Chapter 3

Bases in Banach Spaces

Like every vector space a Banach space $X$ admits an algebraic or Hamel basis, i.e. a subset $B \subseteq X$, so that every $x \in X$ is in a unique way the (finite) linear combination of elements in $B$. This definition does not take into account that we can take infinite sums in Banach spaces and that we might want to represent elements $x \in X$ as converging series (with possibly infinite non zero elements). Hamel bases are also not very useful for Banach spaces, since (see Exercise 1 the coordinate functionals might not be continuous.

3.1 Schauder Bases

Definition 3.1.1. (Schauder bases of Banach Spaces)

Let $X$ be an infinite dimensional Banach space. A sequence $(e_n) \subset X$ is called Schauder basis of $X$, or simply a basis of $X$, if for every $x \in X$, there is a unique sequence of scalars $(a_n) \subset \mathbb{K}$ so that

$$x = \sum_{n=1}^{\infty} a_n e_n.$$ 

Examples 3.1.2. For $n \in \mathbb{N}$ let

$$e_n = (\underbrace{0, \ldots, 0}_n, 1, 0, \ldots) \in \mathbb{K}^N$$

Then $(e_n)$ is a basis of $\ell_p$, $1 \leq p < \infty$ and $c_0$. We call $(e_n)$ the unit vector basis of $\ell_p$ and $c_0$, respectively.

Remarks. Assume that $X$ is a Banach space and $(e_n)$ a basis of $X$. 

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a) \((e_n)\) is linear independent.

b) \(\text{span}(e_n : n \in \mathbb{N})\) is dense in \(X\), in particular \(X\) is separable.

c) Every element \(x\) is uniquely determined by the sequence \((a_n)\) so that
\[x = \sum_{j=1}^{\infty} a_n e_n.\]
So we can identify \(X\) with a space of sequences in \(\mathbb{K}^\mathbb{N}\), for which \(\sum a_n e_n\) converges in \(X\).

**Proposition 3.1.3.** Let \(X\) be a normed linear space and assume that \((e_n) \subset X\) has the property that each \(x \in X\) can be uniquely represented as a series
\[x = \sum_{n=1}^{\infty} a_n e_n, \quad \text{with } (a_n) \subset \mathbb{K}\]
(we could call \((e_n)\) Schauder basis of \(X\) but we want to reserve this term only if \(X\) is a Banach space).

For \(n \in \mathbb{N}\) and \(x \in X\) define \(e^*_n(x) \in \mathbb{K}\) to be the unique element in \(\mathbb{K}\), so that
\[x = \sum_{n=1}^{\infty} e^*_n(x) e_n.\]
Then \(e^*_n : X \to \mathbb{K}\) is linear.

For \(n \in \mathbb{N}\) let
\[P_n : X \to \text{span}(e_j : j \leq n), \quad x \mapsto \sum_{j=1}^{n} e^*_n(x) e_n.\]
Then \(P_n : X \to X\) are linear projections onto \(\text{span}(e_j : j \leq n)\) and the following properties hold:

a) \(\dim(P_n(X)) = n\),

b) \(P_n \circ P_m = P_m \circ P_n = P_{\min(m,n)}\), for \(m, n \in \mathbb{N}\),

c) \(\lim_{n \to \infty} P_n(x) = x\), for every \(x \in X\).

Conversely if \((P_n : n \in \mathbb{N})\) is a sequence of linear projections satisfying (a), (b), and (c), and moreover are bounded, and if \(e_1 \in P_1(X) \setminus \{0\}\) and \(e_n \in P_n(X) \cap \mathcal{N}(P_{n-1})\), with \(e_n \neq 0\), if \(n > 1\), then each \(x \in X\) can be uniquely represented as a series
\[x = \sum_{n=1}^{\infty} a_n e_n, \quad \text{with } (a_n) \subset \mathbb{K},\]
so in particular \((e_n)\) is a Schauder basis of \(X\) in case \(X\) is a Banach space.
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Proof. The linearity of \( e_n^* \) follows from the unique representation of every \( x \in X \) as \( x = \sum_{j=1}^{\infty} e_n^*(x)e_n \), which implies that for \( x \) and \( y \) in \( X \) and \( \alpha, \beta \in \mathbb{K} \),

\[
\alpha x + \beta y = \lim_{n \to \infty} \alpha \sum_{j=1}^{n} e_j^*(x)e_j + \beta \sum_{j=1}^{n} e_j^*(y)e_j = \lim_{n \to \infty} \sum_{j=1}^{n} (\alpha e_j^*(x) + \beta e_j^*(y))e_j = \sum_{j=1}^{\infty} (\alpha e_j^*(x) + \beta e_j^*(y))e_j,
\]

and, on the other hand

\[
\alpha x + \beta y = \sum_{j=1}^{\infty} e_j^*(\alpha x + \beta y)e_j,
\]

thus, by uniqueness, \( e_j^*(\alpha x + \beta y) = \alpha e_j^*(x) + \beta e_j^*(y) \), for all \( j \in \mathbb{N} \). The conditions (a), (b), and (c) are clear.

Conversely, assume that \( (P_n) \) is a sequence of bounded and linear projections satisfying (a), (b), and (c). By (b) \( P_{n-1}(X) = P_n \circ P_{n-1}(X) \subset P_n(X) \), for \( n \in \mathbb{N} \) (put \( P_0 = 0 \)) and, thus, by (a), the codimension of \( P_{n-1}(X) \) inside \( P_n(X) \) is 1. So if \( e_1 \in P_1(X) \setminus \{0\} \) and \( e_n \in P_n(X) \setminus \mathcal{N}(P_{n-1}) \), if \( n > 1 \), then for \( x \in X \), by (b)

\[
P_{n-1}(P_n(x) - P_{n-1}(x)) = P_{n-1}(x) - P_{n-1}(x) = 0,
\]

and thus \( P_n(x) - P_{n-1}(x) \in \mathcal{N}(P_{n-1}) \) and

\[
P_n(x) - P_{n-1}(x) = P_n(P_n(x) - P_{n-1}(x)) \in P_n(X),
\]

and therefore \( P_n(x) - P_{n-1}(x) \in P_n(X) \setminus \mathcal{N}(P_{n-1}) \). Thus, we can write \( P_n(x) - P_{n-1}(x) = a_ne_n \), for \( n \in \mathbb{N} \), and it follows from (c) that (letting \( P_0 = 0 \))

\[
x = \lim_{n \to \infty} P_n(x) = \lim_{n \to \infty} \sum_{j=1}^{n} P_j(x) - P_{j-1}(x) = \lim_{n \to \infty} \sum_{j=1}^{n} a_j e_j = \sum_{j=1}^{\infty} a_j e_j.
\]

In order to show uniqueness of this representation of \( x \) assume \( x = \sum_{j=1}^{\infty} b_j e_j \).

From the continuity of \( P_m - P_{m-1} \), for \( m \in \mathbb{N} \) it follows that

\[
a_m e_m = (P_m - P_{m-1})(x) = \lim_{n \to \infty} (P_m - P_{m-1}) \left( \sum_{j=1}^{n} b_j e_j \right) = b_m e_m,
\]

and thus \( a_m = b_m \). \( \square \)
Definition 3.1.4. (Canonical Projections and Coordinate functionals)
Let $X$ be a normed linear space and assume that $(e_i)$ satisfies the assumptions of Proposition 3.1.3. The linear functionals $(e^*_n)$ as defined in Proposition 3.1.3 are called the Coordinate Functionals for $(e_n)$ and the projections $P_n, n \in \mathbb{N}$, are called the Canonical Projections for $(e_n)$.

Proposition 3.1.5. Suppose $X$ is a normed linear space and assume that $(e_n) \subset X$ has the property that each $x \in X$ can be uniquely represented as a series

$$x = \sum_{n=1}^{\infty} a_n e_n, \text{ with } (a_n) \subset \mathbb{K}.$$ 

If the canonical projections are bounded, and, moreover, $\sup_{n \in \mathbb{N}} \|P_n\| < \infty$ (i.e. uniformly the $P_n$ are bounded), then $(e_i)$ is a Schauder basis of its completion $\tilde{X}$.

Proof. Let $\tilde{P}_n : \tilde{X} \to \tilde{X}, n \in \mathbb{N}$, be the (by Proposition 1.1.5 and Exercise 1 in Section 1.2 uniquely existing) extensions of $P_n$. Since $P_n$ has finite dimensional range it follows that $\tilde{P}_n(\tilde{X}) = P_n(X) = \text{span}(e_j : j \leq n)$ is finite dimensional and, thus, closed. $(\tilde{P}_n)$ satisfies therefore (a) of Proposition 3.1.3. Since the $P_n$ are continuous, and satisfy the equalities in (b) of Proposition 3.1.3 on a dense subset of $\tilde{X}$, (b) is satisfied on all of $\tilde{X}$. Finally, (c) of Proposition 3.1.3 is satisfied on a dense subset of $\tilde{X}$, and we deduce for $\tilde{x} \in \tilde{X}, \tilde{x} = \lim_{k \to \infty} x_k$, with $x_k \in X$, for $k \in \mathbb{N}$, that

$$\|\tilde{x} - \tilde{P}_n(\tilde{x})\| \leq \|\tilde{x} - x_k\| + \sup_{j \in \mathbb{N}} \|P_j\| \|\tilde{x} - x_k\| + \|x_k - P_n(x_k)\|$$

and, since $(P_n)$ is uniformly bounded, we can find for given $\varepsilon > 0$, $k$ large enough so that the first two summands do not exceed $\varepsilon$, and then we choose $n \in \mathbb{N}$ large enough so that the third summand is smaller than $\varepsilon$. It follows therefore that also (c) is satisfied on all of $\tilde{X}$. Thus, our claim follows from the second part of Proposition 3.1.3 applied to $\tilde{X}$. \hfill \Box

Our goal is now to show the converse of Proposition 3.1.3, and prove that if $(e_n)$ is a Schauder basis, then the canonical projections are uniformly bounded, and thus that the coordinate functionals are bounded.

Theorem 3.1.6. Let $X$ be a Banach space with a basis $(e_n)$ and let $(e^*_n)$ be the corresponding coordinate functionals and $(P_n)$ the canonical projections. Then $P_n$ is bounded for every $n \in \mathbb{N}$ and

$$b = \sup_{n \in \mathbb{N}} \|P_n\|_{L(X,X)} < \infty,$$
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and thus \( e_n^* \in X^* \) and

\[
\|e_n^*\|_{X^*} = \left\| \frac{P_n - P_{n-1}}{e_n} \right\| \leq \frac{2b}{\|e_n\|}.
\]

We call \( b \) the basis constant of \((e_j)\). If \( b = 1 \) we say that \((e_i)\) is a monotone basis.

Furthermore there is an equivalent renorming \( \| \cdot \| \) of \((X, \| \cdot \|)\) for which \((e_n)\) is a monotone basis for \((X, \| \cdot \|)\).

**Proof.** For \( x \in X \) we define

\[
\|x\| = \sup_{n \in \mathbb{N}} \|P_n(x)\|,
\]

since \( \|x\| = \lim_{n \to \infty} \|P_n(x)\| \), it follows that \( \|x\| \leq \|x\| < \infty \) for \( x \in X \).

It is clear that \( \| \cdot \| \) is a norm on \( X \). Note that for \( n \in \mathbb{N} \)

\[
\|P_n\| = \sup_{x \in X, \|x\| \leq 1} \|P_n(x)\| = \sup_{x \in X, \|x\| \leq 1} \sup_{m \in \mathbb{N}} \|P_m \circ P_n(x)\| = \sup_{x \in X, \|x\| \leq 1} \sup_{m \in \mathbb{N}} \|P_{\min(m,n)}(x)\| \leq 1.
\]

Thus the projections \( P_n \) are uniformly bounded on \((X, \| \cdot \|)\). Let \( \tilde{X} \) be the completion of \( X \) with respect to \( \| \cdot \| \), \( \tilde{P}_n \), for \( n \in \mathbb{N} \), the (unique) extension of \( P_n \) to an operator on \( \tilde{X} \). We note that the \( \tilde{P}_n \) also satisfy the conditions (a), (b) and (c) of Proposition 3.1.3. Indeed (a) and (b) are purely algebraic properties which are satisfied by the first part of Proposition 3.1.3. Moreover for \( x \in X \) then

\[
(3.1) \quad \|x - P_n(x)\| = \sup_{m \in \mathbb{N}} \|P_m(x) - P_{\min(m,n)}(x)\| = \sup_{m \geq n} \|P_m(x) - P_n(x)\| \to 0 \text{ if } n \to \infty,
\]

which verifies condition (c). Thus, it follows therefore from the second part of Proposition 3.1.3, the above proven fact that \( \|P_n\| \leq 1 \), for \( n \in \mathbb{N} \), and Proposition 3.1.5, that \((e_n)\) is a Schauder basis of the completion of \((X, \| \cdot \|)\) which we denote by \((\tilde{X}, \| \cdot \|)\).

We will now show that actually \( \tilde{X} = X \), and thus that, \((X, \| \cdot \|)\) is already complete. Then it would follow from Corollary 1.3.6 of the Closed Graph Theorem that \( \| \cdot \| \) is an equivalent norm, and thus that

\[
C = \sup_{n \in \mathbb{N}} \sup_{x \in B_X} \|P_n(x)\| = \sup_{x \in B_X} \|x\| < \infty.
\]
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So, let \( \tilde{x} \in \tilde{X} \) and write it (uniquely) as \( \tilde{x} = \sum_{j=1}^{\infty} a_j e_j \), where this convergence happens in \( \| \cdot \| \). Since \( \| \cdot \| \leq \| \cdot \|_\cdot \), and since \( X \) is complete the series \( \sum_{j=1}^{\infty} a_j e_j \) also converges with respect to \( \| \cdot \| \) in \( X \) to say \( x \in X \).

**Important Side Note:** This means that the sequence of partial sums \( \sum_{j=1}^{n} a_j e_j \) converges in \( (X, \| \cdot \|) \) to \( x \), which means that \( (a_n) \) is the unique sequence in \( K \), for which \( x = \sum_{j=1}^{\infty} a_j e_j \). In particular this means that \( P_n(x) = \sum_{j=1}^{n} a_j e_j = \tilde{P}_n(\tilde{x}) \), for all \( n \in \mathbb{N} \).

But now (3.1) yields that \( P_n(x) \) also converges in \( \| \cdot \| \) to \( x \).

