Chapter 4

Convexity and Smoothness

4.1 Strict Convexity, Smoothness, and Gateaux Differentiablity

Definition 4.1.1. Let X be a Banach space with a norm denoted by $\|\cdot\|$. A map

$$f : X \setminus \{0\} \to X^* \setminus \{0\}, \quad f \mapsto f_x$$

is called a *support mapping* whenever.

a) $f(\lambda x) = \lambda f_x$, for $\lambda > 0$ and

b) If $x \in S_X$, then $\|f_x\| = 1$ and $f_x(x) = 1$ (and thus $f_x(x) = \|x\|^2$ for all $x \in X$).

Often we only define f_x for $x \in S_X$ and then assume that $f_x = \|x\| f_{x/\|x\|}$, for all $x \in X \setminus \{0\}$.

For $x \in X$ a *support functional of* x is an element $x^* \in X^*$, with $\|x^*\| = \|x\|$ and $\langle x^*, x\langle = \|x\|^2$. Thus a support map is a map $f_{(\cdot)} : X \to X^*$, which assigns to each $x \in X$ a support functional of x.

We say that X is *smooth at* $x_0 \in S_X$ if there exists a unique $f_x \in S_{X^*}$, for which $f_x(x) = 1$, and we say that X is *smooth* if it is smooth at each point of S_X.

The Banach space X is said to have *Gateaux differentiable norm at* $x_0 \in S_X$, if for all $y \in S_X$

$$\rho'(x_0, y) = \lim_{h \to 0} \frac{\|x_0 + hy\| - \|x_0\|}{h}$$

exists, and we say that $\|\cdot\|$ is *Gateaux differentiable* if it is Gateaux differentiable norm at each $x_0 \in S_X$.

Example 4.1.2. For $X = L_p[0,1]$, $1 < p < \infty$ the function

$$f : L_p[0,1] \to L_q[0,1], \quad f_x(t) = \operatorname{sign}(x(t))\left|\frac{x(t)}{\|x\|_p}\right|^{p/q} = \|x\|_p^{1-\frac{p}{q}}|x(t)|^{\frac{p}{q}}$$

is a (and the only) support function for $L_p[0,1]$.

In order establish a relation between Gateaux differentiability and smoothness we observe the following equalities and inequalities for any $x \in X$, $y \in S_X$, and $h > 0$:

$$\begin{aligned}
\frac{f_x(y)}{\|x\|} &= \frac{f_x(hy)}{h\|x\|} \\
&= \frac{f_x(x) - \|x\|^2 + f_x(hy)}{h\|x\|} \\
&= \frac{f_x(x+hy) - \|x\|^2}{h\|x\|} \\
&\leq \frac{|f_x(x+hy)| - \| x\|^2}{h\|x\|} \\
&\leq \frac{\|f_x\|\|x+hy\| - \|x\|^2}{h\|x\|} \\
&= \frac{\|x+hy\| - \|x\|}{h} \\
&= \frac{\|x+hy\|^2 - \|x+hy\|\|x\|}{h\|x+hy\|} \\
&\leq \frac{\|x+hy\|^2 - |f_{x+hy}(x)|}{h\|x+hy\|} \\
&= \frac{f_{x+hy}(x+hy) - |f_{x+hy}(x)|}{h\|x+hy\|} \\
&= \frac{hf_{x+hy}(y) + f_{x+hy}(x) - |f_{x+hy}(x)|}{h\|x+hy\|} \\
&\leq \frac{hf_{x+hy}(y)}{h\|x+hy\|} = \frac{f_{x+hy}(y)}{\|x+hy\|}
\end{aligned}$$

and thus for any $x \in X$,$y \in S_X$, and $h > 0$:

$$\frac{f_x(y)}{\|x\|} \leq \frac{|f_x(x+hy)| - \| x\|}{h\|x\|} \leq \frac{\|x+hy\| - \|x\|}{h} \leq \frac{f_{x+hy}(y)}{\|x+hy\|}. \tag{4.1}$$

Theorem 4.1.3. *Assume X is a Banach space and $x_0 \in S_X$ The following statements are equivalent:*

a) X is smooth at x_0.

b) Every support mapping $f : x \mapsto f_x$ is norm to w^* continuous from S_X to S_{X^*} at the point x_0.

c) There exists a support mapping $f_{(\cdot)} : x \mapsto f_x$ which is norm to w^* continuous from S_X to S_{X^*} at the point x_0.

d) The norm is Gateaux differentiable at x_0.

