5.3 On “Small Subspaces” of $L_p$

By *small subspaces* of $L_p[0,1]$ we usually mean subspaces which are not isomorphic to the whole space. Khintchine’s theorem, says that $L_p[0,1]$, $1 \leq p \leq \infty$ contains isomorphic copies of $\ell_2$, which are complemented if $1 < p < \infty$.

Note that all the arguments below can be made in a general probability space $(\Omega, \Sigma, \mathbb{P})$ on which a Rademacher sequence $(r_i)$ exists, i.e. $(r_i)$ is an independent sequence of random variables for which $\mathbb{P}(r_i = 1) = \mathbb{P}(r_i = -1) = 1/2$.

**Theorem 5.3.1.** [Khintchine’s Theorem]
$L_p[0,1]$, $1 \leq p \leq \infty$ contains a subspace isomorphic to $\ell_2$, if $1 < p < \infty$ $L_p[0,1]$, contains a complemented subspace isomorphic to $\ell_2$.

**Remark.** By Theorem 5.1.1 the conclusion of Theorem 5.3.1 holds for all spaces $L_p(\mu)$, as long as $\mu$ is a measure on some measurable space $(\Omega, \Sigma)$ for which there is in $\Omega' \subset \Omega$, $\Omega' \in \Sigma$ so that $\mu|_{\Omega'}$ is a non zero atomless measure.

**Definition 5.3.2.** The *Rademacher functions* are the functions:

$$r_n : [0,1] \to \mathbb{R}, \quad t \mapsto \text{sign}(\sin(2^n \pi t)),$$

whenever $n \in \mathbb{N}$.

**Lemma 5.3.3.** [Khintchine inequality]
For every $p \in [1, \infty)$ there are numbers $0 < A_p \leq 1 \leq B_p$ so that for any $m \in \mathbb{N}$ and any scalars $(a_i)_{i=1}^m$.

$$A_p \left( \sum_{i=1}^m |a_i|^2 \right)^{1/2} \leq \left\| \sum_{i=1}^m a_i r_i \right\|_{L_p} \leq B_p \left( \sum_{i=1}^m |a_i|^2 \right)^{1/2}.$$

**Proof.** We prove the claim for Banach spaces over the reals. The complex case can be easily deduced (using some worse constants).

Since for $p > r \geq 1$

$$\left\| \sum_{i=1}^m a_i r_i \right\|_{L_p} \geq \left\| \sum_{i=1}^m a_i r_i \right\|_{L_r},$$

it is enough to prove the right hand inequality for all even integers, and then choose $B_p = B_{p'}$ with $p' = 2\lceil p/2 \rceil$, for $1 \leq p < \infty$ and the left hand inequality for $p = 1$, and take $A_p = A_1$. 
5.3. ON “SMALL SUBSPACES” OF $L_P$

We first show the existence of $B_{2k}$ for any $k \in \mathbb{N}$. For scalars $(a_i)_{i=1}^m$ we deduce

$$
\int_0^1 \left( \sum_{i=1}^m a_i r_i(t) \right)^{2k} dt
$$

$$
= \sum_{(\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}_0^m} A(\alpha_1, \alpha_2, \ldots, \alpha_m) a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_m^{\alpha_m} \int_0^1 r_1^{\alpha_1}(t) r_2^{\alpha_2}(t) \cdots r_m^{\alpha_m}(t) dt
$$

where $A(\alpha_1, \alpha_2, \ldots, \alpha_m) = \frac{\left( \sum_{i=1}^m a_i \right)!}{\prod_{i=1}^m a_i!}$

$$
= \sum_{(\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{N}_0^m} A(2\beta_1, 2\beta_2, \ldots, 2\beta_m) a_1^{2\beta_1} a_2^{2\beta_2} \cdots a_m^{2\beta_m}
$$

[Note that above integral vanishes if one of the exponents is odd, and that it equals otherwise to 1].