This means (since \( (P_n(x) \) cannot converge to two different elements) that \( x = \tilde{x} \), which finishes our proof. \( \square \)

After reading the proof of Theorem 3.1.6 one might ask whether the last part couldn’t be generalized and whether the following could be true: If \( \| \cdot \| \) and \( \| \cdot \|_\cdot \) are two norms on the same linear space \( X \), so that \( \| \cdot \| \leq \| \cdot \|_\cdot \), and so that \( (\| \cdot \|_\cdot, X) \) is complete, does it then follow that \( (X, \| \cdot \|) \) is also complete (and thus \( \| \cdot \| \) and \( \| \cdot \|_\cdot \) are equivalent norms). The answer is negative, as the following example shows.

**Example 3.1.7.** Let \( X = \ell_2 \) with its usual norm \( \| \cdot \|_2 \) and let \((b_\gamma : \gamma \in \Gamma) \subset S_{\ell_2} \) be a Hamel basis of \( \Gamma \) (by Exercise 4 in Section 1.3, \( \Gamma \) is necessarily uncountable). For \( x \in \ell_2 \) define \( \| x \| \),

\[
\| x \| = \sum_{\gamma \in \Gamma} |x_\gamma|,
\]

where \( x = \sum_{\gamma \in \Gamma} x_\gamma b_\gamma \) is the unique representation of \( x \) as a finite linear combination of elements of \( (b_\gamma : \gamma \in \Gamma) \). Since \( \| b_\gamma \|_2 \), for \( \gamma \in \Gamma \), it follows for \( x = \sum_{\gamma \in \Gamma} x_\gamma b_\gamma \in \ell_2 \) from the triangle inequality that

\[
\| x \| = \sum_{\gamma \in \Gamma} |x_\gamma| = \sum_{\gamma \in \Gamma} \| x_\gamma b_\gamma \|_2 \geq \sum_{\gamma \in \Gamma} \| x_\gamma b_\gamma \|_2 = \| x \|_2.
\]

Finally both norms \( \| \cdot \| \) and \( \| \cdot \|_\cdot \), cannot be equivalent. Indeed, for arbitrary \( \varepsilon > 0 \), there is an uncountable set \( \Gamma' \subset \Gamma \), so that \( \| b_\gamma - b_\gamma' \|_2 < \varepsilon, \gamma, \gamma' \in \Gamma' \), (\( \Gamma \) is uncountable but \( S_{\ell_2} \) is in the \( \| \cdot \|_2 \)-norm separable). For any two different elements \( \gamma, \gamma' \in \Gamma' \) it follows that

\[
\| b_\gamma - b_\gamma' \| < \varepsilon < 2 = \| b_\gamma - b_\gamma' \|.
\]

Since \( \varepsilon > 0 \) was arbitrary this proves that \( \| \cdot \| \) and \( \| \cdot \|_\cdot \) cannot be equivalent.
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Definition 3.1.8. (Basic Sequences)
Let \( X \) be a Banach space. A sequence \((x_n) \subset X \setminus \{0\}\) is called basic sequence if it is a basis for \( \text{span}(x_n : n \in \mathbb{N}) \).

If \((e_j)\) and \((f_j)\) are two basic sequences (in possibly two different Banach spaces \( X \) and \( Y \)). We say that \((e_j)\) and \((f_j)\) are isomorphically equivalent if the map

\[
T : \text{span}(e_j : j \in \mathbb{N}) \to \text{span}(f_j : j \in \mathbb{N}), \quad \sum_{j=1}^{n} a_j e_j \mapsto \sum_{j=1}^{n} a_j f_j,
\]

extends to an isomorphism between the Banach spaces between \( \text{span}(e_j : j \in \mathbb{N}) \) and \( \text{span}(f_j : j \in \mathbb{N}) \).

Note that this is equivalent with saying that there are constants \( 0 < c \leq C \) so that for any \( n \in \mathbb{N} \) and any sequence of scalars \((\lambda_j)_{j=1}^{n} \subset \mathbb{K}\) it follows that

\[
c \left\| \sum_{j=1}^{n} \lambda_j e_j \right\| \leq \left\| \sum_{j=1}^{n} \lambda_j f_j \right\| \leq C \left\| \sum_{j=1}^{n} \lambda_j e_j \right\|.
\]

Proposition 3.1.9. Let \( X \) be Banach space and \((x_n : n \in \mathbb{N}) \subset X \setminus \{0\}\). The \((x_n)\) is a basic sequence if and only if there is a constant \( K \geq 1 \), so that for all \( m < n \) and all scalars \((a_j)_{j=1}^{n} \subset \mathbb{K}\) we have

\[
\left\| \sum_{i=1}^{m} a_i x_i \right\| \leq K \left\| \sum_{i=1}^{n} a_i x_i \right\|.
\]

In that case the basis constant is the smallest of all \( K \geq 1 \) so that (3.2) holds.

Proof. “\( \Rightarrow \)” Follows from Theorem 3.1.6, since \( K := \sup_{n \in \mathbb{N}} \|P_n\| < \infty \) and

\[
P_m \left( \sum_{i=1}^{m} a_i x_i \right) = \sum_{i=1}^{m} a_i x_m, \text{ if } m \leq n \text{ and } (a_i)_{i=1}^{n} \subset \mathbb{K}.
\]

“\( \Leftarrow \)” Assume that there is a constant \( K \geq 1 \) so that for all \( m < n \) and all scalars \((a_j)_{j=1}^{n} \subset \mathbb{K}\) we have

\[
\left\| \sum_{i=1}^{m} a_i x_i \right\| \leq K \left\| \sum_{i=1}^{n} a_i x_i \right\|.
\]

We first note that this implies that \((x_n)\) is linear independent. Indeed, if we assume that \( \sum_{j=1}^{n} a_j x_j = 0 \), for some choice of \( n \in \mathbb{N} \) and \((a_j)_{j=1}^{n} \subset \mathbb{K}\), and not all of the \( a_j \) are vanishing, we first observe that at least two of \( a_j's \) cannot be equal to 0 (since \( x_j \neq 0 \), for \( j \in \mathbb{N} \)), thus if we let \( m := \min\{j : a_j \neq 0\} \),
it follows that \( \sum_{j=1}^{m} a_j x_j \neq 0 \), but \( \sum_{j=1}^{n} a_j x_j = 0 \), which contradicts our assumption.

It follows therefore that \((x_n)\) is a Hamel basis for (the vector space) \( \text{span}(x_j : j \in \mathbb{N}) \), which implies that the projections \( P_n \) are well defined on \( \text{span}(x_j : j \in \mathbb{N}) \), and satisfy (a), (b), and (c) of Proposition 3.1.3. Moreover, it follows from our assumption that

\[
\|P_m\| = \sup \left\{ \left\| \sum_{j=1}^{m} a_j x_j \right\| : n \in \mathbb{N}, (a_j)_{j=1}^{n} \subset \mathbb{K}, \left\| \sum_{j=1}^{n} a_j x_j \right\| \leq 1 \right\} \leq K.
\]

Thus, our claim follows from Proposition 3.1.5.

Also note that the proof of “\( \Rightarrow \)” implies that the smallest constant so that 3.2 is at most as big as the basis constant, and the proof of “\( \Leftarrow \)” yielded that it is at least as large as the basis constant. \( \square \)

**Remark.** It was for a long time an open problem whether or not every separable Banach space admits a Schauder basis. 1973 this was solved by Enflo [En] in the negative. He constructed the first separable Banach space which does not admit a Schauder basis.

Every separable Hilbert space has a basis (for example an orthogonal basis). Thus, every subspace of a Hilbert space has also a basis. It was shown [Jo] that only Banach space which in some sense are “very close” to a Hilbert space, have the property that each of their subspaces have bases.

**Exercises**

1. Assume that \((e_j)\) is a basis of a Banach space \(X\). Then \(\left( \frac{e_j}{\|e_j\|} \right)\) is well defined and also a Schauder basis of \(X\).

2. Show that a Banach space \(X\) has basis enjoys the Bounded Approximation Property (Exercise 4 in Section 2.5).

3. Let \((e_\gamma : \gamma \in \Gamma)\) be a Hamel basis of an infinite dimensional Banach space \(X\). Show that some of the coordinate functionals associated with that basis are not continuous. **Hint:** pick a sequence \((\gamma_n) \subset \Gamma\) of pairwise different elements of \(\Gamma\) and consider

\[
x = \sum_{n=1}^{\infty} 2^{-n} \frac{e_{\gamma_n}}{\|e_{\gamma_n}\|}.
\]
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4. For $n \in \mathbb{N}$ define in $c_0$

$$s_n = \sum_{j=1}^{n} e_j = (1, 1, \ldots, 1, 0, \ldots).$$

Prove that $(s_n)$ is a basis for $c_0$, but that one can reorder $(s_n)$ so that it is not a basis of $c_0$.

$(s_n)$ is called the summing basis of $c_0$.

**Hint:** Play around with alternating series of $(s_n)$.

5.* Let $1 < p < \infty$ and assume that $(x_n)$ is a weakly null sequence in $\ell_p$ with $\inf_{n \in \mathbb{N}} \|x_n\| > 0$. Show that $(x_n)$ has a subsequence which is isomorphically equivalent to the unit vector basis of $\ell_p$.

And then deduce from this:

Let $T : \ell_p \to \ell_q$ with $1 < q < p < \infty$, be a bounded linear operator. Show that $T$ is compact, meaning that $T(B_{\ell_p})$ is relatively compact in $\ell_q$. 


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3.2 Bases of $C[0,1]$ and $L_p[0,1]$

In the previous section we introduced the unit vector bases of $\ell_p$ and $c_0$. Less obvious is it to find bases of function spaces like $C[0,1]$ and $L_p[0,1]$.

Example 3.2.1. (The Spline Basis of $C[0,1]$)

Let $(t_n) \subset [0,1]$ be a dense sequence in $[0,1]$, and assume that $t_1 = 0$, $t_2 = 1$. It follows that

\[(3.3) \quad \text{mesh}(t_1, t_2, \ldots , t_n) \to 0, \text{ if } n \to \infty, \text{ where} \]

\[\text{mesh}(t_1, t_2, \ldots , t_n) = \max_{i=1,2,\ldots,n} \{|t_i - t_j| : t_j \text{ is neighbor of } t_i\} .\]

For $f \in C[0,1]$ we let $P_1(f)$ to be the constant function taking the value $f(0)$, and for $n \geq 2$ we let $P_n(f)$ be the piecewise linear function which interpolates the $f$ at the points $t_1, t_2, \ldots , t_n$. More precisely, let $0 = s_1 < s_2 < \ldots < s_n = 1$ be the increasing reordering of $\{t_1, t_2, \ldots , t_n\}$, then define $P_n(f)$ by

\[P_n(f) : [0,1] \to \mathbb{K}, \text{ with} \]

\[P_n(f)(s) = \frac{s_j - s}{s_j - s_{j-1}} f(s_{j-1}) + \frac{s - s_{j-1}}{s_j - s_{j-1}} f(s_j), \text{ for } s \in [s_{j-1}, s_j].\]

We note that $P_n : C[0,1] \to C[0,1]$ is a linear projection and that $\|P_n\| = 1$, and that (a), (b), (c) of Proposition 3.1.3 are satisfied. Indeed, the image of $P_n(C[0,1])$ is generated by the functions $f_1 \equiv 1$, $f_2(s) = s$, for $s \in [0,1]$, and for $n \geq 2$, $f_n(s)$ is the functions with the property $f(t_n) = 1$, $f(t_j) = 0$, $j \in \{1,2,\ldots\} \setminus \{t_n\}$, and is linear between any $t_j$ and the next bigger $t_i$. Thus $\dim(P_n(C[0,1])) = n$. Property (b) is clear, and property (c) follows from the fact that elements of $C[0,1]$ are uniformly continuous, and condition (3.3).

Also note that for $n > 1$ it follows that $f_n \in P_n(C[0,1]) \cap \mathcal{N}(P_{n-1}) \setminus \{0\}$ and thus it follows from Proposition 3.1.3 that $(f_n)$ is a monotone basis of $C[0,1]$.

Now we define a basis of $L_p[0,1]$, the Haar basis of $L_p[0,1]$. Let

\[T = \{(n,j) : n \in \mathbb{N}_0, j = 1,2,\ldots,2^n\} \cup \{0\}.\]

We partially order the elements of $T$ as follows

\[(n_1,j_1) < (n_2,j_2) \iff [(j_2 - 1)2^{-n_2}, j_22^{-n_2}] \subseteq [(j_1 - 1)2^{-n_1}, j_12^{-n_1}]\]
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\[
\iff (j_1 - 1)2^{-n_1} \leq (j_2 - 1)2^{-n_2} < j_22^{-n_2} \leq j_12^{-n_1}, \text{ and } n_1 < n_2
\]

whenever $(n_1, j_1), (n_2, j_2) \in T$

and

\[0 < (n, j), \quad \text{whenever } (n, j) \in T \setminus \{0\}\]

Let $1 \leq p < \infty$ be fixed. We define the Haar basis $(h_t)_{t \in T}$ and the in $L_p$

normalized Haar basis $(h_t^{(p)})_{t \in T}$ as follows.

$h_0 = h_0^{(p)} \equiv 1$ on $[0, 1]$ and for $n \in \mathbb{N}_0$ and $j = 1, 2, \ldots, 2^n$ we put

\[h_{(n,j)} = 1_{[(j-1)2^{-n}, (j-\frac{1}{2})2^{-n})]} - 1_{[(j-\frac{1}{2})2^{-n}, j2^{-n})]}\]

and we let

\[
\Delta_{(n,j)} = \text{supp}(h_{(n,j)}) = [(j - 1)2^{-n}, j2^{-n})],
\]

\[
\Delta^+_{(n,j)} = [(j - 1)2^{-n}, (j - \frac{1}{2})2^{-n})]
\]

\[
\Delta^-_{(n,j)} = [(j - \frac{1}{2})2^{-n}, j2^{-n})]
\]

We let $h_{(n,j)}^{(\infty)} = h_{(n,j)}$. And for $1 \leq p < \infty$

\[h_{(n,j)}^{(p)} = \frac{h_{(n,j)}}{\|h_{(n,j)}\|_p} = 2^{n/p} \frac{1_{[(j-1)2^{-n}, (j-\frac{1}{2})2^{-n})]} - 1_{[(j-\frac{1}{2})2^{-n}, j2^{-n})]}}.
\]

Note that $\|h_t\|_p = 1$ for all $t \in T$ and that $\text{supp}(h_t) \subseteq \text{supp}(h_s)$ if and only if $s \leq t$.

**Theorem 3.2.2.** If one orders $(h_t^{(p)})_{t \in T}$ linearly in any order compatible with the order on $T$ then $(h_t^{(p)})$ is a monotone basis of $L_p[0, 1]$ for all $1 \leq p < \infty$.

