3.4 Unconditional bases

As shown in Exercise 2 of Section 3.2 there are basic sequences which are not longer basic sequences if one reorders them. Unconditional bases are defined to be bases which are bases no matter how one reorders them.

We will first observe the following result on unconditionally converging series.

**Theorem 3.4.1.** For a sequence \((x_n)\) in Banach space \(X\) the following statements are equivalent.

1. For any reordering (also called permutation) \(\sigma\) of \(\mathbb{N}\) (i.e. \(\sigma : \mathbb{N} \to \mathbb{N}\) is bijective) the series \(\sum_{n \in \mathbb{N}} x_{\sigma(n)}\) converges.
2. For any \(\varepsilon > 0\) there is an \(n \in \mathbb{N}\) so that whenever \(M \subset \mathbb{N}\) is finite with \(\min(M) > n\), then \(\|\sum_{n \in M} x_n\| < \varepsilon\).
3. For any subsequence \((n_j)\) the series \(\sum_{j \in \mathbb{N}} x_{n_j}\) converges.
4. For sequence \((\varepsilon_j) \subset \{\pm 1\}\) the series \(\sum_{j=1}^{\infty} \varepsilon_j x_{n_j}\) converges.

In the case that above conditions hold we say that the series \(\sum x_n\) converges unconditionally.

**Proof.** “(a) \(\Rightarrow\) (b)” Assume that (b) is false. Then there is an \(\varepsilon > 0\) and for every \(n \in \mathbb{N}\) there is a finite set \(M \subset \mathbb{N}\), \(n < \min(M)\), so that \(\|\sum_{j \in M} x_j\| \geq \varepsilon\).

We can therefore, recursively choose finite subsets of \(\mathbb{N}\), \(M_1, M_2, M_3\) etc. so that \(\min(M_{n+1}) > \max(M_n)\) and \(\|\sum_{j \in M_n} x_j\| \geq \varepsilon\), for \(n \in \mathbb{N}\). Now consider a bijection \(\sigma : \mathbb{N} \to \mathbb{N}\), which on each interval of the form \([\max(M_{n-1} + 1, \max(M_n))\) (with \(M_0 = 0\)) is as follows: The interval \([\max(M_{n-1} + 1, \max(M_n) + \#M_n)\) will be mapped to \(M_n\), and \([\max(M_{n-1} + \#M_n, \max(M_n)\) will be mapped to \([\max(M_{n-1} + 1, \max(M_n)\) \(\setminus M_n\). It follows then for each \(n \in \mathbb{N}\) that

\[
\left\| \sum_{j=\max(M_{n-1} + 1, \max(M_n))}^{\max(M_{n-1} + \#M_n)} x_{\sigma(j)} \right\| = \left\| \sum_{j \in M_n} x_j \right\| \geq \varepsilon,
\]

and, thus, the series \(\sum x_{\sigma(n)}\) cannot be convergent, which is a contradiction.

“(b) \(\Rightarrow\) (c)” Let \((n_j)\) be a subsequence of \(\mathbb{N}\). For a given \(\varepsilon > 0\), use condition (b) and choose \(n \in \mathbb{N}\), so that \(\|\sum_{j \in M} x_j\| < \varepsilon\), whenever \(M \subset \mathbb{N}\) is finite and \(\min M > n\). This implies that for all \(i_0 \leq i < j\), with \(i_0 = \min\{s : n_s > n\}\), it follows that \(\|\sum_{s=i}^{j-1} x_{n_s}\| < \varepsilon\). Since \(\varepsilon > 0\) was arbitrary this means that the sequence \((\sum_{s=1}^{j} x_{n_s})_{j \in \mathbb{N}}\) is Cauchy and thus convergent.
“(c)⇒(d)” If \((\varepsilon_n)\) is a sequence of \(\pm 1\)’s, let \(N^+ = \{n \in \mathbb{N} : \varepsilon_n = 1\}\) and \(N^- = \{n \in \mathbb{N} : \varepsilon_n = -1\}\). Since

\[
\sum_{j=1}^{n} \varepsilon_j x_j = \sum_{j \in N^+, j \leq n} x_j - \sum_{j \in N^-, j \leq n} x_j, \text{ for } n \in \mathbb{N},
\]

and since \(\sum_{j \in N^+, j \leq n} x_j\) and \(\sum_{j \in N^-, j \leq n} x_j\) converge by (d), it follows that \(\sum_{j=1}^{n} \varepsilon_j x_j\) converges.