On the other hand

$$
\left( \sum |a_i|^2 \right)^k
$$

$$
= \left( \sum_{(\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{N}_0^m} A(\beta_1, \beta_2, \ldots, \beta_m) a_1^{2\beta_1} a_2^{2\beta_2} \cdots a_m^{2\beta_m} \right)
$$

$$
= \sum_{(\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{N}_0^m} A(\beta_1, \beta_2, \ldots, \beta_m) A(2\beta_1, 2\beta_2, \ldots, 2\beta_m) a_1^{2\beta_1} a_2^{2\beta_2} \cdots a_m^{2\beta_m}
$$

$$
\geq \min_{(\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{N}_0^m} A(\beta_1, \beta_2, \ldots, \beta_m) \sum_{(\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{N}_0^m} A(2\beta_1, 2\beta_2, \ldots, 2\beta_m) a_1^{2\beta_1} a_2^{2\beta_2} \cdots a_m^{2\beta_m}
$$

$$
= \min_{(\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{N}_0^m} A(\beta_1, \beta_2, \ldots, \beta_m) \sum_{\beta_i=k}^m \left( \sum_{i=1}^m a_i r_i(t) \right)^{2k} dt
$$

which implies our claim if put

$$
B_{2k} = \min_{(\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{N}_0^m} A(\beta_1, \beta_2, \ldots, \beta_m) \sum_{\beta_i=k}^m \min_{(\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{N}_0^m} A(\beta_1, \beta_2, \ldots, \beta_m) \frac{A(\beta_1, \beta_2, \ldots, \beta_m)}{A(2\beta_1, 2\beta_2, \ldots, 2\beta_m)}.
$$
In order to show that we can choose $A_1 > 0$, to satisfy (5.12) we observe that for $f(t) = \sum_{i=1}^m a_i r_i(t)$

$$
\int_0^1 |f(t)|^2 \, dt = \int_0^1 |f(t)|^{2/3} |f(t)|^{1/3} \, dt \\
\leq \left[ \int_0^1 |f(t)| \, dt \right]^{2/3} \left[ \int_0^1 |f(t)|^4 \, dt \right]^{1/3}
$$

[By H"older's inequality for $p = 3/2$ and $q = 3$]

$$
\leq \left[ \int_0^1 |f(t)| \, dt \right]^{2/3} B_4^{4/3} \left[ \sum_{i=1}^m a_i^2 \right]^{2/3}.
$$

Therefore

$$
\int_0^1 |f(t)| \, dt \geq \left[ B_4^{-4/3} \int_0^1 |f(t)|^2 \, dt \left( \sum_{i=1}^m a_i^2 \right)^{-2/3} \right]^{3/2}
$$

$$
= \left[ B_4^{-4/3} \sum_{i=1}^m |a_i|^2 \left( \sum_{i=1}^m a_i^2 \right)^{-2/3} \right]^{3/2} = B_4^{-2} \left( \sum_{i=1}^m a_i^2 \right)^{1/2}
$$

which proves our claim if we let $A_1 = B_4^{-2}$.

**Proof of Theorem 5.3.1.** Since the Rademacher functions are an orthonormal basis inside $L_2[0, 1]$ it follows from Lemma 5.3.3 that $\ell_2$ is isomorphically embeddable in $L_p[0, 1]$, for $1 \leq p < \infty$. Secondly, for $2 \leq p < \infty$ the formal identity $I : L_p[0, 1] \to L_2[0, 1]$ is bounded and the restriction of $I$ to $\overline{\text{span}(r_i : i \in \mathbb{N})}$ is an isomorphism onto $\overline{\text{span}(r_i : i \in \mathbb{N})}$. We conclude that the map:

$$
P : L_p[0, 1] \to \overline{\text{span}(r_i : i \in \mathbb{N})}, \quad f \mapsto \sum_{n=1}^\infty \left( \int_0^1 f(s) r_n(s) \, ds \right) r_n,
$$

is a projection onto $\overline{\text{span}(r_i : i \in \mathbb{N})}$, which proves that $\ell_2$ is isomorphic to a complemented subspace of $L_p[0, 1]$, if $2 \leq p < \infty$. The same conclusion follows also for $1 < p < 2$ by duality.

The next Theorem on subspaces of $L_p$ is due to Kadets and Pelczynski. We first state the **Extrapolation Principle**.

**Theorem 5.3.4.** [The Extrapolation Principle]

*Let $X \subset L_p[0, 1]$ be a linear subspace on which $\| \cdot \|_{p_1}$ and $\| \cdot \|_{p_2}$, where $p_1 < p_2$, are finite and equivalent. Thus, there is a $C \geq 1$ so that

$$
\|f\|_{p_1} \leq \|f\|_{p_2} \leq C \|f\|_{p_1}, \quad \text{whenever } f \in X.
$$
*
5.3. **ON “SMALL SUBSPACES” OF** $L_p$

(first inequality holds always by Hölder inequality).

