{dist}(x_n - x - u, \mathcal{N}(T + F)) \leq \lim_{n \to \infty} \text{dist}(x_n - x - u, \mathcal{N}(T) \cap \mathcal{N}(F)) = 0.
\]

But this contradicts our assumption on the sequence \( (x_n) \). \( \square \)

**Lemma 2.7.5.** Let \( X \) be a Banach space, \( A = (a_{i,j})_{i \leq m, j \leq n} \) an \( m \) by \( n \) matrix and \( B = (b_{i,j})_{i \leq p, j \leq n} \) a \( p \) by \( n \) matrix, and assume that \( B \) has only real entries (even if \( K = \mathbb{C} \)).
2.7. THE PRINCIPLE OF LOCAL REFLEXIVITY

Suppose that \( y_1, \ldots, y_m \in X, \ y_1^*, \ldots, y_p^* \in X^*, \ \xi_1, \ldots, \xi_p \in \mathbb{R}, \) and \( x_1^*, \ldots, x_n^* \in B_{X^*}^{\circ} \) satisfy the following equations:

\[
\sum_{j=1}^{n} a_{i,j} x_j^* = y_i, \text{ for all } i = 1, 2, \ldots, m, \text{ and }
\]

\[
\langle y_i^*, \sum_{j=1}^{n} b_{i,j} x_j^* \rangle = \xi_i, \text{ for all } i = 1, 2, \ldots, p.
\]

Then there are vectors \( x_1, \ldots, x_n \in B_X^{\circ} \) satisfying:

\[
\sum_{j=1}^{n} a_{i,j} x_j = y_i, \text{ for all } i = 1, 2, \ldots, m, \text{ and }
\]

\[
\langle y_i^*, \sum_{j=1}^{n} b_{i,j} x_j \rangle = \xi_i, \text{ for all } i = 1, 2, \ldots, p.
\]

Proof. Recall from Linear Algebra that we can write the matrix \( A \) as a product \( A = U \circ P \circ V \), where \( U \) and \( V \) are invertible and \( P \) is of the form

\[
P = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},
\]

where \( r \) is the rank of \( A \) and \( I_r \) the identity on \( \mathbb{K}^r \).

For a general \( s \) by \( t \) matrix \( C = (c_{i,j})_{i \leq s, j \leq t} \) consider the operator

\[
T_C : \ell^n_\infty(X) \to \ell^s_\infty(X), \quad (x_1, x_2, \ldots, x_t) \mapsto \left( \sum_{j=1}^{t} c_{i,j} x_j : i = 1, 2, \ldots, m \right).
\]

If \( s = t \) and if \( C \) is invertible then \( T_C \) is an isomorphism. Also if \( C^{(1)} \) and \( C^{(2)} \) are two matrices so that the number of columns of \( C^{(1)} \) is equal to the number of rows of \( C^{(2)} \) one easily computes that \( T_{C^{(1)} \circ C^{(2)}}} = T_{C^{(1)}} \circ T_{C^{(2)}} \).

Secondly it is clear that \( T_P \) is a closed operator (\( P \) defined as above), since \( T_P \) is simply the projection onto the first \( r \) coordinates in \( \ell^r_\infty(X) \).

It follows therefore that \( T_A = T_U \circ T_P \circ T_V \) is an operator with closed range. Secondly define the operator

\[
S_A : \ell^n_\infty(X) \to \ell^n_\infty(X) \oplus \ell^p_\infty,
\]

\[
(x_1, \ldots, x_n) \mapsto \left( T_A(x_1, \ldots, x_n), \left( \langle y_i^*, \sum_{j=1}^{n} b_{i,j} x_j \rangle \right)_{i=1}^{p} \right).
\]
$S_A$ can be written as the sum of $T_A$ and a finite rank operator and has therefore also closed range by Lemma 2.7.4.