“(d)⇒(b)” Assume that (b) is false. Then there is an \(\varepsilon > 0\) and for every \(n \in \mathbb{N}\) there is a finite set \(M \subset \mathbb{N}\), \(n < \min M\), so that \(\|\sum_{j \in M} x_j\| \geq \varepsilon\). As above choose finite subsets of \(\mathbb{N}\), \(M_1, M_2, M_3\) etc. so that \(\min M_{n+1} > \max M_n\) and \(\|\sum_{j \in M_n} x_j\| \geq \varepsilon\), for \(n \in \mathbb{N}\). Assign \(\varepsilon_n = 1\) if \(n \in \bigcup_{k \in \mathbb{N}} M_k\) and \(\varepsilon_n = -1\), otherwise.

Note that the series \(\sum_{n=1}^{\infty} (1 + \varepsilon_n)x_n\) cannot converge because

\[
\sum_{j=1}^{k} \sum_{i \in M_j} x_j = \frac{1}{2} \sum_{n=1}^{\max M_n} (1 + \varepsilon_n)x_n, \text{ for } k \in \mathbb{N}.
\]

Thus at least one of the series \(\sum_{n=1}^{\infty} x_n\) and \(\sum_{n=1}^{\infty} \varepsilon_n x_n\) cannot converge.

“(b)⇒(a)” Assume that \(\sigma : \mathbb{N} \rightarrow \mathbb{N}\) is a permutation for which \(\sum x_{\sigma(j)}\) is not convergent. Then we can find an \(\varepsilon > 0\) and \(0 = k_0 < k_1 < k_2 < \ldots \) so that

\[
\left\| \sum_{j=k_{n-1}+1}^{k_n} x_{\sigma(j)} \right\| \geq \varepsilon.
\]

Then choose \(M_1 = \{\sigma(1), \ldots, \sigma(k_1)\}\) and if \(M_1 < M_2 < \ldots M_n\) have been chosen with \(\min M_{j+1} > \max M_j\) and \(\|\sum_{i \in M_j} x_i\| \geq \varepsilon\), if \(i = 1, 2, \ldots, n\), choose \(m \in \mathbb{N}\) so that \(\sigma(j) > \max M_n\) for all \(j > k_m\) (we are using the fact that for any permutation \(\sigma\), \(\lim_{j \to \infty} \sigma(j) = \infty\)) and let

\[
M_{n+1} = \{\sigma(k_m+1), \sigma(k_m+2), \ldots, \sigma(k_{m+1})\},
\]

then \(\min(M_{n+1}) > \max M_n\) and \(\|\sum_{i \in M_j} x_i\| \geq \varepsilon\). It follows that (b) is not satisfied.

**Proposition 3.4.2.** In case that the series \(\sum x_n\) is unconditionally converging, then \(\sum x_{\sigma(j)} = \sum x_j\) for every permutation \(\sigma : \mathbb{N} \to \mathbb{N}\).

**Definition 3.4.3.** A basis \((e_j)\) for a Banach space \(X\) is called unconditionally, if for every \(x \in X\) the expansion \(x = \sum \langle e_j^*, x \rangle e_j\) converges unconditionally, where \((e_j^*)\) are coordinate functionals of \((e_j)\).