Then for all $0 < p \leq p_1$ and all $x \in X$

$$C^{(p_2/p)(1-(1/\lambda))} \|x\|_{p_1} \leq \|x\|_{p} \leq \|x\|_{p_1},$$

where $\lambda \in (0, 1)$ is defined by $p_1 = \lambda p + (1 - \lambda)p_2$.

**Proof.** Let $0 < p \leq p_1$ and choose $0 < \lambda < 1$ so that $p_1 = \lambda p + (1 - \lambda)p_2$.

For $x \in X$ it follows

$$\|x\|_{p_1} = \left[ \int |x(t)|^{\lambda p} \cdot |x(t)|(1-\lambda)p_2 \,dt \right]^{1/p_1}$$

$$\leq \left[ \int |x(t)|^{p} \,dt \right]^{\lambda/p_1} \cdot \left[ \int |x(t)|^{p_2} \,dt \right]^{(1-\lambda)/p_1} \,dt$$

[Hölder inequality for exponents $1/\lambda$ and $1/(1 - \lambda)$]

$$= \|x\|_{p_1}^{p_2} \|x\|_{p_2}^{(1-\lambda)p_1} \leq C^{p_2/p_1} (1-\lambda) \|x\|_{p_1}^{p_2} \|x\|_{p_1}^{1-\lambda}$$

thus (since $1 - \frac{p_2}{p_1} (1-\lambda) = \frac{\lambda p}{p_1}$)

$$\|x\|_{p_1}^{\lambda p} \leq C^{p_2/p_1} (1-\lambda) \|x\|_{p_1}^{\frac{\lambda p}{p_1}}$$

and thus

$$\|x\|_{p_1} \leq C^{p_2/p_1} (1-\lambda) \|x\|_{p}$$

which yields that

$$C^{(p_2/p)(1-(1/\lambda))} \|x\|_{p_1} \leq \|x\|_{p}.$$

\[\square\]

**Remark.** The Interpolation is obvious, and follows from applying Hölder’s Theorem twice:

Assume as in the previous Theorem that $X \subset L_p[0, 1]$, is a linear subspace on which $\| \cdot \|_{p_1}$ and $\| \cdot \|_{p_2}$, where $p_1 < p_2$, are $C$-equivalent. Thus, there is a $C \geq 1$ so that

$$\|f\|_{p_1} \leq \|f\|_{p_2} \leq C\|f\|_{p_1}$$

whenever $f \in X$.

Then for all $p \in (p_1, p_2)$

$$\|f\|_{p_1} \leq \|f\|_{p} \leq \|f\|_{p_2} \leq C\|f\|_{p_1}.$$
Theorem 5.3.5 (Kadets and Pelczynski). Assume $2 < p < \infty$ and assume that $X$ is a closed subspace of $L_p[0,1]$. Then:

Either there is an $0 < r < p$ so that $\| \cdot \|_r$ and $\| \cdot \|_p$ are equivalent norms. In that case it follows that $X$ is isomorphic to a Hilbert space, $X$ is complemented in $L_p[0,1]$ and the constant of isomorphism as well as the constant of complementation only depend on $r$, $p$ and the equivalence constant between $\| \cdot \|_r$ and $\| \cdot \|_p$ on $X$.

Or $\| \cdot \|_r$ and $\| \cdot \|_p$ are not equivalent on $X$ for some $r < p$. Then $X$ contains for any $\varepsilon > 0$ a sequence which is $(1+\varepsilon)$-equivalent to the $\ell_p$-unit vector basis.