In that case

$$f_x(y) = \rho'(x_0, y) = \lim_{h\to 0} \frac{\|x_0 + hy\| - \|x_0\|}{h} \text{ for all } y \in S_X.$$

Proof. $\neg$(b) $\Rightarrow$ $\neg$(a). Assume that $(x_n) \subset S_X$ is a net, which converges in norm to x_0, but for which f_{x_n} does not converge in w^* to f_{x_0}, where $f_{(\cdot)} : X \to X^*$ is the support map. We can assume that there is a w^* neighborhood U of f_{x_0}, not containing any of the f_{x_n}, and by Alaoglu's Theorem 2.1.8 we can assume that f_{x_n} has an accumulation point $x^* \in U$, which cannot be equal to f_{x_0}.

As

$$\begin{aligned}
&|x^*(x_0) - 1| \\
&\quad = |x^*(x_0) - f_{x_n}(x_n)| \\
&\quad \le |x^*(x_0) - f_{x_n}(x_0)| + |f_{x_n}(x_0 - x_n)| \\
&\quad \le |x^*(x_0) - f_{x_n}(x_0)| + \|x_0 - x_n\| \to_{n\in M, n\to\infty} 0, \text{ for some infinite } M \subset \mathbb{N}
\end{aligned}$$

it follows that $x^*(x_0) = 1$, and since $\|x^*\| \le 1$ we must have $\|x^*\| = 1$. Since $x^* \neq f_{x_0}$, X cannot be smooth at x_0.

(b) $\Rightarrow$ (c) is clear (since by The Theorem of Hahn Banach there is always at least one support map).

(c) $\Rightarrow$ (d) Follows from (4.1), and from the fact that (4.1) implies for $x \in X$, $y \in \S_X$ and $h > 0$, that

$$\frac{\|x - hy\| - \|x\|}{-h\|x\|} = -\frac{\|x + h(-y)\| - \|x\|}{h\|x\|} \ge -\frac{f_x(-y)}{\|x\|} = f_x(y)$$

and

$$\frac{\|x - hy\| - \|x\|}{-h\|x\|} = -\frac{\|x + h(-y)\| - \|x\|}{h\|x\|} \le -\frac{f_{x+h(-y)}(-y)}{\|x + h(-y)\|} = \frac{f_{x+h(-y)}(y)}{\|x + h(-y)\|}.$$

(d) $\Rightarrow$ (a) Let $f \in S_{x^*}$ be such that $f(x) = \|x\| = 1$. Since (4.1) is true for any support function it follows that

$$f(y) \le \frac{\|x_0 + hy\| - \|x_0\|}{h}, \text{ for all } y \in S_X \text{ and } h > 0,$$

and

$$\frac{\|x_0 - hy\| - \|x_0\|}{-h} = -\frac{\|x_0 + (-y)\| - \|x_0\|}{h} \le -f(-y) = f(y)$$
$$\text{for all } y \in S_X \text{ and } h < 0.$$

Thus, by assumption (d), $\rho'(x_0, y) = f(y)$, which proves the uniqueness of $f \in S_{X^*}$ with $f(x_0) = 1$. □

Definition 4.1.4. A Banach space X with norm $\|\cdot\|$ is called *strictly convex* whenever $S(X)$ contains no non-trivial line segement, i.e. if for all $x, y \in S_X$, $x \ne y$ it follows that $\|x + y\| < 2$.