Since the second adjoint of $S_A^{**}$ is the operator

$$S_A^{**} : \ell_n^\infty(X^{**}) \rightarrow \ell_n^\infty(X^{**}) \oplus \ell_\infty^p,$$

$$(x_1^{**}, \ldots, x_n^{**}) \mapsto \left( T_A^{**}(x_1^{**}, \ldots, x_n^{**}), \left( <y_i^*, \sum_{j=1}^n b_{i,j}x_j^{**}> \right)_{i=1}^p \right)$$

with

$$T_A^{**} : \ell_n^\infty(X^{**}) \rightarrow \ell_n^\infty(X^{**}), \ (x_1^{**}, x_2^{**}, \ldots, x_n^{**}) \mapsto \left( \sum_{j=1}^t a_{i,j}x_j^{**} : i = 1, 2, \ldots, m \right),$$

our claim follows from Lemma 2.7.3.

\[\square\]

**Lemma 2.7.6.** Let $E$ be a finite dimensional space and $(x_i)_{i=1}^N$ is an $\varepsilon$-net of $S_F$ for some $0 < \varepsilon < 1/3$. If $T : E \rightarrow E$ is a linear map so that

$$(1 - \varepsilon) \leq \|T(x_j)\| \leq (1 + \varepsilon), \text{ for all } j = 1, 2, \ldots, N.$$

Then

$$\frac{1 - 3\varepsilon}{1 - \varepsilon}\|x\| \leq \|T(x)\| \leq \frac{1 + \varepsilon}{1 - \varepsilon}\|x\|, \text{ for all } x \in E,$$

and thus

$$\|T\| \cdot \|T^{-1}\| \leq \frac{(1 + \varepsilon)^2}{(1 - \varepsilon)(1 - 3\varepsilon)}.$$

We are now ready to proof Theorem 2.7.1.

**Proof of Theorem 2.7.1.** Let $F \subset X^{**}$ and $G \subset X^*$ be finite dimensional subspaces, and let $0 < \varepsilon < 1$. Choose $\delta > 0$, so that $\frac{(1+\delta)^2}{(1-\delta)(1-3\delta)} < \varepsilon$, and a $\delta$-net $(x_j^{**})_{j=1}^N$ of $S_F$. It can be shown that $(x_j^{**})_{j=1}^N$ span all of $F$, but we can also simply assume without loss of generality, that it does, since we can add a basis of $F$.

Let

$$S : \mathbb{R}^N \rightarrow F, \quad (\xi_1, \xi_2, \ldots, \xi_N) \mapsto \sum_{j=1}^N \xi_jx_j^{**},$$

and note that $S$ is surjective.
2.7. THE PRINCIPLE OF LOCAL REFLEXIVITY

Put \( H = S^{-1}(F \cap X) \), and let \( (a^{(i)} : i = 1, 2, \ldots, m) \) be a basis of \( H \), write \( a^{(i)} \) as \( a^{(i)} = (a_{i,1}, a_{i,2}, \ldots a_{i,N}) \), and define \( A \) to be the \( m \) by \( N \) matrix \( A = (a_{i,j})_{i \leq m, j \leq N} \). For \( i = 1, 2, \ldots, m \) put

\[
y_i = S(a^{(i)}) = \sum_{j=1}^{N} a_{i,j} x_j^{**} \in F \cap X,
\]

choose \( x_1^*, x_2^*, \ldots, x_N^* \in S_X^* \) so that \( \langle x_j^{**}, x_j^* \rangle > 1 - \delta \), and pick a basis \( \{g_1^*, g_2^*, \ldots g_N^* \} \) of \( G \).

Consider the following system of equations in \( N \) unknowns \( z_1^{**}, z_2^{**}, \ldots, z_N^{**} \) in \( X^{**} \):

\[
\sum_{j=1}^{N} a_{i,j} z_j^{**} = y_i \quad \text{for } i = 1, 2, \ldots, m
\]

\[
\langle z_j^{**}, x_j^* \rangle = \langle x_j^{**}, x_j^* \rangle \quad \text{for } j = 1, 2, \ldots, N \text{ and }
\]

\[
\langle z_j^{**}, g_k \rangle = \langle x_j^{**}, g_k \rangle \quad \text{for } j = 1, 2, \ldots, N \text{ and } k = 1, 2, \ldots, \ell.
\]

By construction \( z_j^{**} = x_j^{**} \), \( j = 1, 2, \ldots, N \), is a solution to these equations. Since \( \|x_j^{**}\| = 1 < 1 + \delta \), for \( j = 1, 2, \ldots, N \), we can use Lemma 2.7.5 and find \( x_1, x_2, \ldots x_N \in X \), with \( \|x_j\| = 1 < 1 + \delta \), for \( j = 1, 2, \ldots, N \), which solve above equations.