Proof. Let $X$ be (w.l.o.g) an infinite dimensional subspace of $L_p[0,1]$. If for some $r < p$ the norms $\| \cdot \|_p$ and $\| \cdot \|_r$ are equivalent on $X$ it follows from Theorem 5.3.4 and the following remark that $\| \cdot \|_2$ and $\| \cdot \|_p$ are equivalent norms on $X$ and the constant of equivalence only depends on $r$, $p$ and the equivalence constant of $\| \cdot \|_r$ and $\| \cdot \|_p$. Thus, $X$ is isomorphic to a separable Hilbert space. Moreover $X$, seen as a linear subspace of $L_2[0,1]$, is closed and thus complemented. Let $P : L_2[0,1] \rightarrow X$ be the orthogonal projection from $L_2[0,1]$ onto $X$. Then $Q = P \circ I$, where $I : L_p[0,1] \rightarrow L_2[0,1]$ is the formal identity, is a projection from $L_p[0,1]$ onto $X$.

Assume for all $r < p$ the norms $\| \cdot \|_p$ and $\| \cdot \|_r$ are not equivalent on $X$ and let $\varepsilon > 0$. For $n \in \mathbb{N}$, choose inductively $r_n < p$, $M_n > 1$ and $f_n \in X$ so that

\begin{equation}
M_n \geq 2^n \text{ and, } \int_{\{|f| > M_n\}} |f_i(t)|^p \, dt < 2^{-n-1}\varepsilon,
\end{equation}

whenever $1 \leq i < n$ and $f \in B_{L_p[0,1]}$

\begin{equation}
M_n^{p-r_n} = 2
\end{equation}

\begin{equation}
\|f_n\|_{r_n} < 2^{-n-1}\varepsilon, \text{ and } \|f_n\|_p = 1.
\end{equation}

Indeed, for $n = 1$ let $M_1 = 2$ (which satisfies (5.13), since the second condition is vacuous). Then choose $r_1 < p$ close enough to $p$ so that (5.14) holds. Since $\| \cdot \|_{r_1}$ and $\| \cdot \|_p$ are not equivalent on $X$, and we can choose $f_1 \in S_X$ so that (5.15) holds.

Assuming $f_1, f_2, \ldots f_{n-1}, r_1, r_2 \ldots r_{n-1}$, and $M_1, M_2, \ldots M_{n-1}$ have been chosen, we first choose $\eta > 0$ so that for all $i = 1, 2 \ldots n$, and all measurable $A \subset [0,1]$ with $m(A) < \eta$ and all $i = 1, 2 \ldots n - 1$, it follows that

\[ \int_A |f_i(t)|^p \, dt < 2^{-n-1}\varepsilon. \]
Now for any $f \in B_{L_p[0,1]}$ and any $M > 0$ we have
\[ m\{|f| > M\} \leq \frac{1}{M^p} \int |f(t)|^p \, dt \leq \frac{1}{M^p}, \]
So choosing $M_n = \max\{2^n, \frac{1}{n^{1/p}}\}$, we deduce (5.13). We can then choose $r_n \in (0, p)$ close enough to $p$, so that (5.14), and since by assumption $\| \cdot \|_{r_n}$ and $\| \cdot \|_p$ are not equivalent on $X$, we can choose $f_n \in X$ so that (5.15) holds. This finishes the recursion.

Then
\[ \int_{|f_n| < M_n} |f_n(t)|^p \, dt \leq \int M_n^{p-r_n} |f(t)|^{r_n} \, dt \leq 2\|f_n\|_{r_n} < 2^{-n} \varepsilon. \]

For $n \in \mathrm{put} A_n = \{f_n \geq M_n\} \setminus \bigcup_{j > n} \{|f_j| \geq M_j\}$ and $g_n = f_n1_{A_n}$. Then the $g_n$’s have disjoint support and
\[ \|f_n - g_n\|_p^p \leq \int_{|f_n| < M_n} |f_n(t)|^p \, dt + \sum_{j > n} \int_{|f_j| > M_j} |f_n(t)|^p \, dt \]
\[ \leq 2^{-n} \varepsilon + \sum_{j > n} \int_{|f_j| > M_j} |f_n(t)|^p \, dt \leq 2^{-n} \varepsilon + \sum_{j > n} 2^{j-1} \varepsilon = 2^{1-n} \varepsilon, \]

Fix $\delta > 0$. For $\varepsilon$ small enough (depending on $\delta$), it follows that $(g_n)$ is $(1+\delta)$-equivalent to the $\ell_p$-unit vector basis (since the $g_n$ have disjoint support). By choosing $\delta$ small enough we can secondly ensure that
\[ \sum_{n \in \mathbb{N}} \|g_n - f_n\|_p \|g_n^*\|_q < 1, \]
where the $(g_n^*)$ are the coordinate functionals of $(g_n)$. Applying now the Small Perturbation Lemma yields that $(f_n)$ is also equivalent to the $\ell_p$ unit basis.