**Remark.** A linear order compatible with the order on $T$ is for example the lexicographical order

\[h_0, h_{(0,1)}, h_{(1,1)}, h_{(1,2)}, h_{(2,1)}, h_{(2,2)}, \ldots.\]

Important observation: if $(h_t : t \in T)$ is linearly ordered into $h_0, h_1, \ldots$, which is compatible with the partial order of $T$, then the following is true:
If \( n \in \mathbb{N} \) and and if
\[
h = \sum_{j=1}^{n-1} a_j h_j,
\]
is any linear combination of the first \( n - 1 \) elements, then \( h \) is constant on the support of \( h_n \). Moreover, \( h \) can be written as a step function
\[
h = \sum_{j=1}^{N} b_j 1_{[s_{j-1}, s_j)},
\]
with \( 0 = s_0 < s_1 < \ldots s_N \), so that
\[
\int_{s_{j-1}}^{s_j} h_n(t) dt = 0.
\]

As we will see later, if \( 1 < p < \infty \), any linear ordering of \((h_t : t \in T)\) is a basis of \( L_p[0,1] \), but not necessarily a monotone one.

**Proof of Theorem 3.2.2.** First note that the indicator functions on all dyadic intervals are in \( \text{span}(h_t : t \in T) \). Indeed: \( 1_{[0,1/2)} = (h_0 + h_{(0,1)})/2, 1_{[1/2,1]} = (h_0 - h_{(0,1)})/2, 1_{[0,1/4]} = 1/2(1_{[0,1/2]} - h_{(1,1)}), \) etc.

Since the indicator functions on all dyadic intervals are dense in \( L_p[0,1] \), it follows that \( \text{span}(h_t : t \in T) \).

Let \((h_n)\) be a linear ordering of \((h_{(p)}^t)_{t \in T}\) which is compatible with the ordering of \( T \).

Let \( n \in \mathbb{N} \) and let \((a_i)_{i=1}^{n}\) be a scalar sequence. We need to show that
\[
\left\| \sum_{i=1}^{n-1} a_i h_i \right\| \leq \left\| \sum_{i=1}^{n} a_i h_i \right\|.
\]

As noted above, on the set \( A = \text{supp}(h_n) \) the function \( f = \sum_{i=1}^{n-1} a_i h_i \) is constant, say \( f(x) = a \), for \( x \in A \). Therefore we can write
\[
1_A(f + a_n h_n) = 1_{A^+}(a + a_n) + 1_{A^-}(a - a_n),
\]
where \( A^+ \) is the first half of the interval \( A \) and \( A^- \) the second half. From the convexity of \([0,\infty) \ni r \mapsto r^p\), we deduce that
\[
\frac{1}{2} |a + a_n|^p + |a - a_n|^p \geq |a|^p,
\]
3.2. BASES OF $C[0, 1]$ AND $L_P[0, 1]$

and thus

$$\int |f + a_n h_n|^p dx = \int_{A^c} |f|^p dx + \int_A |a + a_n|^p 1_{A^+} + |a - a_n|^p 1_{A^-} dx$$

$$= \int_{A^c} |f|^p dx + \frac{1}{2} m(A) [ |a + a_n|^p + |a - a_n|^p ]$$

$$\geq \int_{A^c} |f|^p dx + m(A) |a|^p = \int |f|^p dx$$

which implies our claim. \qed

**Proposition 3.2.3.** Since for $1 \leq p < \infty$, and $1 < q \leq \infty$, with $\frac{1}{p} + \frac{1}{q}$ it is easy to see that for $s \hookrightarrow t \in T$

$$\langle h_s^{(p)}, h_t^{(q)} \rangle = \delta(s, t),$$

we deduce that $(h_t^{(q)})_{t \in T}$ are the coordinate functionals of $(h_t^{(p)})_{t \in T}$.

**Exercises**

1. Decide whether or not the monomials $1, x, x^n$ are a Schauder basis of $C[0, 1]$.

2. Show that the Haar basis in $L_1[0, 1]$ can be reordered in such a way that it is not a a Schauder basis anymore.
3.3 Shrinking, and boundedly complete bases

Proposition 3.3.1. Let \((e_n)\) be a Schauder basis of a Banach space \(X\), and let \((e^*_n)\) be the coordinate functionals and \((P_n)\) the canonical projections for \((e_n)\).

Then

\[ P^*_n(x^*) = \sum_{j=1}^{n} \langle x^*, e_j \rangle e^*_j = \sum_{j=1}^{n} \langle \chi(e_j), x^* \rangle e^*_j, \text{ for } n \in \mathbb{N} \text{ and } x^* \in X^*. \]

b) \(x^* = \sigma(X^*, X) - \lim_{n \to \infty} P^*_n(x^*), \text{ for } x^* \in X^*.\)

c) \((e^*_n)\) is a Schauder basis of \(\text{span}(e^*_n : n \in \mathbb{N})\) whose coordinate functionals are \((e_n)\).

Proof. (a) For \(n \in \mathbb{N}, x^* \in X^* \) and \(x = \sum_{j=1}^{\infty} \langle e^*_j, x \rangle e_j \in X\) it follows that

\[ \langle P^*_n(x^*), x \rangle = \langle x^*, P_n(x) \rangle = \sum_{j=1}^{n} \langle \chi(e_j), x \rangle e_j = \sum_{j=1}^{n} \langle x^*, e_j \rangle e^*_j, x \]

and thus

\[ P^*_n(x^*) = \sum_{j=1}^{n} \langle x^*, e_j \rangle e^*_j. \]

(b) For \(x \in X \) and \(x^* \in X^*\)

\[ \langle x^*, x \rangle = \lim_{n \to \infty} \langle x^*, P_n(x) \rangle = \lim_{n \to \infty} \langle P^*_n(x^*), x \rangle. \]

(c) It follows for \(m \leq n \) and \((a_i)_{i=1}^{n} \subset \mathbb{K}, \) that

\[ \left\| \sum_{i=1}^{m} a_i e^*_i \right\| = \sup_{x \in B_X} \left| \sum_{i=1}^{m} a_i \langle e^*_i, x \rangle \right| = \sup_{x \in B_X} \left| \sum_{i=1}^{n} a_i \langle e^*_i, P_m(x) \rangle \right| \leq \left\| \sum_{i=1}^{n} a_i e^*_i \right\| \|P_m\| \leq \sup_{j \in \mathbb{N}} \|P_j\| \cdot \left\| \sum_{i=1}^{n} a_i e^*_i \right\|. \]

It follows therefore from Proposition 3.1.9 that \((e^*_n)\) is a basic sequence, thus, a basis of \(\text{span}(e^*_n)\). Since \(\langle \chi(e_j), e^*_i \rangle = \langle e^*_i, e_j \rangle = \delta_{i,j} \), it follows that \((\chi(e_n))\) are the coordinate functionals for \((e^*_n)\). \(\Box\)
3.3. SHRINKING, AND BOUNDEDLY COMPLETE BASES

Remark. If $X$ is a space with basis $(e_n)$ one can identify $X$ with a vector space of sequences $x = (\xi_n) \subset K$. If $(e_n^*)$ are coordinate functionals for $(e_n)$ we can also identify the subspace $\text{span}(e_n^* : n \in \mathbb{N})$ with a vector space of sequences $x^* = (\eta_n) \subset K$. The way such a sequence $x^* = (\eta_n) \in X^*$ acts on elements in $X$ is via the infinite scalar product:

$$\langle x^*, x \rangle = \left\langle \sum_{n \in \mathbb{N}} \eta_n e_n^*, \sum_{n \in \mathbb{N}} \xi_n e_n \right\rangle = \sum_{n \in \mathbb{N}} \eta_n \xi_n.$$

We want to address two questions for a basis $(e_n)$ of a Banach space $X$ and its coordinate functionals $(e_j^*)$:

1. Under which conditions does it follow that $X^* = \overline{\text{span}(e_n^*)}$?

2. Under which condition does it follow that the map $J : X \to \overline{\text{span}(e_n^*)}$, with

   $$J(x)(z^*) = \langle z^*, x \rangle,$$

   for $x \in X$ and $z^* \in \overline{\text{span}(e_n^*)},$

   an isomorphy or even an isometry?

We need first the following definition and some observations.

Definition 3.3.2. [Block Bases]
Assume $(x_n)$ is a basic sequence in Banach space $X$, a block basis of $(x_n)$ is a sequence $(z_n) \subset X \setminus \{0\}$, with

$$z_n = \sum_{j=k_{n-1}+1}^{k_n} a_j x_j, \text{ for } n \in \mathbb{N}, \text{ where } 0 = k_0 < k_1 < k_2 < \ldots \text{ and } (a_j) \subset K.$$

We call $(z_n)$ a convex block of $(x_n)$ if the $a_j$ are non negative and $\sum_{j=k_{n-1}+1}^{k_n} a_j = 1$.

Proposition 3.3.3. The block basis $(z_n)$ of a basic sequence $(x_n)$ is also a basic sequence, and the basis constant of $(z_n)$ is smaller or equal to the basis constant of $(x_n)$.

Proof. Let $K$ be the basis constant of $(x_n)$, let $m \leq n$ in $\mathbb{N}$, and $(b_i)_{i=1}^{n} \subset K$. Then

$$\left\| \sum_{i=1}^{m} b_i z_i \right\| = \left\| \sum_{i=1}^{m} \sum_{j=k_{i-1}+1}^{k_i} b_i a_j x_j \right\|.$$
CHAPTER 3. BASES IN BANACH SPACES

\[ \leq K \left\| \sum_{i=1}^{n} \sum_{j=k_{i-1}+1}^{k_{i}} b_i a_j x_j \right\| = K \left\| \sum_{i=1}^{n} b_i z_i \right\| \]

\[ \square \]

**Theorem 3.3.4.** For a Banach space with a basis \((e_n)\) and its coordinate functionals \((e_n^\ast)\) the following are equivalent.

\[ a) \quad X^\ast = \overline{\text{span}}(e_n^\ast : n \in \mathbb{N}) \quad \text{(and, thus, by Proposition 3.3.1, \((e_n^\ast)\) is a basis of} \ X^\ast \quad \text{whose canonical projections are} \ P_n^\ast. \]

\[ b) \quad \text{For every} \ x^\ast \in X^\ast, \lim_{n \to \infty} \left\| x^\ast \right\| = \lim_{n \to \infty} \sup_{x \in \text{span}(e_j : j > n), \|x\| \leq 1} |\langle x^\ast, x \rangle| = 0. \]

\[ c) \quad \text{Every bounded block basis of} \ (e_n) \quad \text{is weakly convergent to} \ 0. \]

We call the basis \((e_n)\) shrinking if these conditions hold.

**Remark.** Recall that by Corollary 2.2.6 the condition \((c)\) is equivalent with

\[ c') \quad \text{Every bounded block basis of} \ (e_n) \quad \text{has a further convex block which converges to} \ 0 \ \text{in norm.} \]

**Proof of Theorem 3.3.4.** “(a)⇒(b)” Let \( x^\ast \in X^\ast \) and, using \((a)\), write it as \( x^\ast = \sum_{j=1}^{\infty} a_j e_j^\ast. \) Then

\[ \lim_{n \to \infty} \sup_{x \in \text{span}(e_j : j > n), \|x\| \leq 1} |\langle x^\ast, x \rangle| = \lim_{n \to \infty} \sup_{x \in \text{span}(e_j : j > n), \|x\| \leq 1} |\langle x^\ast, (I - P_n)(x) \rangle| \]

\[ = \lim_{n \to \infty} \sup_{x \in \text{span}(e_j : j > n), \|x\| \leq 1} |\langle (I - P_n^\ast)(x^\ast), x \rangle| \]

\[ \leq \lim_{n \to \infty} \| (I - P_n^\ast)(x^\ast) \| = 0. \]

“(b)⇒(c)” Let \((z_n)\) be a bounded block basis of \((x_n)\), say

\[ z_n = \sum_{j=k_{n-1}+1}^{k_n} a_j x_j, \quad \text{for} \ n \in \mathbb{N}, \quad \text{with} \ 0 = k_0 < k_1 < k_2 < \ldots \ \text{and} \ (a_j) \subset \mathbb{K}. \]

and \( x^\ast \in X^\ast. \) Then, letting \( C = \sup_{j \in \mathbb{N}} \| z_j \|, \)

\[ |\langle x^\ast, z_n \rangle| \leq \sup_{z \in \text{span}(e_j : j > k_{n-1}) \cap \{\|z\| \leq C\}} |\langle x^\ast, z \rangle| \to_{n \to \infty} 0, \ \text{by condition} \ (b), \]
thus, \((z_n)\) is weakly null.

\(- (a) \Rightarrow - (c)\)” Assume there is an \(x^* \in S_{X^*}\), with \(x^* \notin \overline{\text{span}(e^*_j : j \in \mathbb{N})}\). It follows for some \(0 < \varepsilon \leq 1\)

\[
(3.5) \quad \varepsilon = \limsup_{n \to \infty} \|x^* - P_n(x^*)\| > 0.
\]

By induction we choose \(z_1, z_2, \ldots \) in \(B_X\) and \(0 = k_0 < k_1 < \ldots\), so that \(z_n = \sum_{j=k_n-1+1}^{k_n} a_j e_j\), for some choice of \((a_j)_{j=k_n-1+1}^{k_n}\) and \(\langle x^*, z_n \rangle \geq \varepsilon/2(1 + K)\), where \(K = \sup_{j \in \mathbb{N}} \|P_j\|\). Indeed, let \(z_1 \in B_X \cap \text{span}(e_j)\), so that \(\langle x^*, z_1 \rangle \geq \varepsilon/2(1 + K)\) and let \(k_1 = \min\{k : z_1 \in \text{span}(e_j : j \leq k)\}\). Assuming \(z_1, z_2, \ldots, z_n\) and \(k_1 < k_2 < \ldots k_n\) has been chosen. Using (3.5) we can choose \(m > k_n\) so that \(\|x^* - P_m(x^*)\| > \varepsilon/2\) and then we let \(\tilde{z}_{n+1} \in B_X \cap \text{span}(e_i : i \in \mathbb{N})\) with

\[
\langle x^* - P_m(x^*), \tilde{z}_{n+1} \rangle = \langle x^*, \tilde{z}_{n+1} - P_m(\tilde{z}_{n+1}) \rangle > \varepsilon/2.
\]

Finally choose

\[
z_{n+1} = \frac{\tilde{z}_{n+1} - P_m(\tilde{z}_{n+1})}{1 + K} \in B_X
\]

and

\[
k_{n+1} = \min\{k : z_{n+1} \in \text{span}(e_j : j \leq k)\}.
\]

It follows that \((z_n)\) is a bounded block basis of \((e_n)\) which is not weakly null.