Theorem 4.1.5. *If X^* is strictly convex then X is smooth, and if X^* is smooth the X is strictly convex.*

Proof. If X is not smooth then there exists an $x_0 \in S_X$, and two functionals $x^* \ne y^*$ in S_{X^*} with $x^*(x_0) = y^*(x_0) = 1$ but this means that

$$\|x^* + y^*\| \ge (x^* + y^*)(x_0) = 2,$$

which implies that X^* is not strictly convex. If X is not strictly convex then there exist $x \ne y$ in S_X so that $\|\lambda x + (1-\lambda)y\| = 1$, for all $0 \le \lambda \le 1$. So let $x^* \in S_{X^*}$ such that

$$x^*\Big(\frac{x+y}{2}\Big) = 1.$$

But this implies that

$$1 = x^*\Big(\frac{x+y}{2}\Big) = \frac{1}{2}x^*(x) + \frac{1}{2}x^*(y) \le \frac{1}{2} + \frac{1}{2} = 1,$$

which implies that $x^*(x) = x^*(y) = 1$, which by viewing x and y to be elements in X^{**}, implies that X^* is not smooth. □

Exercises

1. Show that ℓ_1 admits an equivalent norm $\||\cdot\||$ which is strictly convex and $(\ell_1, \||\cdot\||)$ is (isometrically) the dual of c_0 with some equivalent norm.

2. Assume that $T : X \to Y$ is a linear, bounded, and injective operator between two Banach spaces and assume that Y is strictly convex. Show that X admits an equivalent norm for which X is strictly convex.

4.2 Uniform Convexity and Uniform Smoothness

Definition 4.2.1. Let X be a Banach space with norm $\|\cdot\|$. We say that *the norm of X is Fréchet differentiable at $x_0 \in S_X$* if

$$\lim_{h\to 0} \frac{\|x_0 + hy\| - \|x_0\|}{h}$$

exists uniformly in $y \in S_X$.

We say that *the norm of X is Fréchet differentiable* if the norm of X is Fréchet differentiable at each $x_0 \in S_X$.

Remark. By Theorem 4.1.3 it follows from the Frechét differentiability of the norm at x_0 that there a unique support functional $f_{x_0} \in S_X^*$ and

$$\lim_{h\to 0} \left| \frac{\|x_0 + hy\| - \|x_0\|}{h} - f_{x_0}(y) \right| = 0,$$

uniformly in y and thus that (put $z = hy$)

$$\lim_{z\to 0} \frac{\|x_0 + z\| - \|x_0\| - f_{x_0}(z)}{\|z\|} = 0.$$

In particular, if X has a Fréchet differentiable norm it follows from Theorem 4.1.3 that there is a unique support map $x \to f_x$.

Proposition 4.2.2. *Let X be a Banach space with norm $\|\cdot\|$. Then the norm is Fréchet differentiable if and only if the support map is norm-norm continuous.*

Proof. (We assume that $\mathbb{K} = \mathbb{R}$) "$\Rightarrow$" Assume that $(x_n) \subset S_X$ converges to x_0 and put $x_n^* = f_{x_n}$, $n \in \mathbb{N}$, and $x_0^* = f_{x_0}$. It follows from Theorem 4.1.3 that $x_n^*(x_0) \to 1$, for $n \to \infty$. Assume that our claim were not true, and we can assume that for some $\varepsilon > 0$ we have $\|x_n^* - x_0^*\| > 2\varepsilon$, and therefore we can choose vectors $z_n \in S_X$, for each $n \in \mathbb{N}$ so that $(x_n^* - x_0^*)(z_n) > 2\varepsilon$. But then