\[ \square \]

**Remark.** The Theorem of Kadets and Pelczynski started the investigation of complemented subspaces of $L_p[0, 1]$, $2 < p < \infty$. Here are some results:

Johnson-Odell 1974: Every complemented subspace of $L_p[0, 1]$ which does not contain $\ell_2$, must be a subspace of $\ell_p$. In other words if $X$ is an infinite dimensional complemented subspace of $L_p[0, 1]$ it must be either $\ell_2$ or $\ell_p$ or contain $\ell_p \oplus \ell_2$ (we are using here also that $\ell_p$ is prime, i.e. that every infinite dimensional complemented subspace of $\ell_p$ is isomorphic to $\ell_p$).
Bourgain-Rosenthal-Schechtman 1981: There are uncountable many non isomorphic complemented subspaces of $L_p[0, 1]$.

Haydon-Odell-Schlumprecht 2011: If $X$ is a complemented subspace of $L_p[0, 1]$ which does not isomorphically embed into $\ell_2 \oplus \ell_p$ then it must contain $\ell_p(\ell_2)$.

Next Question: Assume that $X$ is a complemented subspace of $L_p[0, 1]$ which is not contained in an isomorphic copy of $\ell_p(\ell_2)$. What can we say about $X$?

Exercises

1. Show that for $p \neq 2$, $\ell_2$ is not isomorphic to a subspace of $\ell_p$
5.4. THE SPACES $\ell_p$, $1 \leq p < \infty$, AND $c_0$ ARE PRIME SPACES

5.4 The Spaces $\ell_p$, $1 \leq p < \infty$, and $c_0$ are Prime Spaces

The main goal of this section is show that the spaces $\ell_p$, $1 \leq p < \infty$, and $c_0$ are prime spaces.

Definition 5.4.1. A Banach space $X$ is said to be prime if every complemented subspace of $X$ is isomorphic to $X$.

The following Theorem is due to Pelczynski.

Theorem 5.4.2. The spaces $\ell_p$, $1 \leq p < \infty$, and $c_0$ are prime.

We will prove this theorem using the Pelczynski Decomposition Method, an argument which is important in its own right and also very pretty. Before doing that we need some lemmas. The first one was, up to the “moreover part” a homework problem and can be easily deduced from the Small Perturbation Lemma.

Lemma 5.4.3. [The Gliding Hump Argument]

Let $X$ be a Banach space with a basis $(e_i)$ and $Y$ an infinite dimensional closed subspace of $X$. Let $\varepsilon > 0$. Then $Y$ contains a normalized sequence $(y_n)$ which is basic and $(1 - \varepsilon)^{-1}$-equivalent to some normalized block basis $(u_n)$.

Moreover, if the span of $(u_n)$ is complemented in $X$, so is the span of $(y_n)$.

Proof. Without loss of generality we can assume that $\|e_n\| = 1$, for $n \in \mathbb{N}$.

Let $b$ be the basis constant, and $(e_i^*)$ the coordinated functionals of $(e_n)$. Let $\delta_n \subset (0, 1)$ a null sequence, with $\sum_{n=1}^{\infty} \delta_n \leq \varepsilon/2b$. By induction we choose for every $n \in \mathbb{N}$ $y_n, u_n \in S_X$ and $k_n \in \mathbb{N}$, so that:

a) $0 = k_0 < k_1 < k_2 < \ldots$,

b) $u_n \in \text{span}(e_j : k_{n-1} + 1 \leq j \leq k_n)$, and

c) $y_n \in Y$, and $\|u_n - y_n\| < \delta_n$.

For $n = 1$ we simply choose any $y_1 \in S_Y$, and then by density of $\text{span}(e_j : j \in \mathbb{N})$ in $X$ an element $x_1 \in \text{span}(e_j : j \in \mathbb{N})$, with $\|x_1\| = 1$ and choose $k_1 \in \mathbb{N}$ so that $x_1 \in \text{span}(e_j \in \mathbb{N})$.