**Examples 3.3.5.** Note that the unit vector bases of \(l_p, 1 < p < \infty\), and \(c_0\) are shrinking. But the unit vector basis of \(l_1\) is not shrinking (consider \((1, 1, 1, 1, 1, \ldots) \in l_1^* = l_\infty\)).

**Proposition 3.3.6.** Let \((e_j)\) be a shrinking basis for a Banach space \(X\) and \((e^*_j)\) its coordinate functionals. Put

\[
Y = \{(a_i) \subset \mathbb{K} : \sup_n \left\| \sum_{j=1}^{n} a_j e_j \right\| < \infty\}.
\]

Then \(Y\) with the norm

\[
\|(a_i)\| = \sup_{n \in \mathbb{N}} \left\| \sum_{j=1}^{n} a_j e_j \right\|,
\]

is a Banach space and

\[
T : X^{**} \to Y, \quad x^{**} \mapsto (\langle x^{**}, e^*_j \rangle)_{j \in \mathbb{N}}
\]

is an isomorphism between \(X^{**}\) and \(Y\).

If \((e_n)\) is monotone then \(T\) is an isometry.
Remark. Note that if \( a_j = 1 \), for \( j \in \mathbb{N} \), then in \( c_0 \)
\[
\sup_{n \in \mathbb{N}} \left\| \sum_{j=1}^{n} a_j e_j \right\|_{c_0} = 1,
\]
but the series \( \sum_{j \in \mathbb{N}} a_j e_j \) does not converge in \( c_0 \).

Considering \( X \) as a subspace of \( X^{**} \) (via the canonical embedding) the image of \( X \) under \( T \) is the space of sequences
\[
Z := \left\{ (a_i) \in Y : \sum_{j=1}^{\infty} a_j e_j \text{ converges in } X \right\}.
\]

Proof of Proposition 3.3.6. Let \( K \) denote the basis constant of \( (e_n), (e^*_n) \) the coordinate functionals, and \( (P_n) \) the canonical projections. It is straightforward to check that \( Y \) is a vector space and that \( \| \cdot \| \) is a norm on \( Y \).

For \( x^* \in X^* \) and \( x^{**} \in X^{**} \) we have by Proposition 3.3.1
\[
P_n^*(x^*) = \sum_{j=1}^{n} \langle x^*, e_j \rangle e_j^* \quad \text{and}
\]
\[
\langle P_n^{**}(x^{**}), x^* \rangle = \left\langle x^{**}, \sum_{j=1}^{n} \langle x^*, e_j \rangle e_j^* \right\rangle = \sum_{j=1}^{n} \langle x^{**}, e_j^* \rangle \langle x^*, e_j \rangle = \left\langle x^*, \sum_{j=1}^{n} \langle x^{**}, e_j^* \rangle e_j \right\rangle,
\]
which implies that
\[
\|T(x^{**})\| = \sup_{n \in \mathbb{N}} \left\| \sum_{j=1}^{n} \langle x^{**}, e_j^* \rangle e_j \right\| = \sup_{n \in \mathbb{N}} \| P_n^{**}(x^{**}) \| \leq K \|x^{**}\|.
\]

Thus \( T \) is bounded and \( \|T\| \leq K \).

Assume that \( (a_n) \in Y \). We want to find \( x^{**} \in X^{**} \), so that \( T(x^{**}) = (a_n) \). Put
\[
x_n^{**} = \sum_{j=1}^{n} a_j e_j, \text{ for } n \in \mathbb{N}.
\]
(where we identify \( X \) with its canonical image in \( X^{**} \) and, thus, \( e_j \) with \( \chi(e_j) \in X^{**} \)) Since
\[
\|x_n^{**}\|_{X^{**}} = \left\| \sum_{j=1}^{n} a_j e_j \right\|_{X} \leq \|a_i\|, \text{ for all } n \in \mathbb{N},
\]
and since $X^*$ is separable (and thus $(B_{X^*}, \sigma(X^*, X^*))$ is metrizable by Exercise 8 in Chapter 2) $(x_n^{**})$ has a $w^*$-converging subsequence $x_{n_j}^{**}$ to an element $x^{**}$ with

$$
\|x^{**}\| \leq \limsup_{n \to \infty} \|x_{n_j}^{**}\| \leq \|(a_j)\|.
$$

It follows for $m \in \mathbb{N}$ that

$$
\langle x^{**}, e_{m}^* \rangle = \lim_{j \to \infty} \langle x_{n_j}^{**}, e_{m}^* \rangle = a_{m},
$$

and thus it follows that $T(x^{**}) = (a_j)$, and thus that $T$ is surjective.

Finally, since $(e_n^*)$ is a basis for $X^*$ it follows for any $x^{**}$

$$
\|T(x^{**})\| = \sup_{n \in \mathbb{N}} \left\| \sum_{j=1}^{n} \langle x^{**}, e_{j}^* \rangle e_{j} \right\|
$$

$$
= \sup_{n \in \mathbb{N}, x^* \in B_{X^*}} \left| \sum_{j=1}^{n} \langle x^{**}, e_{j}^* \rangle \langle x^*, e_{j} \rangle \right|
$$

$$
= \sup_{x^* \in B_{X^*}, n \in \mathbb{N}} \sup \langle x^{**}, P_n^*(x^*) \rangle
$$

$$
\geq \|x^{**}\| \text{ (since } P_n^*(x^*) \to x^* \text{ if } n \to \infty),
$$

which proves that $T$ is an isomorphism, and, that $\|T(x^{**})\| \geq \|x^{**}\|$, for $x^{**} \in X^{**}$. Together with (3.6) that shows $T$ is an isometry if $K = 1$. 

**Lemma 3.3.7.** Let $X$ be a Banach space with a basis $(e_n)$, with basis constant $K$ and let $(e_n^*)$ be its coordinate functionals. Let $Z = \text{span}(e_n^*: n \in \mathbb{N}) \subset X^*$ and define the operator

$$
S : X \to Z^*, \quad x \mapsto \chi(x)|_Z \quad \text{i.e. } S(x)(z) = \langle z, x \rangle, \text{ for } z \in Z \text{ and } x \in X.
$$

Then $S$ is an isomorphic embedding of $X$ into $Z^*$ and for all $x \in X$,

$$
\frac{1}{K} \|x\| \leq \|S(x)\| \leq \|x\|.
$$

Moreover, the sequence $(S(e_n)) \subset Z^*$ are the coordinate functionals of $(e_n^*)$ (which by Proposition 3.3.1 is a basis of $Z$).

**Proof.** For $x \in X$ note that

$$
\|S(x)\| = \sup_{z \in Z, \|z\|_{X^*} \leq 1} |\langle z, x \rangle| \leq \sup_{x^* \in B_{X^*}} |\langle x^*, x \rangle| = \|x\|,
$$

By Corollary 1.4.6 of the Hahn Banach Theorem.
On the other hand, again by using that Corollary of the Hahn Banach Theorem, we deduce that

$$
\|x\| = \sup_{w^* \in B_{X^*}} |\langle w^*, x \rangle|
= \sup_{w^* \in B_{X^*}} \lim_{n \to \infty} |\langle w^*, P_n(x) \rangle|
= \sup_{w^* \in B_{X^*}} \lim_{n \to \infty} |\langle P_n^*(w^*), x \rangle|
\leq \sup_{n \in \mathbb{N}} \sup_{w^* \in B_{X^*}} |\langle P_n^*(w^*), x \rangle|
\leq \sup_{n \in \mathbb{N}} \sup_{z \in \text{span}(e_j^* : j \leq n), \|z\| \leq K} |\langle z, x \rangle|
= K \|S(x)\|.
$$

Theorem 3.3.8. Let $X$ be a Banach space with a basis $(e_n)$, and let $(e_n^*)$ be its coordinate functionals. Let $Z = \text{span}(e_n^* : n \in \mathbb{N}) \subset X^*$. Then the following are equivalent

a) $X$ is isomorphic to $Z^*$, via the map $S$ as defined in Lemma 3.3.7

b) $(e_n^*)$ is a shrinking basis of $Z$.

c) If $(a_j) \subset \mathbb{K}$, with the property that

$$
\sup_{n \in \mathbb{N}} \left\| \sum_{j=1}^n a_j e_j \right\| < \infty,
$$

then $\sum_{j=1}^\infty a_j e_j$ converges.

In that case we call $(e_n)$ boundedly complete.

Proof. “(a)$\Rightarrow$(b)” Assuming condition (a) we will verify condition (b) of Theorem 3.3.4 for $Z$ and its basis $(e_n^*)$. So let $z^* \in Z^*$. By (a) we can write $z^* = S(x)$ for some $x \in X$. Since $x = \lim_{n \to \infty} P_n(x)$, where $(P_n)$ are the canonical projection for $(e_n)$, we deduce that

$$
\sup_{w \in \text{span}(e_j^* : j > n), \|w\| \leq 1} \langle z^*, w \rangle = \sup_{w \in \text{span}(e_j^* : j > n), \|w\| \leq 1} \langle S(x), w \rangle
= \sup_{w \in \text{span}(e_j^* : j > n), \|w\| \leq 1} \langle w, x \rangle
= \sup_{w \in \text{span}(e_j^* : j > n), \|w\| \leq 1} \langle w, (I - P_n)(x) \rangle
$$

3.3. SHRINKING, AND BOUNDEDLY COMPLETE BASES

\[
\leq \|(I - P_n)(x)\| \to_{n \to \infty} 0.
\]

It follows now from Theorem 3.3.4 that \((e_j^*)\) is a shrinking basis of \(Z\).

“(b)⇒(c)” Assume (b) and let \((a_j) \subset \mathbb{K}\) so that

\[
\|(a_j)\| = \sup_{n \in \mathbb{N}} \left\| \sum_{j=1}^{n} a_j e_j \right\| = \sup_{n \in \mathbb{N}} \left\| \sum_{j=1}^{n} a_j \chi(e_j) \right\| < \infty.
\]

The sequence \((x_n^*) \subset X^{**}\), with \(x_n^* = \sum_{j=1}^{n} a_j \chi(e_j)\), is bounded in \(X^{**}\) and must therefore have an \(\sigma(X^{**}, X^*)\)-converging subnet whose limit we denote by \(x^*\). It follows that \(a_j = \langle x^*, e_j^* \rangle\), for all \(j \in \mathbb{N}\).

Let \(z^*\) be the restriction of \(x^*\) to the space \(Z\) (which is a subspace of \(X^*\)). Since by assumption \((e_j^*)\) is a shrinking basis of \(Z\) and since by Lemma 3.3.7 \((S(e_j))_{j \in \mathbb{N}}\) are the coordinate functionals we can write \(z^*\) in a unique way as

\[
z^* = \sum_{j=1}^{\infty} b_j S(e_j).
\]

But this means that \(a_j = \langle x^*, e_j^* \rangle = \langle z^*, e_j^* \rangle = b_j\), for all \(j \in \mathbb{N}\). and since \(S\) is an isomorphism between \(X\) and its image it follows that \(\sum_{j=1}^{\infty} a_j e_j\) converges in norm in \(X\).

“(c)⇒(a)” By Lemma 3.3.7 it is left to show that the operator \(S\) is surjective. Thus, let \(z^* \in Z^*\). Since \((e_n^*)\) is a basis of \(Z\) and \((S(e_n)) \subset Z^*\) are the coordinate functionals of \((e_n^*)\), it follows from Proposition 3.3.1 that \(z^*\) is the \(w^*\) limit of \((z_n^*)\) where

\[
z_n^* = \sum_{j=1}^{n} \langle z^*, e_j^* \rangle S(e_j).
\]

Since \(w^*\)-converging sequences are bounded it follows that

\[
\sup_{n \in \mathbb{N}} \left\| \sum_{j=1}^{n} \langle z^*, e_j^* \rangle S(e_j) \right\| < \infty
\]

and, thus, by Lemma 3.3.7

\[
\sup_{n \in \mathbb{N}} \left\| \sum_{j=1}^{n} \langle z^*, e_j^* \rangle e_j \right\| < \infty.
\]
By our assumption (c) it follows therefore that $x = \sum_{j=1}^{n} \langle z^*, e_j^* \rangle e_j$ converges in $X$, and moreover

$$S(x) = \lim_{n \to \infty} \sum_{j=1}^{n} \langle z^*, e_j^* \rangle S(e_j) = z^*,$$

which proves our claim.

**Theorem 3.3.9.** Let $X$ be a Banach space with a basis $(e_n)$. Then $X$ is reflexive if and only if $(e_j)$ is shrinking and boundedly complete, or equivalently if $(e_j)$ and $(e_j^*)$ are shrinking.

**Proof.** Let $(e_n^*)$ be the coordinate functionals of $(e_n)$ and $(P_n)$ be the canonical projections for $(e_n)$.

$\Rightarrow$ Assume that $X$ is reflexive. By Proposition 3.3.1 it follows for every $x^* \in X^*$

$$x^* = w^* - \lim_{n \to \infty} P_n^*(x^*) = w - \lim_{n \to \infty} P_n^*(x^*),$$

which implies that $x^* \in \overline{\text{span}(e_n^* : n \in \mathbb{N})}$, and thus, by Proposition 2.2.5 $x^* \in \overline{\text{span}(e_n^* : n \in \mathbb{N})}$. It follows therefore that $x^* = \overline{\text{span}(e_n^* : n \in \mathbb{N})}$ and thus that $(e_j)$ is shrinking (by Proposition 3.3.1).

Thus $X^*$ is a Banach space with a basis $(e_j^*)$ which is also reflexive. We can therefore apply to $X^*$ what we just proved for $X$ and deduce that $(e_n)$ is a shrinking basis for $X^*$. But, by Theorem 3.3.8 (in this case $Z = X^*$) this means that $(e_n)$ is boundedly complete.

$\Leftarrow$ Assume that $(e_n)$ is shrinking and boundedly complete, and let $x^{**} \in X^{**}$. Then

$$x^{**} = \sigma(X^{**}, X^*) - \lim_{n \to \infty} \sum_{j=1}^{n} \langle x^{**}, e_j^* \rangle \chi(e_j)$$

By Proposition 3.3.1 and the fact that $X^* = \overline{\text{span}(e_j^* : j \in \mathbb{N})}$ has $(e_j^*)$ as a basis, since $(e_j)$ is shrinking

$$= \| \cdot \| - \lim_{n \to \infty} \sum_{j=1}^{n} \langle x^{**}, e_j^* \rangle \chi(e_j) \in \chi(X)$$

[Since $\sup_{n \in \mathbb{N}} \| \sum_{j=1}^{n} \langle P^{**}(x^{**}), e_j^* \rangle e_j \| < \infty$, and since $(e_j)$ is boundedly complete]

which proves our claim.
3.3. SHRINKING, AND BOUNDEDLY COMPLETE BASES

The last Theorem in this section describes how much one can perturb a basis of a Banach space $X$ and still have a basis of $X$.