$$\begin{aligned}
x_0^*(x_0) - x_n^*(x_0) &\le \big(x_0^*(x_0) - x_n^*(x_0)\big)\Big(\frac{1}{\varepsilon}\big(\underbrace{x_n^*(z_n) - x_0^*(z_n)}_{>2\varepsilon}\big) - 1\Big) \\
&= \big(x_n^*(x_0) - x_0^*(x_0)\big) + \frac{1}{\varepsilon}\big(x_0^*(z_n) - x_n^*(z_n)\big)\big(x_n^*(x_0) - x_0^*(x_0)\big) \\
&= \big(x_n^* - x_0^*\big)\Big(x_0 + z_n \frac{1}{\varepsilon}\big(x_0^*(x_0) - x_n^*(x_0)\big)\Big)
\end{aligned}$$

$$\leq \Big| x_n^*\Big(x_0 + z_n \frac{1}{\varepsilon}\big(x_0^*(x_0) - x_n^*(x_0)\big)\Big)\Big| - \\ - x_0^*\Big(x_0 + z_n \frac{1}{\varepsilon}\big(x_0^*(x_0) - x_n^*(x_0)\big)\Big)$$

$$\leq \Big\| x_0 + z_n \frac{1}{\varepsilon}\big(x_0^*(x_0) - x_n^*(x_0)\big)\Big\| \\ - \|x_0\| - x_0^*\Big(z_n \frac{1}{\varepsilon}\big(x_0^*(x_0) - x_n^*(x_0)\big)\Big).$$

Thus if we put

$$y_n = z_n \frac{1}{\varepsilon}\big(x_0^*(x_0) - x_n^*(x_0)\big),$$

it follows that $\|y_n\| \to 0$, if $n \to \infty$, and, using the Fréchet differentiability of the norm that (note that $\big(x_0^*(x_0) - x_n^*(x_0)\big)/\|y_n\| = \varepsilon$) we deduce that

$$0 < \varepsilon = \frac{x_0^*(x_0) - x_n^*(x_0)}{\|y_n\|} \leq \frac{\|x_0 + y_n\| - \|x_0\| - x_0^*(y_n)}{\|y_n\|} \to_{n\to\infty} 0,$$

which is a contradiction.
"⇐" From (4.1) it follows that for $x, y \in S_X$, and $h \in \mathbb{R}$

$$\Big|\frac{\|x+hy\| - \|x\|}{h} - f_x(y)\Big|$$

$$\leq \Big|\frac{f_{x+hy}(y)}{\|x+hy\|} - f_x(y)\Big|$$

$$\leq \big|f_{x+hy}(y) - f_x(y)\big| + \Big|\frac{f_{x+hy}(y)}{\|x+hy\|} - f_{x+hy}(y)\Big|$$

$$\leq \|f_{x+hy} - f_x\| + \Big|\frac{1}{1+|h|} - 1\Big|\|f_{x+hy}\|,$$

which converges uniformly in y to 0 and proves our claim. □

Definition 4.2.3. Let X be a Banach space with norm $\|\cdot\|$.

We say that the norm is *uniformly Fréchet differentiable on* S_X if

$$\lim_{h\to 0}\Big|\frac{\|x+hy\| - \|x\|}{h} - f_x(y)\Big|,$$

uniformly in $x \in S_X$ and $y \in S_X$. In other words if for all $\varepsilon > 0$ there is a $\delta > 0$ so that for all $x, y \in S_X$ and all $h \in \mathbb{R}$, $0 < |h| < \delta$

$$\Big|\frac{\|x+hy\| - \|x\|}{h} - f_x(y)\Big| < \varepsilon.$$

X is *uniformly convex* if for all $\varepsilon > 0$ there is a $\delta > 0$ so that for all $x, y \in S_X$ with $\|x - y\| \geq \varepsilon$ it follows that $\|(x + y)/2\| < 1 - \delta$.

We call

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in S_X, \|x - y\| \geq \varepsilon\right\}, \text{ for } \varepsilon \in [0, 2]$$

the *modulus of uniform convexity of* X.