Define

\[
S_1 : \mathbb{R}^N \to X, \quad (\xi_1, \xi_2, \ldots, \xi_N) \mapsto \sum_{j=1}^{N} \xi_j x_j.
\]

We claim that the null space of \( S \) is contained in the null space of \( S_1 \). Indeed if we assumed that \( \xi \in \mathbb{K}^N \), and \( \sum_{j=1}^{N} \xi_j x_j = 0 \), but \( \sum_{j=1}^{N} x_j^{**} \neq 0 \), then, Lemma 2.7.6 (consider the operator \( F \to \mathbb{R}^N, x^{**} \mapsto \langle x^{**}, x_j^* \rangle \)) there is an \( i \in \{1, 2, \ldots, N \} \) so that

\[
\langle x_i^*, \sum_{j=1}^{N} x_j^{**} \rangle \neq 0,
\]

but since \( \langle x_j^{**}, x_j^* \rangle = \langle x_j^*, x_j^* \rangle \) this is a contradiction.

It follows therefore that we can find a linear map \( T : F \to X \) so that \( S_1 = TS \). Denoting the standard basis of \( \mathbb{R}^N \) by \( (e_i)_{i \leq N} \) we deduce that \( x_i = S_1(e_i) = T \circ S(e_i) = T(x_i^{**}) \), and thus

\[
1 + \delta > \|x_i\| = \|T(x_i^{**})\| \geq \|\langle x_i^*, x_i \rangle\| = \langle x_i^*, x_i \rangle > 1 - \delta.
\]
CHAPTER 2. WEAK TOPOLOGIES AND REFLEXIVITY

By Lemma 2.7.6 and the choice of \( \delta \) it follows therefore that \( \|T\| \cdot \|T^{-1}\| \leq 1 + \varepsilon \).

Note that for \( \xi \in H = S^{-1}(F \cap X) \), say \( \xi = \sum_{i=1}^{m} \beta_i a^{(i)} \), we compute

\[
S_1(\xi) = \sum_{i=1}^{m} \beta_i S_1(a^{(i)}) = \sum_{i=1}^{m} \beta_i \sum_{j=1}^{N} a_{i,j} x_j \\
= \sum_{i=1}^{m} \beta_i \sum_{j=1}^{N} a_{i,j} x_j^{**} = \sum_{i=1}^{m} \beta_i S(a^{(i)}) = S(\xi).
\]

We deduce therefore for \( x \in F \cap X \), that \( T(x) = x \).

Finally from the third part of the system of equations it follows, that

\[
\langle x^*, T(x^{**}) \rangle = \langle x^*, x_j \rangle = \langle x^*, x_j \rangle, \text{ for all } j = 1, 2, \ldots, N \text{ and } x^* \in G, \text{ and, thus (since the } x_j^{**} \text{ span all of } F), \text{ that} \\
\langle x^*, T(x^{**}) \rangle = \langle x^{**}, x^* \rangle, \text{ for all } x^{**} \in F \text{ and } x^* \in G.
\]

Exercises

1. Prove Proposition 2.7.2.

2. Prove Lemma 2.7.6.

3. A Banach space \( X \) is said to have the Bounded Approximation Property (BAP), if there is a sequence of finite rank operators \( (T_n) \), \( T_n : X \rightarrow X \), so that for all \( x \in X \), \( x = \lim_{n \rightarrow \infty} T_n(x) \).

   a) Show that \( \ell_p, 1 \leq p < \infty \) has (BAP).

   And if you know a bit of probability theory: \( L_p[0,1], 1 \leq p < \infty \), has also (BAP).

   b) Show that if \( X^* \) is separable and has (BAP), so does \( X \).

   **Hint:** There are finite rank operators \( T_n : X^* \rightarrow X^*, n \in \mathbb{N} \), which point wise converge to the identity. Consider the adjoints, but notice that they might not map \( X \) to \( X \).