A sequence \((x_n) \subset X\) is called unconditionally basic sequence if \((x_n)\) is an unconditional basis of \(\text{span}(a_j : j \in \mathbb{N})\).
Proposition 3.4.4. For a sequence of non zero elements \((x_j)\) in a Banach space \(X\) the following are equivalent.

a) \((a_j)\) is an unconditional basic sequence,

b) There is a constant \(C\), so that for all \(n \in \mathbb{N}\), all \(A \subset \{1, 2, \ldots, n\}\) and all scalars \((a_j)_{j=1}^n \subset \mathbb{K}\),

\[
\left\| \sum_{j \in A} a_j x_j \right\| \leq C \left\| \sum_{j=1}^n a_j x_j \right\|.
\]  

(3.8)

c) There is a constant \(C'\), so that for all \(n \in \mathbb{N}\), all \((\varepsilon_j)_{j=1}^n \subset \{\pm 1\}\) and all scalars \((a_j)_{j=1}^n \subset \mathbb{K}\),

\[
\left\| \sum_{j=1}^n \varepsilon_j a_j x_j \right\| \leq C' \left\| \sum_{j=1}^n a_j x_j \right\|.
\]  

(3.9)

In that case we call the smallest constant \(C = K_u\) which satisfies (3.8) the supression-unconditional constant of \((x_n)\) for all \(n, A \subset \{1, 2, \ldots, n\}\) and all scalars \((a_j)_{j=1}^n \subset \mathbb{K}\) and we call the smallest constant \(C' = K_u\) so that (3.9) holds for all \(n, (\varepsilon_j)_{j=1}^n \subset \{\pm 1\}\) and all scalars \((a_j)_{j=1}^n \subset \mathbb{K}\) the unconditional constant of \((x_n)\).

Moreover, it follows

\[
K_s \leq K_u \leq 2K_s.
\]  

(3.10)

Proof. “\((a) \Rightarrow (b)\)” Assume that \((b)\) does not hold. We can assume that \((x_n)\) is a basic sequence with constant \(b\). Then (see Exercise 4) we choose recursively \(k_0 < k_1 < k_2, \ldots, A_n \subset \{k_{n-1} + 1, k_{n-1} + 1, \ldots k_n\}\), and scalars \((a_j)_{j=k_{n-1}+1}^{k_n}\) so that

\[
\left\| \sum_{j \in A_n} a_j x_j \right\| \geq 1 \text{ and } \left\| \sum_{j=k_{n-1}}^{k_n} a_j x_j \right\| \leq \frac{1}{n^2} \text{ for all } n \in \mathbb{N}.
\]

For any \(l < m\), we can choose \(i \leq j\) so that \(k_{i-1} < l \leq k_i\) and \(k_{j-1} < m \leq k_j\), and thus

\[
\left\| \sum_{s=l}^m a_s x_s \right\| \leq \left\| \sum_{s=l}^{k_i} a_s x_s \right\| \left\| \sum_{t=i+1}^{k_l} x_t \right\| \left\| \sum_{s=k_{l-1}+1}^{k_{l-1}} x_s \right\| + \left\| \sum_{s=k_{j-1}+1}^m x_s \right\|.
\]
where the second term is defined to be 0, if \( i \geq j - 1 \)

\[
\leq \frac{2b}{(i-1)^2} + \sum_{t=i+1}^{j-1} \frac{1}{t^2} + \frac{2b}{(j-1)^2}
\]

It follows therefore that \( x = \sum_{j=1}^{\infty} a_j x_j \) converges, but by Theorem 3.4.1 (b) it is not unconditionally.

“(b) \iff (c)” and (3.10) follows from the following estimates for \( n \in \mathbb{N} \), \( (a_j)_{j=1}^{n} \subset \mathbb{K} \), \( A \subset \{1, 2, \ldots, n\} \) and \( (\varepsilon_j)_{j=1}^{n} \subset \{\pm\} \)

\[
\left\| \sum_{j=1}^{n} \varepsilon_j a_j x_j \right\| \leq \left\| \sum_{j=1, \varepsilon_j=1}^{n} a_j x_j \right\| + \left\| \sum_{j=1, \varepsilon_j=-1}^{n} a_j x_j \right\|
\]

It follows that \( x = \sum_{j=1}^{\infty} a_j x_j \) converges, but by Theorem 3.4.1 (b) it is not unconditionally converging.