**Theorem 3.3.10.** (The small Perturbation Lemma)

Let $(x_n)$ be a basic sequence in a Banach space $X$, and let $(x^*_n)$ be the coordinate functionals (they are elements of $\text{span}(x_j : j \in \mathbb{N})$) and assume that $(y_n)$ is a sequence in $X$ such that

$$c = \sum_{n=1}^{\infty} \|x_n - y_n\| \cdot \|x^*_n\| < 1.$$

Then

a) $(y_n)$ is also basic in $X$ and isomorphically equivalent to $(x_n)$, more precisely

$$\left(1 - c\right) \sum_{n=1}^{\infty} a_n x_n \leq \sum_{n=1}^{\infty} a_n y_n \leq \left(1 + c\right) \sum_{n=1}^{\infty} a_n x_n,$$

for all in $X$ converging series $x = \sum_{n \in \mathbb{N}} a_n x_n$.

b) If $\text{span}(x_j : j \in \mathbb{N})$ is complemented in $X$, then so is $\text{span}(y_j : j \in \mathbb{N})$.

c) If $(x_n)$ is a Schauder basis of all of $X$, then $(y_n)$ is also a Schauder basis of $X$ and it follows for the coordinate functionals $(y^*_n)$ of $(y_n)$, that $y^*_n \in \text{span}(x^*_j : j \in \mathbb{N})$, for $n \in \mathbb{N}$.

**Proof.** By Corollary 1.4.4 of the Hahn Banach Theorem we extend the functionals $x^*_n$ to functionals $\tilde{x}^*_n \in X^*$, with $\|\tilde{x}^*_n\| = \|x^*_n\|$, for all $n \in \mathbb{N}$.

Consider the operator:

$$T : X \to X, \quad x \mapsto \sum_{n=1}^{\infty} \langle \tilde{x}^*_n, x \rangle (x_n - y_n).$$

Since $\sum_{n=1}^{\infty} \|x_n - y_n\| \cdot \|x^*_n\| < 1$, $T$ is well defined, linear and bounded and $\|T\| \leq c < 1$. It follows $S = \text{Id} - T$ is an isomorphism between $X$ and itself. Indeed, for $x \in X$ we have, $\|S(x)\| \geq \|x\| - \|T\| \cdot \|x\| \geq (1 - c)\|x\|$ and if $y \in X$, define $x = \sum_{n=0}^{\infty} T^n(y)$ ($T^0 = \text{Id}$) then

$$(\text{Id} - T)(x) = \sum_{n=0}^{\infty} T^n(y) - T\left(\sum_{n=0}^{\infty} T^n(y)\right) = \sum_{n=0}^{\infty} T^n(y) - \sum_{n=1}^{\infty} T^n(y) = y.$$
Thus \( I - T \) is surjective, and, it follows from Corollary 1.3.6 that \( I - T \) is an isomorphism between \( X \) and itself.

(a) We have \((I - T)(x_n) = y_n\), for \( n \in \mathbb{N} \), this means in particular that \((y_n)\) is basic and \((x_n)\) and \((y_n)\) are isomorphically equivalent.

(b) Let \( P : X \to \text{span}(x_n : n \in \mathbb{N}) \) be a bounded linear projection onto \( \text{span}(x_n : n \in \mathbb{N}) \). Then it is easily checked that

\[
Q : X \to \text{span}(y_n : n \in \mathbb{N}), \quad x \mapsto (I - T) \circ P \circ (I - T)^{-1}(x),
\]

is a linear projection onto \( \text{span}(y_n : n \in \mathbb{N}) \).

(c) If \( X = \text{span}(x_n : n \in \mathbb{N}) \), then, since \( I - T \) is an isomorphism, \((y_n) = ((I - T)(x_n))\) is also a Schauder basis of \( X \).

Moreover define for \( k \) and \( i \) in \( \mathbb{N} \),

\[
y^*_i = \sum_{j=1}^{k} \langle y^*_i, x_j \rangle x^*_j = \sum_{j=1}^{n} \langle \chi(x_j), y^*_i \rangle x^*_j \in \text{span}(x^*_j : j \in \mathbb{N}).
\]

It follows from Proposition 3.3.1, part (b), that \( w^* - \lim_{k \to \infty} y^*_{(i,k)} = y^*_i \), which implies that \( y^*_i(x) = \sum_{j=1}^{\infty} \langle y^*_i, x_j \rangle \langle x^*_j, x \rangle \), for all \( x \in X \), and thus for \( k \geq i \)

\[
\|y^*_i - y^*_{(i,k)}\| = \sup_{x \in B_X} |\langle y^*_i - y^*_{(i,k)}, x \rangle| \leq \|y^*_i\| \sum_{j=k+1}^{\infty} \|x^*_j\| \to 0, \text{ if } k \to \infty.
\]

so it follows that \( y^*_i = \|\cdot\| - \lim_{k \to \infty} y^*_{(k,i)} \in \text{span}(x^*_j : j \in \mathbb{N}) \) for every \( i \in \mathbb{N} \), which finishes the proof of our claim (c).

\( \square \)

Exercises
1. Prove that $Y$ with $\| \cdot \|$, as defined in Proposition 3.3.6 is a normed linear space.

2. A Banach space $X$ is said to have the Approximation Property if for every compact set $K \subset X$ and every $\varepsilon > 0$ there is a finite rank operator $T$ so that $\|x - T(x)\| < \varepsilon$ for all $x \in K$.

Show that the bounded approximation property (Exercise 4 in Section 2.5) implies the approximation property.

3. Show that $(e_i)$ is a shrinking basis of a Banach space $X$, then the coordinate functionals $(e_i^*)$ are boundedly complete basis of $X^*$.

4.* A Banach space is called $\mathcal{L}_{(p,\lambda)}$-space, for some $1 \leq p \leq \infty$ and some $\lambda \geq 1$, if for every finite dimensional subspace $F$ of $X$ and every $\varepsilon > 0$ there is a finite dimensional subspace $E$ of $X$ which contains $F$ and so that $d_{BM}(E, \ell^p_{\dim(E)}) < \lambda + \varepsilon$.

Show that $L_p[0,1], 1 \leq p < \infty$ is a $\mathcal{L}_{(p,1)}$-space.

**Hint:** Firstly, the span of the first $n$ elements of the Haar basis is isometrically isomorphically to $\ell^p_n$ (why?), secondly consider the Small Perturbation Lemma.
3.4 Unconditional Bases

As shown in Exercise 2 of Section 3.2 there are basic sequences which are no longer basic sequences if one reorders them. Unconditional bases are defined to be bases which are bases no matter how one reorders them.

We will first observe the following result on unconditionally converging series.

**Theorem 3.4.1.** For a sequence \((x_n)\) in Banach space \(X\) the following statements are equivalent.

a) For any reordering (also called permutation) \(\sigma\) of \(\mathbb{N}\) (i.e. \(\sigma : \mathbb{N} \to \mathbb{N}\) is bijective) the series \(\sum_{n \in \mathbb{N}} x_{\sigma(n)}\) converges.

b) For any \(\varepsilon > 0\) there is an \(n \in \mathbb{N}\) so that whenever \(M \subset \mathbb{N}\) is finite with \(\min(M) > n\), then \(\left\| \sum_{n \in M} x_n \right\| < \varepsilon\).

c) For any subsequence \((n_j)\) the series \(\sum_{j \in \mathbb{N}} x_{n_j}\) converges.

d) For sequence \((\varepsilon_j) \subset \{\pm 1\}\) the series \(\sum_{j=1}^{\infty} \varepsilon_j x_{n_j}\) converges.

In the case that above conditions hold we say that the series \(\sum x_n\) converges unconditionally.

**Proof.** “(a)⇒(b)” Assume that (b) is false. Then there is an \(\varepsilon > 0\) and for every \(n \in \mathbb{N}\) there is a finite set \(M \subset \mathbb{N}\), \(n < \min(M)\), so that \(\left\| \sum_{j \in M} x_j \right\| \geq \varepsilon\).

We can therefore, recursively choose finite subsets of \(\mathbb{N}\), \(M_1, M_2, M_3\) etc. so that \(\min M_{n+1} > \max M_n\) and \(\left\| \sum_{j \in M_n} x_j \right\| \geq \varepsilon\), for \(n \in \mathbb{N}\). Now consider a bijection \(\sigma : \mathbb{N} \to \mathbb{N}\), which on each interval of the form \([\max M_n + 1, \max M_n]\) (with \(M_0 = 0\)) is as follows: The interval \([\max M_n + 1, \#M_n]\) will be mapped to \(M_n\), and \([\max M_{n-1} + \#M_n, \max M_n]\) will be mapped to \([\max M_{n-1} + 1, \max M_n] \setminus M_n\). It follows then for each \(n \in \mathbb{N}\) that

\[
\left\| \sum_{j=\max M_{n-1}+1}^{\max M_{n-1}+\#M_n} x_{\sigma(j)} \right\| = \left\| \sum_{j \in M_n} x_j \right\| \geq \varepsilon,
\]

and, thus, the series \(\sum x_{\sigma(n)}(n)\) cannot be convergent, which is a contradiction.

“(b)⇒(c)” Let \((n_j)\) be a subsequence of \(\mathbb{N}\). For a given \(\varepsilon > 0\), use condition (b) and choose \(n \in \mathbb{N}\), so that \(\left\| \sum_{j \in M} x_j \right\| < \varepsilon\), whenever \(M \subset \mathbb{N}\) is finite and \(\min M > n\). This implies that for all \(i_0 \leq i < j\), with \(i_0 = \min\{s : n_s > n\}\), it follows that \(\left\| \sum_{s=i_0}^{j} x_{n_s} \right\| < \varepsilon\). Since \(\varepsilon > 0\) was arbitrary this means that the sequence \((\sum_{s=i_0}^{j} x_{n_s})_{j \in \mathbb{N}}\) is Cauchy and thus convergent.
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“(c)⇒(d)” If \((\varepsilon_n)\) is a sequence of \(\pm 1\)’s, let \(N^+ = \{n \in \mathbb{N} : \varepsilon_n = 1\}\) and \(N^- = \{n \in \mathbb{N} : \varepsilon_n = -1\}\). Since

\[
\sum_{j=1}^{n} \varepsilon_j x_j = \sum_{j \in N^+, j \leq n} x_j - \sum_{j \in N^-, j \leq n} x_j, \text{ for } n \in \mathbb{N},
\]

and since \(\sum_{j \in N^+, j \leq n} x_j\) and \(\sum_{j \in N^-, j \leq n} x_j\) converge by (c), it follows that \(\sum_{j=1}^{n} \varepsilon_j x_j\) converges.

“(d)⇒(b)” Assume that (b) is false. Then there is an \(\varepsilon > 0\) and for every \(n \in \mathbb{N}\) there is a finite set \(M \subset \mathbb{N}\), \(n < \min M\), so that \(\|\sum_{j \in M} x_j\| \geq \varepsilon\). As above choose finite subsets of \(\mathbb{N}\), \(M_1, M_2, M_3\) etc. so that \(\min M_{n+1} > \max M_n\) and \(\|\sum_{j \in M_n} x_j\| \geq \varepsilon\), for \(n \in \mathbb{N}\). Assign \(\varepsilon_n = 1\) if \(n \in \bigcup_{k \in \mathbb{N}} M_k\) and \(\varepsilon_n = -1\), otherwise.

Note that the series \(\sum_{n=1}^{\infty} (1 + \varepsilon_n)x_n\) cannot converge because

\[
\sum_{j=1}^{k} \sum_{i \in M_j} x_i = \frac{1}{2} \sum_{n=1}^{\max M_k} (1 + \varepsilon_n)x_n, \text{ for } k \in \mathbb{N}.
\]

Thus at least one of the series \(\sum_{n=1}^{\infty} x_n\) and \(\sum_{n=1}^{\infty} \varepsilon_n x_n\) cannot converge.

“(b)⇒(a)” Assume that \(\sigma : \mathbb{N} \to \mathbb{N}\) is a permutation for which \(\sum x_{\sigma(j)}\) is not convergent. Then we can find an \(\varepsilon > 0\) and \(0 = k_0 < k_1 < k_2 < \ldots\) so that

\[
\left\| \sum_{j=k_{n-1}+1}^{k_n} x_{\sigma(j)} \right\| \geq \varepsilon.
\]

Then choose \(M_1 = \{\sigma(1), \ldots \sigma(k_1)\}\) and if \(M_1 < M_2 < \ldots M_n\) have been chosen with \(\min M_{j+1} > \max M_j\) and \(\|\sum_{i \in M_j} x_i\| \geq \varepsilon\), if \(i = 1, 2, \ldots, n\), choose \(m \in \mathbb{N}\) so that \(\sigma(j) > \max M_n\) for all \(j > k_m\) (we are using the fact that for any permutation \(\sigma\), \(\lim_{j \to \infty} \sigma(j) = \infty\)) and let

\[
M_{n+1} = \{\sigma(k_m + 1), \sigma(k_m + 2), \ldots \sigma(k_{m+1})\},
\]

then \(\min M_{n+1} > \max M_n\) and \(\|\sum_{i \in M_j} x_i\| \geq \varepsilon\). It follows that (b) is not satisfied. \(\Box\)

**Proposition 3.4.2.** In case that the series \(\sum x_n\) is unconditionally converging, then \(\sum x_{\sigma(j)} = \sum x_j\) for every permutation \(\sigma : \mathbb{N} \to \mathbb{N}\).

**Definition 3.4.3.** A basis \((e_j)\) for a Banach space \(X\) is called unconditional, if for every \(x \in X\) the expansion \(x = \sum (e^*_j, x)e_j\) converges unconditionally, where \((e^*_j)\) are coordinate functionals of \((e_j)\).