Assuming $k_n$ has been chosen we can choose $y_{n+1} \in \bigcap_{i \leq k_n} N(e_i^*) \cap S_X$. Since $\text{span}(e_j : j \in \mathbb{N}, j > k_n)$ is dense in $\bigcap_{i \leq k_n} N(e_i^*)$, we can choose
\( u_{n+1} \in \text{span}(e_j : j \in \mathbb{N}, j > k_n) \cap S_X \) so that \( \|x_{n+1} - u_{n+1}\| < \delta_{n+1} \), and finally choose \( k_{n+1} \), so that \( u_{n+1} \in \text{span}(e_j : j \in \mathbb{N}, k_n < j \leq k_{n+1}) \).

Since the basis constant of \((u_n)\) does not exceed \( b \) (Proposition 3.3.3) we deduce for the coordinate functionals \((u_n^*)\) of \((u_n)\) that

\[
\sup_{n \in \mathbb{N}} \|u_n^*\| \leq \sup_{n \in \mathbb{N}} \frac{2b}{\|u_n\|} = 2b,
\]

and thus

\[
\sum_{j=1}^{n} \|y_n - u_n\| \cdot \|u_n^*\| \leq 2b \sum_{j=1}^{\infty} \delta_n \leq \varepsilon,
\]

and we conclude therefore our claim from the Small Perturbation Lemma 3.3.10.

\[\square\]

**Proposition 5.4.4.** The closed span of block bases in \(\ell_p\) and \(c_0\) are isometrically equivalent to the unit vector basis and are 1-complemented in \(\ell_p\), or \(c_0\).

**Proof.** We only present the proof for \(\ell_p\), \(1 \leq p < \infty\), the \(c_0\) case works in the same way.

Let \((u_n)\) be a normalized block basis, and write \(u_n, n \in \mathbb{N}\), as

\[
u_n = \sum_{j=k_{n-1}+1}^{k_n} a_j e_j, \text{ with } 0 = k_0 < k_1 < k_2 < \ldots \text{ and } (a_n) \subset \mathbb{K}.
\]

It follows for \(m \in \mathbb{N}\) and \((b_n)_{n=1}^{m} \subset \mathbb{K}\), that

\[
\left\| \sum_{n=1}^{m} b_n u_n \right\|_p = \sum_{n=1}^{m} \sum_{j=k_{n-1}+1}^{k_n} |b_n|^p |a_j|^p = \sum_{j=1}^{m} |b_j|^p,
\]

and thus \((u_n)\) is isometrically equivalent to \((e_n)\).

For \(n \in \mathbb{N}\) choose \(u_n^* \in \ell_q\), \(u_n^* \in \text{span}(e_j^* : k_{n-1} < j \leq k_n)\), \(\|u_n^*\|_q = 1\), so that \(\langle u_n^*, u_n \rangle = 1\), and define

\[
P : \ell_p \to \text{span}(u_n : j \in \mathbb{N}), x \mapsto \sum \langle x, u_n^* \rangle u_n.
\]

For \(x = \sum_{j=1}^{\infty} x_j e_j \in \ell_p\) it follows that

\[
\|\langle u_n^*, x \rangle\|_p = \left| \left\langle u_n^*, \sum_{j=k_{n-1}+1}^{k_n} x_j e_j \right\rangle \right| \leq \sum_{j=k_{n-1}+1}^{k_n} |x_j|^p,
\]
and, thus, that
\[
\|P(x)\|_p^p = \sum_{n=1}^{\infty} \sum_{j=k_{n-1}+1}^{k_n} |a_j|^p |\langle u_n^*, x \rangle|^p \leq \sum_{n=1}^{\infty} \langle u_n^*, x \rangle |x_j|^p \leq \sum_{n=1}^{\infty} \sum_{j=k_{n-1}+1}^{k_n} |x_j|^p = \|x\|_p^p.
\]
This shows that \(\|P\| \leq 1\), and, since moreover \(P(u_n) = u_n\), and \(P(X) \subset \text{span}(u_j : j \in \mathbb{N})\), it follows that \(P\) is a projection onto \(\text{span}(u_j : j \in \mathbb{N})\) of norm 1.