“(b) \Rightarrow (a)” First, note that (b) implies by Proposition 3.1.9 that \( \{x_n\} \) is a basic sequence. Now assume that for some \( x = \sum_{j=1}^{\infty} a_j x_j \in \text{span}(e_j : j \in \mathbb{N}) \) is converging but not unconditionally converging. It follows from the equivalences in Theorem 3.4.1 that there is some \( \varepsilon > 0 \) and of \( \mathbb{N} \), \( M_1 \), \( M_2 \), \( M_3 \) etc. so that \( \min M_{n+1} > \max M_n \) and \( \| \sum_{j \in M_n} a_j x_j \| \geq \varepsilon \), for \( n \in \mathbb{N} \). On the other hand it follows from the convergence of the series \( \sum_{j=1}^{\infty} a_j x_j \) that

\[
\limsup_{n \to \infty} \left\| \sum_{j=1}^{\max(M_n)} a_j x_j \right\| = 0,
\]

and thus

\[
\sup_{n \to \infty} \left\| \sum_{j=1}^{\max(M_n)} a_j x_j \right\| = \infty,
\]

which is a contradiction to condition (b).

\( \Box \)

**Proposition 3.4.5.** Assume that \( X \) is a Banach space over the field \( \mathbb{C} \) with an unconditional basis \( (e_n) \), then it follows if \( \sum_{j=1}^{\infty} \alpha_n e_n \) is convergent and \( \{\beta_n \subset \mathbb{C} : |\beta| = 1\} \) that \( \sum_{j=1}^{\infty} \beta_n \alpha_n e_n \) is also converging and

\[
\left\| \sum_{n \in \mathbb{N}} \beta_n \alpha_n e_n \right\| \leq 2K \left\| \sum_{n \in \mathbb{N}} \alpha_n e_n \right\|.
\]

**Proof.** See Exercise 1.  \( \Box \)
Proposition 3.4.6. If $X$ is a Banach space with an unconditional basis, then the coordinate functionals $(e^*_n)$ are also a unconditional basic sequence, with the same unconditional constant and the same suppression-unconditional constant.

Proof. Let $K_u$ and $K_s$ the unconditional and suppression unconditional constant of $X$.

Let $x^* = \sum_{n \in \mathbb{N}} \eta_n e^*_n$ and $(\varepsilon_n) \subset \{\pm 1\}$ then

$$\left\| \sum_{n \in \mathbb{N}} \varepsilon_n \eta_n e^*_n \right\|_{X^*} = \sup_{x = \sum_{n=1}^{\infty} \xi_n e_n \in B_X} \left\langle \sum_{n \in \mathbb{N}} \varepsilon_n \eta_n e^*_n, \sum_{n=1}^{\infty} \xi_n e_n \right\rangle$$

$$= \sup_{x = \sum_{n=1}^{\infty} \xi_n e_n \in B_X} \sum_{n \in \mathbb{N}} \varepsilon_n \eta_n \xi_n$$

$$= \sup_{x = \sum_{n=1}^{\infty} \xi_n e_n \in B_X} \left\| \sum_{n \in \mathbb{N}} \eta_n e^*_n \right\| \left\| \sum_{n \in \mathbb{N}} \xi_n e_n \right\| \leq K_u \left\| \sum_{n \in \mathbb{N}} \eta_n e^*_n \right\|.$$

Using the Hahn Banach Theorem we can similarly show that if $K_u^*$ is the unconditional constant of $(e^*_n)$ then

$$\left\| \sum_{n \in \mathbb{N}} \xi_n e_n \right\|_X \leq K^*_u \leq \left\| \sum_{n \in \mathbb{N}} \xi_n e_n \right\|_X.$$

Thus $K_u = K_u^*$. A similar argument works to show that $K_s$ is equal to the suppression unconditional constant of $(e^*_n)$. \qed

The following Theorem about spaces with unconditional basic sequences was shown By James [Ja]

Theorem 3.4.7. Let $X$ be a Banach space with an unconditional basis $(e_j)$. Then either $X$ contains a copy of $c_0$, or a copy of $\ell_1$ or $X$ is reflexive.