A sequence \((x_n) \subset X\) is called unconditional basic sequence if \((x_n)\) is an unconditional basis of \(\text{span}(x_j : j \in \mathbb{N})\).
Proposition 3.4.4. For a sequence of non zero elements \((x_j)\) in a Banach space \(X\) the following are equivalent.

a) \((x_j)\) is an unconditional basic sequence,

b) There is a constant \(C\), so that for all \(n \in \mathbb{N}\), all \(A \subset \{1, 2, \ldots, n\}\) and all scalars \((a_j)_{j=1}^n \subset \mathbb{K}\),

\[
\left\| \sum_{j \in A} a_j x_j \right\| \leq C \left\| \sum_{j=1}^n a_j x_j \right\|. \tag{3.8}
\]

c) There is a constant \(C'\), so that for all \(n \in \mathbb{N}\), all \((\varepsilon_j)_{j=1}^n \subset \{\pm 1\}\) and all scalars \((a_j)_{j=1}^n \subset \mathbb{K}\),

\[
\left\| \sum_{j=1}^n \varepsilon_j a_j x_j \right\| \leq C' \left\| \sum_{j=1}^n a_j x_j \right\|. \tag{3.9}
\]

In that case we call the smallest constant \(C = K_s\) which satisfies (3.8) the supression-unconditional constant of \((x_n)\) for all \(n\), \(A \subset \{1, 2, \ldots, n\}\) and all scalars \((a_j)_{j=1}^n \subset \mathbb{K}\) and we call the smallest constant \(C' = K_u\) so that (3.9) holds for all \(n\), \((\varepsilon_j)_{j=1}^n \subset \{\pm 1\}\) and all scalars \((a_j)_{j=1}^n \subset \mathbb{K}\) the unconditional constant of \((x_n)\).

Moreover, it follows

\[
K_s \leq K_u \leq 2K_s. \tag{3.10}
\]

Proof. “(a)⇒(b)” Assume that (b) does not hold. We can assume that \((x_n)\) is a basic sequence with constant \(b\). Then (see Exercise 5) we choose recursively \(k_0 < k_1 < k_2, \ldots, A_n \subset \{k_{n-1} + 1, k_{n-1} + 1, \ldots, k_n\}\), and scalars \((a_j)_{j=k_{n-1}+1}^{k_n}\) so that

\[
\left\| \sum_{j \in A_n} a_j x_j \right\| \geq 1 \quad \text{and} \quad \left\| \sum_{j=k_{n-1}+1}^{k_n} a_j x_j \right\| \leq \frac{1}{n^2} \quad \text{for all } n \in \mathbb{N}.
\]

For any \(l < m\), we can choose \(i \leq j\) so that \(k_{i-1} < l \leq k_i\) and \(k_{j-1} < m \leq k_j\), and thus

\[
\left\| \sum_{s=l}^{m} a_s x_s \right\| \leq \left\| \sum_{s=l}^{k_i} a_s x_s \right\| + \sum_{t=i+1}^{j-1} \left\| \sum_{s=k_{t-1}+1}^{k_t} a_s x_s \right\| + \left\| \sum_{s=k_{j-1}+1}^{m} a_s x_s \right\|.
\]
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(where the second term is defined to be 0, if \(i \geq j - 1\))

\[
\leq \frac{2b}{(i-1)^2} + \sum_{t=i+1}^{j-1} \frac{1}{t^2} + \frac{2b}{(j-1)^2}
\]

It follows therefore that \(x = \sum_{j=1}^{\infty} a_j x_j\) converges, but by Theorem 3.4.1 (b) it is not unconditionally.

\(\text{“(b) } \iff \text{ (c)” and (3.10) follows from the following estimates for } n \in \mathbb{N}, \ (a_j)_{j=1}^{n} \subset K, A \subset \{1, 2, \ldots, n\} \text{ and } (\varepsilon_j)_{j=1}^{n} \subset \{\pm 1\}\)

\[
\left\| \sum_{j=1}^{n} \varepsilon_j a_j x_j \right\| \leq \left\| \sum_{j=1, \varepsilon_j = 1}^{n} a_j x_j \right\| + \left\| \sum_{j=1, \varepsilon_j = -1}^{n} a_j x_j \right\|
\]

\[
\left\| \sum_{j \in A} a_j x_j \right\| \leq \frac{1}{2} \left[ \left\| \sum_{j \in A} a_j x_j + \sum_{j \in \{1, 2, \ldots \} \setminus A} a_j x_j \right\| + \left\| \sum_{j \in A} a_j x_j - \sum_{j \in \{1, 2, \ldots \} \setminus A} a_j x_j \right\| \right].
\]

\(\text{“(b) } \Rightarrow \text{ (a) First, note that (b) implies by Proposition 3.1.9 that } (x_n) \text{ is a basic sequence. Now assume that for some } x = \sum_{j=1}^{\infty} a_j x_j \in \text{span}(x_j : j \in \mathbb{N}) \text{ is converging but not unconditionally converging. It follows from the equivalences in Theorem 3.4.1 that there is some } \varepsilon > 0 \text{ and of } N, M_1, M_2, M_3 \text{ etc. so that } min M_{n+1} > \max M_n \text{ and } \left\| \sum_{j \in M_n} a_j x_j \right\| \geq \varepsilon, \text{ for } n \in \mathbb{N}. \text{ On the other hand it follows from the convergence of the series } \sum_{j=1}^{\infty} a_j x_j \text{ that}
\]

\[
\limsup_{n \to \infty} \left\| \sum_{j=1}^{\max(M_n)} a_j x_j \right\| = 0,
\]

and thus

\[
\sup_{n \to \infty} \left\| \sum_{j \in M_n} a_j x_j \right\| = \infty,
\]

which is a contradiction to condition (b).

\(\square\)

**Proposition 3.4.5.** Assume that \(X\) is a Banach space over the field \(\mathbb{C}\) with an unconditional basis \((e_n)\), then it follows if \(\sum_{j=1}^{\infty} \alpha_n e_n\) is convergent and \((\beta_n) \subset \{\beta \in \mathbb{C} : |\beta| = 1\}\) that \(\sum_{j=1}^{\infty} \beta_n \alpha_n e_n\) is also converging and

\[
\left\| \sum_{n \in \mathbb{N}} \beta_n \alpha_n e_n \right\| \leq 2K_u \left\| \sum_{n \in \mathbb{N}} \alpha_n e_n \right\|.
\]

**Proof.** See Exercise 1. \(\square\)
Proposition 3.4.6. If $X$ is a Banach space with an unconditional basis, then the coordinate functionals $(e^*_n)$ are also an unconditional basic sequence, with the same unconditional constant and the same suppression-unconditional constant.

Proof. Let $K_u$ and $K_s$ be the unconditional and suppression unconditional constant of $X$.

Let $x^* = \sum_{n \in \mathbb{N}} \eta_n e^*_n$ and $(\varepsilon_n) \subset \{\pm 1\}$ then

$$
\left\| \sum_{n \in \mathbb{N}} \varepsilon_n \eta_n e^*_n \right\|_{X^*} = \sup_{x = \sum_{n=1}^{\infty} \xi_n e_n \in B_X} \left\langle \sum_{n \in \mathbb{N}} \varepsilon_n \eta_n e^*_n, \sum_{n=1}^{\infty} \xi_n e_n \right\rangle
= \sup_{x = \sum_{n=1}^{\infty} \xi_n e_n \in B_X} \sum_{n \in \mathbb{N}} \varepsilon_n \eta_n \xi_n
= \sup_{x = \sum_{n=1}^{\infty} \xi_n e_n \in B_X} \left\| \sum_{n \in \mathbb{N}} \eta_n e^*_n \right\| \cdot \left\| \sum_{n \in \mathbb{N}} \varepsilon_n \xi_n e_n \right\|
\leq K_u \left\| \sum_{n \in \mathbb{N}} \eta_n e^*_n \right\|.
$$

Using the Hahn Banach Theorem we can similarly show that if $K_u^*$ is the unconditional constant of $(e^*_n)$ then

$$
\left\| \sum_{n \in \mathbb{N}} \xi_n \varepsilon_n e_n \right\|_X \leq K_u^* \leq \left\| \sum_{n \in \mathbb{N}} \xi_n e_n \right\|_X.
$$

Thus $K_u = K_u^*$. A similar argument works to show that $K_s$ is equal to the suppression unconditional constant of $(e^*_n)$. \qed

The following Theorem about spaces with unconditional basic sequences was shown by James [Ja]

Theorem 3.4.7. Let $X$ be a Banach space with an unconditional basis $(e_j)$. Then either $X$ contains a copy of $c_0$, or a copy of $\ell_1$ or $X$ is reflexive.

We will need first the following Lemma (See Exercise 2)

Lemma 3.4.8. Let $X$ be a Banach space with an unconditional basis $(e_n)$ and let $K_u$ its constant of unconditionality. Then it follows for any converging series $\sum_{n \in \mathbb{N}} a_n e_n$ and a bounded sequence of scalars $(b_n) \subset \mathbb{K}$, that $\sum_{n \in \mathbb{N}} a_n b_n e_n$ is also converging and

$$
\left\| \sum_{n \in \mathbb{N}} a_n b_n e_n \right\| \leq K \sup_{n \in \mathbb{N}} |b_n| \left\| \sum_{n=1}^{\infty} a_n e_n \right\|,
$$

where $K = K_u$, if $\mathbb{K} = \mathbb{R}$, and $K = 2K_u$, if $\mathbb{K} = \mathbb{C}$.
3.4. UNCONDITIONAL BASES

Proof of Theorem 3.4.7. We will prove the following two statements for a space $X$ with unconditional basis $(e_n)$.

**Claim 1:** If $(e_n)$ is not boundedly complete then $X$ contains a copy of $c_0$.

**Claim 2:** If $(e_n)$ is not shrinking then $X$ contains a copy of $\ell_1$.

Together with Theorem 3.3.9, this yields the statement of Theorem 3.4.7.

Let $K_u$ be the constant of unconditionality of $(e_n)$ and let $K'_{u} = K_u$, if $\mathbb{K} = \mathbb{R}$, and $K'_u = 2K_u$, if $\mathbb{K} = \mathbb{C}$.

**Proof of Claim 1:** If $(e_n)$ is not boundedly complete there is, by Theorem 3.3.8, a sequence of scalars $(a_n)$ such that

$$\sup_{n \in \mathbb{N}} \left\| \sum_{j=1}^{n} a_j e_j \right\| = C_1 < \infty,$$

but $\sum_{j=1}^{\infty} a_j e_j$ does not converge.

This implies that there is an $\varepsilon > 0$ and sequences $(m_j)$ and $(n_j)$ with $1 \leq m_1 < n_1 < m_2 < n_2 < \ldots$ in $\mathbb{N}$ so that if we put $y_k = \sum_{j=m_k}^{n_k} a_j e_j$, for $k \in \mathbb{N}$, it follows that $\|y_k\| \geq \varepsilon$, and also

$$\|y_k\| \leq \left\| \sum_{j=1}^{n_k} a_j e_j \right\| + \left\| \sum_{j=1}^{m_k-1} a_j e_j \right\| \leq 2C_1.$$

For any $k \in \mathbb{N}$ and any sequence of scalars $(\lambda_j)_{j=1}^{k}$ it follows therefore from Lemma 3.4.8, that

$$\left\| \sum_{j=1}^{k} \lambda_j y_j \right\| \leq 2K_u \max_{j \leq k} |\lambda_j| \left\| \sum_{j=1}^{k} y_j \right\| \leq 2K_u K_s \max_{j \leq k} |\lambda_j| \left\| \sum_{i=1}^{n_k} a_i e_j \right\| \leq 2K_u K_s C_1 \max_{j \leq k} |\lambda_j|.$$

On the other hand for every $j_0 \leq n$ that

$$\left\| \sum_{j=1}^{n} \lambda_j y_j \right\| \geq \frac{1}{K_s} \|\lambda_{j_0} y_{j_0}\| \geq \frac{\varepsilon}{K_s} \max_{j \leq n} |\lambda_j|.$$

Letting $c = \varepsilon / K_u$ and $C = 2K_u K_s C_1$, it follows therefore for any $n \in \mathbb{N}$ and any sequence of scalars $(\lambda_j)_{j=1}^{n}$ that

$$c \|(\lambda)_{j=1}^{n} \|_{c_0} \leq \left\| \sum_{j=1}^{n} \lambda_j y_j \right\| \leq C \|(\lambda)_{j=1}^{n} \|_{c_0},$$

which means that $(y_j)$ and the unit vector basis of $c_0$ are isomorphically equivalent.
**Proof of Claim 2.** \((e_n)\) is not shrinking then there is by Theorem 3.3.4 a bounded block basis \((y_n)\) of \((e_n)\) which is not weakly null. After passing to a subsequence we can assume that there is a \(x^* \in X^*, \|x^*\| = 1\), so that

\[
\varepsilon = \inf_{n \in \mathbb{N}} |\langle x^*, y_n \rangle| > 0.
\]

We also can assume that \(\|y_n\| = 1\), for \(n \in \mathbb{N}\) (otherwise replace \(y_n\) by \(y_n/\|y_n\|\) and change \(\varepsilon\) accordingly).

We claim that \((y_n)\) is isomorphically equivalent to the unit vector basis of \(\ell_1\). Let \(n \in \mathbb{N}\) and \((a_j)_{j=1}^n \subset \mathbb{K}\). By the triangle inequality we have

\[
\left\| \sum_{j=1}^n a_j y_j \right\| \leq \sum_{j=1}^n |a_j|,
\]

On the other hand we can choose for \(j = 1, 2, \ldots, n\) \(\varepsilon_j = \text{sign}(a_j\langle x^*, y_j \rangle)\) if \(\mathbb{K} = \mathbb{R}\) and \(\varepsilon_j = a_j\langle x^*, y_j \rangle/|a_j\langle x^*, y_j \rangle|\), if \(\mathbb{K} = \mathbb{C}\) (if \(a_j = 0\), simply let \(\varepsilon = 1\)) and deduce from Lemma 3.4.8

\[
\left\| \sum_{j=1}^n a_j y_j \right\| \geq \frac{1}{K} \left\| \sum_{j=1}^n \varepsilon_j a_j y_j \right\| \geq \left| \sum_{j=1}^n \varepsilon_j a_j \langle x^*, y_j \rangle \right| \geq \varepsilon \sum_{j=1}^n |a_j|,
\]

which implies that \((y_n)\) is isomorphically equivalent to the unit vector basis of \(\ell_1\).

**Remark.** It was for long time an open problem whether or not every infinite dimensional Banach space contains an unconditional basis sequence. If this were so, every infinite dimensional Banach space would contain a copy of \(c_0\) or a copy of \(\ell_1\), or has an infinite dimensional reflexive subspace space. In [GM], Gowers and Maurey proved the existence of a Banach space which does not contain any unconditional basic sequences. Later then Gowers [Go] constructed a space which does not contain any copy of \(c_0\) or \(\ell_1\), and has no infinite dimensional reflexive subspace.

**Exercises.**

1. Prove Proposition 3.4.5.
2. Prove Lemma 3.4.8
3. Prove that a block sequence (of non zero vectors) of an unconditional basic sequence is also a unconditional basic sequence.