**Remark.** It follows from Lemma 5.4.3 and Proposition 5.4.4 for \(X = \ell_p\) or \(c_0\) that every subspace \(Y\) of \(X\) has a further subspace \(Z\) which is complemented in \(X\) and isomorphic to \(X\). We call a space \(X\) which has this property *complementably minimal*, a notion introduced by Casazza. In particular if \(Y\) is any complemented subspace of \(X\) the pair \((Y, X)\) has the *Schröder Bernstein property*, which means that \(X\) is isomorphic to a subspace \(Y\), and \(Y\) is isomorphic to a complemented subspace of \(X\).

It was for long time an open question whether a complementably minimal space is prime, and an even longer open problem was the question whether or not \(\ell_p\) and \(c_0\) are the only separable prime spaces. The first question would have a positive answer if all Banach spaces \(X\) and \(Y\) for which \((X, Y)\) has the Schröder Bernstein property then it follows that \(X\) and \(Y\) are isomorphic. It is also open if complementably minimal spaces have to be prime.

Then Gowers and Maurey [GM2] constructed a space \(X\) (this is a variation of the space cited in [GM] and also does not contain any unconditional basis sequence) which only has trivial complemented subspaces, namely the finite and cofinite dimensional subspaces which has the property that all the cofinite dimensional subspaces are isomorphic to \(X\). Thus, this space is prime, but not \(\ell_p\) or \(c_0\).

Then Gowers [Go2] also found a counterexample to the Schröder Bernstein problem, which also does not contain any unconditional basic sequence.

Both questions are still open for spaces with unconditional basic sequences, and thus spaces with *lots of complemented subspaces*. In [Sch] a space with a 1-unconditional space was constructed which is complementably minimal (shown in [AS]) but does not contain \(\ell_p\) or \(c_0\). This space together with some complemented subspace \(Y\) must either be a counterexample to the Schröder Bernstein Problem, or it is a new prime space.

The *Pelczynski Decomposition Method* now proves that a complementably minimal space is prime, if you assume some additional assumptions which are all satisfied by \(\ell_p\) or \(c_0\).

Let’s start with a very easy and general observation.
Proposition 5.4.5. If $X$ and $Y$ are Banach spaces, with the property that $X$ is isomorphic to a complemented subspace of $Y$ and if $X$ is isomorphic to its square, i.e. $X \sim X \oplus X$, then $Y$ is isomorphic to $X \oplus Y$.

In particular if $X$ and $Y$ are isomorphic to their squares, isomorphic to complemented subspaces of each other, then it follows that $X \sim X \oplus Y \sim Y$.

Proof. Let $Z$ be a complemented subspace of $Y$ so that $Y \sim X \oplus Z$. Then

\[ Y \sim X \oplus Z \sim (X \oplus X) \oplus Z \sim X \oplus (X \oplus Z) \sim X \oplus Y. \]

Remark. It is easy to see that $\ell_p \sim \ell_p \oplus \ell_p$, $1 \leq p < \infty$ and $c_0 \sim c_0 \oplus c_0$, but it is not clear how to show directly that any complemented subspace of $\ell_p$ or $c_0$ is isomorphic to its square. So we will need an additional property of $\ell_p$ and $c_0$. Nevertheless we can easily deduce the following Corollary from Proposition 5.4.5 and Khintchine’s Theorem 5.2.1.

Corollary 5.4.6. For $1 < p < \infty$ it follows that $L_p[0,1]$ is isomorphic to $L_p[0,1] \oplus L_2[0,1]$.

Proof of Theorem 5.4.2. Let $X = \ell_p$ or $c_0$. From now on we consider on all complemented sums the $\ell_p$-sum, respectively $c_0$-sum. Note that $X \sim (\oplus_{j \in \mathbb{N}} X)_X$ (actually isometrically)

Let $Y$ be a complemented subspace of $X$, by Proposition 5.4.5 we only need to show that $X \sim X \oplus Y$, and that can be seen as follows: we let $Z$ be a subspace of $X$ so that $X \sim Y \oplus Z$, then

\[ Y \oplus X \sim Y \oplus (\oplus_{n \in \mathbb{N}} X)_X \]
\[ \sim Y \oplus (\oplus_{n \in \mathbb{N}} (Z \oplus Y))_X \]
\[ \sim Y \oplus Z \oplus (\oplus_{n \in \mathbb{N}} (Y \oplus Z))_X \]
\[ \text{(consider } (y_1, (z_1, x_1, z_2, x_2, \ldots)) \mapsto ((y_1, z_1), (x_1, z_2, x_2, \ldots))\text{)} \]
\[ \sim (\oplus_{n \in \mathbb{N}} (Y \oplus Z))_X \]
\[ \sim (\oplus_{n \in \mathbb{N}} X)_X \sim X. \]