We will need first the following Lemma (See Exercise 2)

Lemma 3.4.8. Let $X$ be a Banach space with an unconditional basis $(e_n)$ and let $K_u$ its constant of unconditionality. Then it follows for any converging series $\sum_{n \in \mathbb{N}} a_n x_n$ and a bounded sequence of scalars $(b_n) \subset \mathbb{K}$, that $\sum_{n \in \mathbb{N}} a_n b_n x_n$ is also converging and

$$\left\| \sum_{n \in \mathbb{N}} a_n b_n e_n \right\| \leq K \sup_{n \in \mathbb{N}} |b_n| \left\| \sum_{n=1}^{\infty} a_n e_n \right\|,$$

where $K = K_u$, if $\mathbb{K} = \mathbb{R}$, and $K = 2K_u$, if $\mathbb{K} = \mathbb{C}$.
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Proof of Theorem 3.4.7. We will prove the following two statements for a space \( X \) with unconditional basis \( (e_n) \).

**Claim 1:** If \( (e_n) \) is not boundedly complete then \( X \) contains a copy of \( c_0 \).

**Claim 2:** If \( (e_n) \) is not shrinking then \( X \) contains a copy of \( \ell_1 \).

Together with Theorem 3.3.9, this yields the statement of Theorem 3.4.7.

Let \( K_u \) be the constant of unconditionality of \( (e_n) \) and let \( K_u^* = K_u \), if \( K = \mathbb{R} \), and \( K_u^* = 2K_u \), if \( K = \mathbb{C} \).

**Proof of Claim 1:** If \( (e_n) \) is not boundedly complete, there is by Theorem 3.4.8 a sequence of scalars \( (a_n) \) such that

\[
\sup_{n \in \mathbb{N}} \left\| \sum_{j=1}^{n} a_j e_j \right\| = C_1 < \infty, \quad \text{but} \quad \sum_{j=1}^{\infty} a_j e_j \text{ does not converge.}
\]

This implies that there is an \( \varepsilon > 0 \) and a sequence \( n_1 < n_2 < \ldots \) in \( \mathbb{N} \) so that if we put \( y_k = \sum_{j=n_{k-1}+1}^{n_k} a_j e_j \), for \( k \in \mathbb{N} \), it follows that \( \|y_k\| \geq \varepsilon \), and also

\[
\|y_k\| \leq \left( \sum_{j=1}^{n_k} |a_j| \right) + \left( \sum_{j=1}^{n_k} a_j e_j \right) \leq 2C_1.
\]

For any \( n \in \mathbb{N} \) and any sequence of scalars \( \lambda_j \) it follows therefore from Lemma 3.4.8, that

\[
\| \sum_{j=1}^{n} \lambda_j y_j \| \leq 2K_u \max_{j \leq n} |\lambda_j| \left\| \sum_{j=1}^{n} y_j \right\| \leq 4K_u C_1 \sup_{j \leq n} |\lambda_j|.
\]

Since the supression-unconditional constant does not exceed the unconditional constant, it follows on the other hand for every \( j_0 \leq n \) that

\[
\| \sum_{j=1}^{n} \lambda_j y_j \| \geq \frac{1}{K_u} \| \lambda_{j_0} y_{j_0} \| \geq \frac{\varepsilon}{K_u} \max_{j \leq n} |\lambda_j|.
\]