4. Show that every separable $X$ Banach space is isomorphic to the quotient space of $\ell_1$.

5. Assume that $(x_n)$ is a basic sequence in a Banach space $X$ for which (b) of Proposition 3.4.4 does not hold. Show that there is a sequence of scalars $(a_j)$ and a subsequence $(k_n)$ of $\mathbb{N}$, so that

\[
\left\| \sum_{j \in A_n} a_j x_j \right\| \geq 1 \quad \text{and} \quad \left\| \sum_{j=k_{n-1}}^{k_n} a_j x_j \right\| \leq \frac{1}{n^2} \quad \text{for all } n \in \mathbb{N}.
\]

6.* Let $1 < p < \infty$ and assume that $(x_n)$ is a weakly null sequence in $\ell_p$ with $\inf_{n \in \mathbb{N}} \|x_n\| > 0$. Show that $(x_n)$ has a subsequence which is isomorphically equivalent to the unit vector basis of $\ell_p$.

Let $T : \ell_p \to \ell_q$ with $1 < q < p < \infty$, be a bounded linear operator. Show that $T$ is compact., meaning that $T(B_{\ell_p})$ is relatively compact in $\ell_q$. 


3.5 James’ Space

The following space \( J \) was constructed by R. C. James [Ja]. It is a space which is not reflexive and does not contain a subspace isomorphic to \( c_0 \) or \( \ell_1 \). By Theorem 3.4.7 it does not have an unconditional basis. Moreover we will prove that \( J^{**/\mathcal{X}(J)} \) is one dimensional and that \( J \) is isomorphically isometric to \( J^{**} \) (but of course not via the canonical mapping).

We will define the space \( J \) over the real numbers \( \mathbb{R} \).

For a sequence \( (\xi_n) \subset \mathbb{R} \) we define the quadratic variation to be

\[
\| (\xi_n) \|_{qv} = \sup \left\{ \left( \sum_{j=1}^{l} |\xi_{n_j} - \xi_{n_{j-1}}|^2 \right)^{1/2} : l \in \mathbb{N} \text{ and } 1 \leq n_0 < n_1 < \ldots n_l \right\}
\]

and the cyclic quadratic variation norm to be

\[
\| (\xi_n) \|_{cqv} = \sup \left\{ \left( \frac{1}{\sqrt{2}} \left( |\xi_{n_0} - \xi_{n_l}|^2 + \sum_{j=1}^{l} |\xi_{n_j} - \xi_{n_{j-1}}|^2 \right)^{1/2} : l \in \mathbb{N} \text{ and } 1 \leq n_0 < n_1 < \ldots n_l \right\}
\]

Note that for a bounded sequences \( (\xi_n), (\eta_n) \subset \mathbb{R} \)

\[
\| (\xi_n + \eta_n) \|_{qv} = \sup \left\{ \| (\xi_n + \eta_n) - (\xi_{n_{i-1}} - \eta_{n_{i-1}}) \|_2 : l \in \mathbb{N}, n_0 < n_1 < \ldots n_l \right\}
\]

\[
\leq \sup \left\{ \| (\xi_n) - (\xi_{n_{i-1}}) \|_2 + \| (\eta_n) - (\eta_{n_{i-1}}) \|_2 : l \in \mathbb{N}, n_0 < n_1 < \ldots n_l \right\}
\]

\[
= \| (\xi_n) \|_{qv} + \| (\eta_n) \|_{qv}
\]

and similarly

\[
\| (\xi_n + \eta_n) \|_{cqv} \leq \| (\xi_n) \|_{cqv} + \| (\eta_n) \|_{cqv}
\]

and we note that

\[
\frac{1}{\sqrt{2}} \| (\xi_n) \|_{qv} \leq \| (\xi_n) \|_{cqv} \leq \sqrt{2} \| (\xi_n) \|_{qv}.
\]

Thus \( \| \cdot \|_{qv} \) and \( \| \cdot \|_{cqv} \) are two equivalent semi norms on the vector space

\[
\tilde{J} = \{ (\xi_n) \subset \mathbb{R} : \| (\xi_n) \|_{qv} < \infty \}
\]
and since
\[ \|\xi_n\|_{qv} = 0 \iff \|\xi_n\|_{cqv} = 0 \iff (\xi_n) \text{ is constant} \]
\[ \|\cdot\|_{qv} \text{ and } \|\cdot\|_{cqv} \text{ are two equivalent norms on the vector space} \]
\[ J = \{ (\xi_n) \in \mathbb{R} : \lim_{n \to \infty} \xi_n = 0 \text{ and } \|\xi_n\|_{qv} < \infty \} . \]

**Proposition 3.5.1.** The space \( J \) with the norms \( \|\cdot\|_{qv} \) and \( \|\cdot\|_{cqv} \) is complete and, thus, a Banach space.

**Proof.** The proof is similar to the proof of showing that \( \ell_p \) is complete. Let \((x_k)\) be a sequence in \( J \) with \( \sum_{k \in \mathbb{N}} \|x_k\|_{qv} < \infty \) and write \( x_k = (\xi_{(k,j)})_{j \in \mathbb{N}} \), for \( k \in \mathbb{N} \). Since for \( j, k \in \mathbb{N} \) it follows that
\[ |\xi_{(k,j)}| = \lim_{n \to \infty} |\xi_{(k,j)} - \xi_{(k,n)}| \leq \|x_k\|_{qv} \]
it follows that
\[ \xi_j = \sum_{k \in \mathbb{N}} \xi_{(k,j)} \]
exists and for \( x = (\xi_j) \) it follows that \( x \in c_0 \) (\( c_0 \) is complete) and
\[ \|x\|_{qv} = \sup \left\{ \left( \sum_{j=1}^{l} |\xi_{n_j} - \xi_{n_{j-1}}|^2 \right)^{1/2} : l \in \mathbb{N} \text{ and } 1 \leq n_0 < n_1 < \ldots < n_l \right\} \]
\[ \leq \sup \left\{ \sum_{k \in \mathbb{N}} \left( \sum_{j=1}^{l} |\xi_{(k,n_j)} - \xi_{(k,n_{j-1})}|^2 \right)^{1/2} : l \in \mathbb{N} \text{ and } 1 \leq n_0 < \ldots < n_l \right\} \]
\[ \leq \sum_{k \in \mathbb{N}} \|x_k\|_{qv} < \infty \]
and for \( m \in \mathbb{N} \)
\[ \left\| x - \sum_{k=1}^{m} x_k \right\|_{qv} \]
\[ = \sup \left\{ \left( \sum_{j=1}^{l} \left( \sum_{k=m+1}^{\infty} |\xi_{(k,n_j)} - \xi_{(k,n_{j-1})}|^2 \right)^{1/2} : l \in \mathbb{N} \text{ and } 1 \leq n_0 \leq \ldots \leq n_l \right\} \]
\[ \leq \sup \left\{ \sum_{k=m+1}^{\infty} \left( \sum_{j=1}^{l} |\xi_{(k,n_j)} - \xi_{(k,n_{j-1})}|^2 \right)^{1/2} : l \in \mathbb{N} \text{ and } 1 \leq n_0 \leq \ldots \leq n_l \right\} \]
(By the triangle inequality in \( \ell_2 \))
\\[\leq \sum_{k=m+1}^{\infty} \|x_k\|_q v \rightarrow 0 \text{ for } m \rightarrow \infty.\]

\[\square\]

**Proposition 3.5.2.** The unit vector basis \((e_i)\) is a monotone basis of \(J\) for both norms, \(\| \cdot \|_q v\) and \(\| \cdot \|_{c q v}\).

**Proof.** First we claim that \(\text{span}(e_j:j \in \mathbb{N}) = J\). Indeed, if \(x = (\xi_n) \in J\), and \(\varepsilon > 0\) we choose \(l\) and \(1 \leq n_0 < n_1 < \ldots n_l\) in \(\mathbb{N}\) so that

\[\sum_{j=1}^{l} |\xi_{n_j} - \xi_{n_{j-1}}|^2 > \|x\|_q^2 - \varepsilon.\]

But this implies that

\[\|x - \sum_{j=1}^{n_l+1} \xi_j e_j\| = \|(0,0,\ldots,0,\xi_{n_{l+2}},\xi_{n_{l+3}},\ldots)\| < \varepsilon.\]

In order to show monotonicity, assume \(m < n\) are in \(\mathbb{N}\) and \((a_i)_{i=1}^{n} \subset \mathbb{R}\). For \(i \in \mathbb{N}\) let

\[\xi_i = \begin{cases} a_i & \text{if } i \leq m \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \eta_i = \begin{cases} a_i & \text{if } i \leq n \\ 0 & \text{otherwise}. \end{cases}\]

For \(x = \sum_{i=1}^{\infty} \xi_i e_i\) and \(y = \sum_{i=1}^{\infty} \eta_i e_i\) we need to show that \(\|x\|_q v \leq \|y\|_q v\) and \(\|x\|_{c q v} \leq \|y\|_{c q v}\). So choose \(l\) and \(n_0 < n_1 < \ldots n_l\) in \(\mathbb{N}\) so that

\[\|x\|_q^2 = \sum_{j=1}^{l} |\xi_{n_j} - \xi_{n_{j-1}}|^2.\]

Then we can assume that \(n_l > n\) (otherwise replace \(l\) by \(l+1\) and add \(n_{l+1} = n + 1\)) and we can assume that \(n_{l-1} \leq m\) (otherwise we drop all the \(n_j\)'s in \((m,n]\)), and thus

\[\|x\|_q^2 = \sum_{j=1}^{l} |\xi_{n_j} - \xi_{n_{j-1}}|^2 = \sum_{j=1}^{l} |\eta_{n_j} - \eta_{n_{j-1}}|^2 \leq \|y\|_q v.\]

The argument for the cyclic variation norm is similar. \(\square\)
Our next goal is to show that \((e_n)\) is a shrinking basis of \(J\). We need the following lemma

**Lemma 3.5.3.** For any normalized block basis \((u_i)\) of \(e_i\) in \(J\), and \(m \in \mathbb{N}\) and any scalars \((a_i)_{i=1}^m\) it follows that

\[
\left\| \sum_{i=1}^m a_i u_i \right\| \leq \sqrt{5}\| (a_i)_{i=1}^n \|_2.
\]

**Proof.** Let \((\eta_j) \subset \mathbb{R}\) and \(k_0 = 0 < k_1 < k_2 < \ldots \) in \(\mathbb{N}\) so that for \(i \in \mathbb{N}\)

\[
u_i = \sum_{j=k_{i-1}+1}^{k_i} \eta_j e_j.
\]

Let for \(i = 1, 2, 3 \ldots m\) and \(j = k_{i-1} + 1, k_{i-1} + 2, \ldots k_i\) put \(\xi_j = a_i \cdot \eta_j\), and

\[
x = \sum_{i=1}^n a_i u_i = \sum_{j=1}^{k_n} \xi_j e_j.
\]

For given \(l \in \mathbb{N}\) and \(1 \leq n_0 < n_1 < \ldots < n_l\) we need to show that

\[
\sum_{j=1}^l |\xi_{n_j} - \xi_{n_{j-1}}|^2 \leq 5 \sum_{i=1}^m a_i^2.
\]

For \(i = 1, 2, \ldots, m\) define \(A_i = \{ j \geq 1 : k_{i-1} < n_{j-1} < n_j \leq k_i \}\). It follows that

\[
\sum_{j \in A_i} |\xi_j - \xi_{j-1}|^2 = a_i^2 \sum_{j \in A_i} |\eta_j - \eta_{j-1}|^2 \leq a_i^2 \| u_i \|_q^2,
\]

and thus

\[
\sum_{j \in \bigcup_{i=1}^n A_i} |\xi_j - \xi_{j-1}|^2 \leq \sum_{i=1}^n a_i^2.
\]

Now let \(A = \bigcup_{i=1}^n A_i\) and \(B = \{ j \leq l : j \notin A \}\). For each \(j \in B\) there must exist \(l(j)\) and \(m(j)\) in \(\{1, 2, \ldots, m\}\) so that

\[
k_{l(j)-1} < n_{j-1} \leq k_{l(j)} \leq k_{m(j)} < n_j k_{m(j)+1}
\]

and thus

\[
|\xi_{n_j} - \xi_{n_{j-1}}|^2 = |a_{m(j)} \eta_{n_j} - a_{l(j)} \eta_{n_{j-1}}|^2.
\]
\[ \leq 2a_{m(j)}^2h_j^2 + 2a_{l(j)}^2\eta_{n_{j-1}}^2 \leq 2a_{m(j)}^2 + 2a_{l(j)}^2 \]

(for the last inequality note that \(|\eta_k| \leq 1\) since \(\|u_j\| = 1\)). For \(j, j' \in B\) it follows that \(l(j) \neq l(j')\) and \(m(j) \neq m(j')\), \(j \neq j'\) and thus

\[
\sum_{j=1}^{l} |\xi_{n_j} - \xi_{n_{j-1}}|^2 = \sum_{j \in A} |\xi_{n_j} - \xi_{n_{j-1}}|^2 + \sum_{j \in B} |\xi_{n_j} - \xi_{n_{j-1}}|^2 \\
\leq \sum_{i=1}^{n} a_i^2 + 2\sum_{j \in B} a_{l(j)}^2 + 2\sum_{j \in B} a_{m(j)}^2 \leq 5\sum_{i=1}^{n} a_i^2,
\]

which finishes the proof of our claim. \(\square\)

**Corollary 3.5.4.** The unit vector basis \((e_n)\) is shrinking in \(J\).

**Proof.** Let \((u_n)\) be any block basis of \((e_n)\), which is w.l.o.g. normalized. Then by Lemma 3.5.3

\[
\frac{1}{n} \left\| \sum_{j=1}^{n} u_j \right\|_{qu} \leq \frac{\sqrt{5}}{\sqrt{n}} \rightarrow 0 \text{ if } n \rightarrow \infty.
\]

By Corollary 2.2.6 \((u_n)\) is therefore weakly null. Since \((u_n)\) was an arbitrary block basis of \((e_n)\) this yields by Theorem 3.3.8 that \((e_n)\) is shrinking. \(\square\)

**Definition 3.5.5.** (Skipped Block Bases)
Assume \(X\) is a Banach space with basis \((e_n)\). A *Skipped Block Basis of \((e_n)\)* is a sequence \((u_n)\) for which there are \(0 = k_0 < k_1 < k_2 < \ldots \) in \(\mathbb{N}\), and \((a_j) \subset \mathbb{K}\) so that

\[
u_n = \sum_{j=k_{n-1}+1}^{k_n-1} a_je_j, \text{ for } n \in \mathbb{N}.
\]

(i.e. the \(k_n\)’s are skipped).