X is called *uniform smooth* if for all $\varepsilon > 0$ there exists a $\delta > 0$ so that for all $x, y \in S_X$ and all $h \in (0, \delta]$

$$\|x + hy\| + \|x - hy\| < 2 + \varepsilon h.$$

The *modulus of uniform smoothness of* X is the map $\rho : [0, \infty) \to [0, \infty)$

$$\rho_X(\tau) = \sup\left\{\frac{\|x + z\|}{2} + \frac{\|x - z\|}{2} - 1 : x, z \in X, \|x\| = 1, \|z\| \leq \tau\right\}.$$

Remark. X is uniformly convex if and only if $\delta_X(\varepsilon) > 0$ for all $\varepsilon > 0$. X is uniformly smooth if and only if $\lim_{\tau \to 0} \rho_X(\tau)/\tau = 0$.

Theorem 4.2.4. *For a Banach space X the following statements are equivalent.*

a) *There exists a support map $x \to f_x$ which uniformly continuous on S_X with respect to the norms.*

b) *The norm on X is uniformly Fréchet differentiable on S_X.*

c) *X is uniformly smooth.*

d) *X^* is uniformly convex.*

e) *Every support map $x \to f_x$ is uniformly continuous on S_X with respect to the norms.*

Proof. "(a)⇒(b)" We proceed as in the proof of Proposition 4.2.2. From (4.1) it follows that for $x, y \in S_X$, and $h \in \mathbb{R}$

$$\left|\frac{\|x + hy\| - \|x\|}{h} - f_x(y)\right|$$
$$\leq \left|\frac{f_{x+hy}(y)}{\|x + hy\|} - f_x(y)\right|$$

$$\leq \left|f_{x+hy}(y) - f_x(y)\right| + \left|\frac{f_{x+hy}(y)}{\|x+hy\|} - f_{x+hy}(y)\right|$$
$$\leq \|f_{x+hy} - f_x\| + \left|\frac{1}{1+|h|} - 1\right|\|f_{x+hy}\|$$

which converges by (a) uniformly in x and y, to 0.
"(b)⇒(c)". Assuming (b) we can choose for $\varepsilon > 0$ a $\delta > 0$ so that for all $h \in (0, \delta)$ and all $x, y \in S_X$

$$\left|\frac{\|x+hy\| - \|x\|}{h} - f_x(y)\right| < \varepsilon/2.$$

But this implies that for all $h \in (0, \delta)$ and all $x, y \in S_X$ we have

$$\begin{aligned}
&\|x+hy\| + \|x-hy\| \\
&\quad = 2 + h\left(\frac{\|x+hy\| - \|x\|}{h} - f_x(y) + \left(\frac{\|x+h(-y)\| - \|x\|}{h} - f_x(-y)\right)\right) \\
&\quad \leq 2 + \varepsilon h,
\end{aligned}$$

which implies our claim.
"(c)⇒(d)". Let $\varepsilon > 0$. By (c) we can find $\delta > 0$ such that for all $x \in S_X$ and $z \in X$, with $\|z\| \leq \delta$, we have $\|x+z\| + \|x-z\| \leq 2 + \varepsilon\|z\|/4$.

Let $x^*, y^* \in S_{X^*}$ with $\|x^* - y^*\| \geq \varepsilon$. There is a $z \in X$, $\|z\| \leq \delta/2$ so that $(x^* - y^*)(z) \geq \varepsilon\delta/2$. This implies

$$\begin{aligned}
\|x^* + y^*\| &= \sup_{x \in S_X} \left|(x^* + y^*)(x)\right| \\
&= \sup_{x \in S_X} \left|x^*(x+z) + y^*(x-z) - (x^* - y^*)(z)\right| \\
&\leq \sup_{x \in S_X} \|x+z\| + \|x-z\| - \varepsilon\delta/2 \\
&\leq 2 + \varepsilon\|z\|/4 - \varepsilon\delta/2 < 2 - \varepsilon\delta/4.
\end{aligned}$$

"(d)⇒(e)". Let $x \mapsto f_x$ be a support functional. By (d) we can choose for $\varepsilon >$ a δ so that for all $x^*, y^* \in S_{X^*}$ we have $\|x^* - y^*\| < \varepsilon$, whenever $\|x^* + y^*\| > 2 - \delta$.