Letting \( c = \varepsilon/K_u \) and \( C = 4K_u C_1 \), it follows therefore for any \( n \in \mathbb{N} \) and any sequence of scalars \( \lambda_j \) that

\[
c \left\| \lambda_j \right\| c_0 \leq \left\| \sum_{j=1}^{n} \lambda_j y_j \right\| \leq C \left\| \lambda_j \right\| c_0,
\]

which means that \( (y_j) \) and the unit vector basis of \( c_0 \) are isomorphically equivalent.
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**Proof of Claim 2.** \((e_n)\) is not shrinking then there is by Theorem 3.3.4 a bounded block basis \((y_n)\) of \((e_n)\) which is not weakly null. After passing to a subsequence we can can assume that there is a \(x^* \in X^*\), \(\|x^*\| = 1\), so that

\[
\varepsilon = \inf_{n \in \mathbb{N}} |\langle x^*, y_n \rangle| > 0.
\]

We also can assume that \(\|y_n\| = 1\), for \(n \in \mathbb{N}\) (otherwise replace \(y_n\) by \(y_n/\|y_n\|\) and change \(\varepsilon\) accordingly).

We claim that \((y_n)\) is isomorphically equivalent to the unit vector basis of \(\ell_1\). Let \(n \in \mathbb{N}\) and \((a_j)_{j=1}^n \subset K\). By the triangle inequality we have

\[
\left\| \sum_{j=1}^n a_j y_j \right\| \leq \sum_{j=1}^n |a_j|,
\]

On the other hand we can choose for \(j = 1, 2 \ldots n\) \(\varepsilon_j = \text{sign}(a_j \langle x^*, y_j \rangle)\) if \(K = \mathbb{R}\) and \(\varepsilon_j = a_j \langle x^*, y_j \rangle / |a_j \langle x^*, y_j \rangle|\), if \(K = \mathbb{C}\) (if \(a_j = 0\), simply let \(\varepsilon_j = 1\)) and deduce from Lemma 3.4.8

\[
\left\| \sum_{j=1}^n a_j y_j \right\| \geq \frac{1}{K^2} \left\| \sum_{j=1}^n \varepsilon_j a_j y_j \right\| \geq \left\| \sum_{j=1}^n \varepsilon_j a_j \langle y_j, x^* \rangle \right\| \geq \varepsilon \sum_{j=1}^n |a_j|,
\]

which implies that \((y_n)\) is isomorphically equivalent to the unit vector basis of \(\ell_1\).

**Remark.** It was for long time an open problem whether or not every infinite dimensional Banach space contains an unconditional basis sequence. If this were so, every infinite dimensional Banach space would contain a copy of \(c_0\) or a copy of \(\ell_1\), or has an infinite dimensional reflexive subspace space. In \([GM]\), Gowers and Maurey proved the existence which do not any unconditional basic sequences. Later then Gowers \([Go]\) constructed a space which does not contain any copy of \(c_0\) or \(\ell_1\), and has no infinite dimensional reflexive subspace.

**Exercises.**

1. Prove Proposition 3.4.5.
2. Prove Lemma 3.4.8
3. Show that every separable $X$ Banach space is isomorphic to the quotient space of $\ell_1$.

4. Assume that $(x_n)$ is a basic sequence in a Banach space $X$ for which (b) of Proposition 3.4.4 does not hold. Show that there is a sequence of scalars $(a_j)$ and a subsequence $(k_n)$ of $\mathbb{N}$, so that

$$\left\| \sum_{j \in A_n} a_j x_j \right\| \geq 1$$

and

$$\left\| \sum_{j=k_{n-1}}^{k_n} a_j x_j \right\| \leq \frac{1}{n^2}$$

for all $n \in \mathbb{N}$.

5.* Let $1 < p < \infty$ and assume that $(x_n)$ is a weakly null sequence in $\ell_p$ with $\inf_{n \in \mathbb{N}} \|x_n\| > 0$. Show that $(x_n)$ has a subsequence which is isomorphically equivalent to the unit vector basis of $\ell_p$.

Let $T : \ell_p \to \ell_q$ with $1 < q < p < \infty$, be a bounded linear operator. Show that $T$ is compact, meaning that $T(B_{\ell_p})$ is relatively compact in $\ell_q$. 