**Proposition 3.5.6.** Every normalized skipped block sequence of the unit vector basis in \(J\) is isomorphically equivalent to the unit vector basis in \(\ell_2\). Moreover the constant of equivalence is \(\sqrt{5}\).

**Proof.** Assume that

\[
u_n = \sum_{j=k_{n-1}+1}^{k_n-1} a_je_j, \text{ for } n \in \mathbb{N}.
\]
with 0 = k_0 < k_1 < k_2 < \ldots \in \mathbb{N}, and (a_j) \subset \mathbb{K}, and a_{k_n} = 0, for n \in \mathbb{N}.

For n \in \mathbb{N} we can find l_n and k_{n-1} = p_0^{(n)} < p_1^{(n)} < \ldots p_{l_n} = k_n in \mathbb{N} so that

$$
\|u_n\|^2_{q^v} = \sum_{j=1}^{l_n} (a_{p_j^{(n)}} - a_{p_{j-1}^{(n)}})^2 = 1.
$$

Now let m \in \mathbb{N} and (b_j)_{j=1}^m \subset \mathbb{R} we can string the p_j^{(n)}'s together and deduce:

$$
\left\| \sum_{n=1}^{m} b_n u_n \right\|^2_{q^v} \geq \sum_{i=1}^{m} b_i^2 \sum_{j=1}^{l_n} (a_{p_j^{(n)}} - a_{p_{j-1}^{(n)}})^2 = \sum_{i=1}^{m} b_i^2.
$$

On the other hand it follows from Lemma 3.5.3 that

$$
\left\| \sum_{n=1}^{m} b_n u_n \right\|^2_{q^v} \leq 5 \sum_{i=1}^{m} b_i^2.
$$

\[\square\]

**Corollary 3.5.7.** \(J\) is hereditarily \(\ell_2\), meaning every infinite dimensional subspace of \(J\) has a further subspace which is isomorphic to \(\ell_2\).

**Proof.** Let \(Z\) be an infinite dimensional subspace of \(J\). By induction we choose for each \(n \in \mathbb{N}\), \(z_n \in Z\), \(u_n \in J\) and \(k_n \in \mathbb{N}\), so that

\begin{align}
(3.13) \quad \|z_n\|_{q^v} = \|u_n\|_{q^v} = 1 \text{ and } \|z_n - u_n\|_{q^v} < 2^{-4-n}, \\
(3.14) \quad u_n \in \text{span}(e_j : k_{n-1} < j < k_n)
\end{align}

Having accomplished that, \((u_n)\) is a skipped block basis of \((e_n)\) and by Proposition 3.5.6 isomorphically equivalent to the unit vector basis of \(\ell_2\). Letting \((u^*_n)\) be the coordinate functionals of \((u_n)\) it follows that \(\|u^*_n\| \leq \sqrt{5}\), for \(n \in \mathbb{N}\), and thus, by the third condition in (3.13),

$$
\sum_{n=1}^{\infty} \|u^*_n\| \|u_n - z_n\| \leq \sqrt{5} 2^{-4} < 1,
$$

which implies by the Small Perturbation Lemma, Theorem 3.3.10, that \((z_n)\) is also isomorphically equivalent to unit vector basis in \(\ell_2\).

We choose \(z_1 \in S_Z\) arbitrarily, and then let \(u_1 \in \text{span}(e_j : j \in \mathbb{N})\), with \(\|u_1\|_{q^v} = 1\) and \(\|u_1 - z_1\|_{q^v} < 2^{-4}\). Then let \(k_1 \in \mathbb{N}\) so that \(u_1 \in \text{span}(e_j : j < k_1)\). If we assume that \(z_1, z_2, \ldots, z_n, u_1, u_2, \ldots, u_n\), and \(k_1 < k_2 < \ldots k_n\)
have been chosen we choose \( z_{n+1} \in Z \cap \{ e_1^*, \ldots, e_{k_n}^* \} \perp \) (note that this space is infinite dimensional and a subspace of \( \text{span}(e_j : j > k_{n+1}) \)) and then choose \( u_{n+1} \in \text{span}(e_j : j > k_{n+1}) \), \( \| u_{n+1} \|_{q^v} = 1 \), with \( \| u_{n+1} - z_{n+1} \|_{q^v} < 2^{\frac{1}{2}} 2^{-n-1} \) and let \( k_{n+1} \in \mathbb{N} \) so that \( u_{n+1} \in \text{span}(e_j : j < k_{n+1}) \).

Using the fact that \( (e_n) \) is a monotone and shrinking basis of \( J \) (see Proposition 3.5.2 and Corollary 3.5.4) we can use Proposition 3.3.6 to represent the bidual \( J^{**} \) of \( J \). We will now use the cyclic variation norm.

\[
J^{**} = \left\{ (\xi_n) \subset \mathbb{R} : \sup_{n \in \mathbb{N}} \left\| \sum_{j=1}^{n} \xi_j e_j \right\|_{q^v} < \infty \right\}
\]

and for \( x^{**} = (\xi_n) \in J^{**} \)

\[
\|x^{**}\|_{J^{**}} = \sup_{n \in \mathbb{N}} \| (\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots) \|_{q^v}
\]

\[
= \sup_{l \in \mathbb{N}, k_0 < k_1 < \ldots < k_l} \max \left( \left( \frac{(\xi_{k_0} - \xi_{k_l})^2 + \sum_{j=1}^{l} (\xi_{k_{j-1}} - \xi_{k_j})^2}{2} \right)^{1/2}, \right)
\]

\[
\left( \xi_{k_0}^2 + \xi_{k_l}^2 + \sum_{j=1}^{l} (\xi_{k_{j-1}} - \xi_{k_j})^2 \right)^{1/2}
\]

The second equality in (3.16) can be seen as follows: Fix an \( n \in \mathbb{N} \) and consider

\( x^{(n)} = (\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots) \), thus \( x^{(n)} = (\xi_j^{(n)}) \), with \( \xi_j^{(n)} = \begin{cases} \xi_j & \text{if } j \leq n \\ 0 & \text{else} \end{cases} \).

Now we let \( l \) and \( 1 \leq k_1 < k_2 < \ldots < k_l \) in \( \mathbb{N} \) be chosen so that

\[
\|x^{(n)}\|_{q^v}^2 = \frac{1}{2} \left( (\xi_{k_0}^{(n)} - \xi_{k_l}^{(n)})^2 + \sum_{j=1}^{l} (\xi_{k_{j-1}}^{(n)} - \xi_{k_j}^{(n)})^2 \right).
\]

There are two cases: Either \( k_l \leq n \). In this case \( \xi_{k_j}^{(n)} = \xi_{k_j} \), for all \( j \leq l \), and thus

\[
\|x^{(n)}\|_{q^v}^2 = \left( (\xi_{k_0} - \xi_{k_l})^2 + \sum_{j=1}^{l} (\xi_{k_{j-1}} - \xi_{k_j})^2 \right)^{1/2},
\]

which leads to the first term in above “max”. Or \( k_l > n \). Then we can assume without loss of generality that \( k_{l-1} \leq n \) (otherwise we can drop
3.5. JAMES’ SPACE

$k_{l-1}$ and we note that $\xi_{k_l}^{(n)} = 0$, while $\xi_{k_j}^{(n)} = \xi_{k_j}$ for all $j \leq l - 1$, and thus

$$\|x^{(n)}\|_{cqv}^2 = \frac{1}{2} \left( \xi_{k_0}^2 + \sum_{j=1}^{l} (\xi_{k_{j-1}} - \xi_{k_j})^2 \right)^{1/2} = \left( \xi_{k_0}^2 + \xi_{k_{l-1}}^2 + \sum_{j=1}^{l-1} (\xi_{k_{j-1}} - \xi_{k_j})^2 \right)^{1/2},$$

which, after renaming $l - 1$ to be $l$, leads to the second term above “max”.

**Remark.** Note that there is a difference between

$$\sup_{n \in \mathbb{N}} \| (\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots) \|_{cqv}$$

and

$$\sup_{n \in \mathbb{N}} \| (\xi_1, \xi_2, \ldots, \xi_n) \|_{cqv}$$

and there is only equality if $\lim_{n \to \infty} \xi_n = 0$.

It follows that for all $x^{**} = (\xi_n) \in J^{**}$, that $e_{\infty}^*(x) = \lim_{n \to \infty} \xi_n$ exists, that $(1, 1, 1, 1, \ldots) \in J^{**} \setminus J$, and that

$$x^{**} - e_{\infty}^*(x)(1, 1, 1, \ldots) \in J.$$

**Theorem 3.5.8.** $J$ is not reflexive, does not contain an isomorphic copy of $c_0$ or $\ell_1$ and the codimension of $J$ in $J^{**}$ is 1.

**Proof.** We only need to observe that it follows from the above that

$$J^{**} = \left\{ (\xi_j) \subset \mathbb{R} : \| (\xi_j) \|_{cqv} < \infty \right\}$$

$$= \frac{1}{\sqrt{2}} (\xi_j + \xi_{\infty}(1, 1, 1 \ldots) \subset \mathbb{R} : \| (\xi_j) \|_{cqv} < \infty \lim_{j \to \infty} \xi_j = 0 \xi_{\infty} \in \mathbb{R} \right\},$$

where the second equality follows from the fact that if $(\xi_n)$ has finite quadratic variation then $\lim_{j \to \infty} \xi_j$ exists.

It follows therefore from Theorem 3.4.7

**Corollary 3.5.9.** $J$ does not have an unconditional basis.

**Theorem 3.5.10.** The operator

$$T : J^{**} \to J, \quad x^{**} = (\xi_j) \mapsto (\eta_j) = (-e_{\infty}^*(x^{**}), \xi_1-e_{\infty}^*(x^{**}), \xi_2-e_{\infty}^*(x^{**}), \ldots)$$

is an isometry between $J^{**}$ and $J$ with respect to the cyclic quadratic variation.
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Proof. Let \( x^{**} = (\xi_j) \in J^{**} \) and
\[
    z = (\eta_j) = (-e^*_\infty(x^{**}), \xi_1 - e^*_\infty(x^{**}), \xi_2 - e^*_\infty(x^{**}), \ldots.
\]

By (3.16)
\[
    \sqrt{2}||x^{**}||
\]
\[
= \sup_{l \in \mathbb{N}, k_0 < k_1 < \ldots < k_l} \max \left( \left( \xi_{k_0} - \xi_{k_1} \right)^2 + \sum_{j=1}^{l} \left( \xi_{k_j} - \xi_{k_{j-1}} \right)^2 \right)^{1/2},
\]
\[
    \left( \xi_{k_0}^2 + \xi_{k_l}^2 + \sum_{j=1}^{l} \left( \xi_{k_j} - \xi_{k_{j-1}} \right)^2 \right)^{1/2}
\]
\[
= \sup_{l \in \mathbb{N}, k_0 < k_1 < \ldots < k_l} \max \left( \left( \eta_{k_0+1} - \eta_{k_l+1} \right)^2 + \sum_{j=1}^{l} \left( \eta_{k_j+1} - \eta_{k_{j-1}+1} \right)^2 \right)^{1/2},
\]
\[
    \left( \eta_{k_0+1}^2 + \eta_{k_l+1}^2 + \sum_{j=1}^{l} \left( \eta_{k_j+1} - \eta_{k_{j-1}+1} \right)^2 \right)^{1/2}
\]
\[
= \sup_{l \in \mathbb{N}, k_0 < k_1 < \ldots < k_l} \max \left( \left( \eta_{k_0+1} - \eta_{k_l+1} \right)^2 + \sum_{j=1}^{l} \left( \eta_{k_j+1} - \eta_{k_{j-1}+1} \right)^2 \right)^{1/2},
\]
\[
    \left( \eta_{k_0+1}^2 + \eta_{k_l+1}^2 + \sum_{j=1}^{l} \left( \eta_{k_j+1} - \eta_{k_{j-1}+1} \right)^2 \right)^{1/2}
\]
\[
= \max \left( \sup_{l \in \mathbb{N}, 1 < k_0 < k_1 < \ldots < k_l} \left( \left( \eta_{k_0} - \eta_{k_l} \right)^2 + \sum_{j=1}^{l} \left( \eta_{k_j} - \eta_{k_{j-1}} \right)^2 \right)^{1/2},
\]
\[
    \sup_{l \in \mathbb{N}, 1 = k_0 < k_1 < \ldots < k_l} \left( \left( \eta_{k_0} - \eta_{k_l} \right)^2 + \sum_{j=1}^{l} \left( \eta_{k_j} - \eta_{k_{j-1}} \right)^2 \right)^{1/2}
\]

(For the first part we rename \( k_j + 1 \) to be \( k_j \), for the second part, we rename 1 to be \( k_0 \), \( k_0 + 1 \) to be \( k_1 \), \ldots, and \( k_l + 1 \) to be \( k_{l+1} \), and then we rename \( l + 1 \) to be \( l \))

\[
= \sup_{l \in \mathbb{N}, k_0 < k_1 < \ldots < k_l} \left( \left( \eta_{k_0} - \eta_{k_l} \right)^2 + \sum_{j=1}^{l} \left( \eta_{k_j} - \eta_{k_{j-1}} \right)^2 \right)^{1/2},
\]
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\[ = \sqrt{2\|z\|_{qv}}. \]

Since \( T \) is surjective this implies the claim.

### Exercises

1. Define the *James Function space* as

\[ JF = \{ f \in C[0, 1] : f(0) = 0 \text{ and } \|f\|_{qv} < \infty \}, \]

where

\[ \|f\|_{qv} = \sup_{t_0 < t_1 < t_2 < \ldots < t_l} \left( \sum_{j=1}^{l} \left| f(t_j) - f(t_{j-1}) \right|^2 \right)^{1/2}. \]

Show that \( \| \cdot \|_{qv} \) is a norm on \( JF \), and that \( (JF, \| \cdot \|_{qv}) \) is a Banach space.

2. Show that the unit vector basis in \( J \) is also monotone with respect to the cyclic quadratic variation \( \| \cdot \|_{cqv} \).

3. *(The Gliding Hump Argument)* Assume that \( Y \) is an infinite dimensional subspace of a Banach space \( X \) with a basis \( (e_i) \) and \( \varepsilon > 0 \). Show that there is an infinite dimensional subspace \( Z \) of \( Y \) which has a normalized basis \( (z_n) \), which is \((1 + \varepsilon)\)-equivalent to a block basis of \( (e_i) \).

4. *Show that \( J \) has a boundedly complete basis.*