Assume now that $x, y \in S_X$ with $\|x - y\| < \delta$. Then

$$\|f_x + f_y\| \geq \frac{1}{2}(f_x + f_y)(x + y)$$

$$= f_x(x) + f_y(y) + \frac{1}{2} f_x(y-x) + \frac{1}{2} f_y(x-y)$$
$$\geq 2 - \|x-y\| \geq 2-\delta,$$

which implies that $\|f_x - f_y\| < \varepsilon$, which proves our claim.
"(e)⇒(a)". Clear. □

Theorem 4.2.5. *Every uniformly convex and every uniformly smooth Banach space is reflexive.*

Proof. Assume that X is uniformly convex, and let $x^{**} \in S_{X^{**}}$. Since B_X is w^*-dense in $B_{X^{**}}$ we can find a net $(x_i)_{i\in I}$ which converges with respect to w^* to x^{**}. Since for every $\eta > 0$ there is a $x^* \in S_{X^*}$ with $\lim_{i\in I} x^*(x_i) = x^{**}(x^*) > 1-\eta$, it follows that $\lim_{i\in I}\|x_i\| = 1$ and we can therefore assume that $\|x_i\| = 1$, $i \in I$. We claim that $\chi(x_i)$ is a Cauchy net with respect to the norm to x^{**}, which would finish our proof.

So let $\varepsilon > 0$ and choose δ so that $\|x+y\| > 2-\delta$ implies that $\|x-y\| < \varepsilon$, for any $x, y \in S_X$. Then choose $x^* \in S_{X^*}$, so that $x^{**}(x^*) > 1-\delta/4$, and finally let $i_0 \in I$ so that $x^* * (x_i) \geq 1-\delta/2$, for all $i \geq i_0$. It follows that

$$\|x_i + x_j\| \geq x^*(x_i + x_j) \geq 2-\delta \text{ whenever } i, j \geq i_0,$$

and thus $\|x_i - x_j\| < \varepsilon$, which verifies our claim.

If X is uniformly smooth it follows from Theorem 4.2.4 that X^* is uniformly convex. The first part yields that X^* is reflexive, which implies that X is reflexive. □

Exercises

1) Show that for there is a constant $c > 0$ so that for all $\varepsilon > 0$,

$$\delta_{\ell_2}(vp) \geq c\varepsilon^2.$$

(Here δ_{ℓ_2} is the modulus of uniform convexity of ℓ_2).

2) Prove that for every $\varepsilon > 0$, $C > 1$ and any $n \in \mathbb{N}$ there is an $N = (n, \varepsilon, C)$ so that the following holds:
If X is an N dimensional space which is C-isomorphic to ℓ_1^N then X has an n-dimensional subspace Y which is $(1+\varepsilon)$ ismorphic to ℓ_1^n.

Hint: prove first the following: Assume that $|||\cdot|||$ is a norm on $\ell_1^{n^2}$ so that

$$\frac{1}{C}\|x\|_1 \leq |||x||| \leq \|x\| < \text{ for all } x \in \ell_1^{n^2},$$

then there is a $|||\cdot|||$-normalized block sequence $(x_1, x_2, \ldots x_n)$ so that :

$$\frac{1}{\sqrt{C}}\sum_{i=1}^{n}|b_i| \leq \Big|\Big|\Big|\sum_{i=1}^{n} b_i x_i\Big|\Big|\Big| \leq \sum_{i=1}^{n}|b_i|.$$

3) Show that $\left(\oplus_{n=1}^{\infty} \ell_1^n\right)_{\ell_2}$ does not admit a norm which is uniformly convex.