Remark. $L_1[0,1]$ cannot be prime since $\ell_1$ is isomorphic to complemented subspaces of $L_1[0,1]$, but is this only other complemented subspace? Are all the complemented subspaces of $L_1[0,1]$ either isomorphic to $\ell_1$ or to $L_p[0,1]$?
5.5. THE HAAR BASIS IS UNCONDITIONAL IN $L_p[0, 1], 1 < P < \infty$}

5.5 The Haar basis is Unconditional in $L_p[0, 1], 1 < P < \infty$

S:5.3

**Theorem 5.5.1.** [Unconditionality of the Haar basis in $L_p$]

Let $1 < p < \infty$. Then $(h_t^{(p)})$ is an unconditional basis of $L_p[0, 1]$. More precisely, for any two families $(a_t)_{t \in T}$ and $(b_t)_{t \in T}$ in $c_00(T)$ with $|a_t| \leq |b_t|$, for all $t \in T$, it follows that

$$
\left\| \sum_{t \in T} a_t h_t^{(p)} \right\| \leq (p^* - 1) \left\| \sum_{t \in T} b_t h_t^{(p)} \right\|,
$$

where

$$
p^* = \max \left( p, \frac{p}{p - 1} \right) = \begin{cases} p & \text{if } p \geq 2 \\ p(p - 1) & \text{if } p \leq 2 \end{cases}
$$

We will prove the theorem for $2 < p < \infty$. For $p = 2$ it is clear since $(h_t^{(2)})$ is orthonormal and for $1 < p < 2$ it follows from Proposition 3.4.5 by duality (note that $p^* = q^*$ if $\frac{1}{p} + \frac{1}{q} = 1$).

We first need the following Lemma

**Lemma 5.5.2.** Let $2 < p < \infty$ and define

$$
(5.17) \quad v : \mathbb{C} \times \mathbb{C} \to [0, \infty), \quad (x, y) \mapsto |y|^p - (p - 1)^p |x|^p, \quad \text{and}
$$

$$
(5.18) \quad u : \mathbb{C} \times \mathbb{C} \to [0, \infty), \quad (x, y) \mapsto \alpha_p(|x| + |y|)^{p-1}(|y| - (p - 1)|x|)
$$

with $\alpha_p = p \left(1 - \frac{1}{p}\right)^{p-1}$.

Then it follows for $x, y, a, b \in \mathbb{C}$, with $|a| \leq |b|$

$$
(5.19) \quad v(x, y) \leq u(x, y)
$$

$$
(5.20) \quad u(-x, -y) = u(x, y)
$$

$$
(5.21) \quad u(0, 0) = 0
$$

$$
(5.22) \quad u(x + a, y + b) + u(x - a, y - b) \leq 2u(x, y)
$$

**Proof.** Let $x, y, a, b \in \mathbb{C}, |a| \leq |b|$ be given. (5.20) and (5.21) are trivially satisfied. Since $u$ and $v$ are both $p$-homogeneous (i.e. $u(\alpha \cdot x, \alpha y) = |\alpha|^p u(x, y)$ for $\alpha \in \mathbb{C}$) we can assume that $|x| + |y| = 1$ in order to show (5.19). Thus the inequality (put $s = |x|$) reduces to show

$$
(5.23) \quad F(s) = \alpha_p(1 - ps) - (1-s)^p + (p-1)s^p \geq 0 \quad \text{for } 0 \leq s \leq 1 \text{ and } 2 \leq p.
$$
In order to verify (5.23), first show that \( F(0) > 0 \). Indeed, by concavity of \( \ln x \) it follows that

\[
\ln p = \ln ((p-1)+1) < \ln(p-1) + \frac{1}{p-1},
\]

and, thus,

\[
\ln(p-1) + 1 = \ln(p-1) + \frac{1}{p-1} + \frac{p-2}{p-1} > \ln p + \frac{p-2}{p-1} > \ln p + \frac{p-2}{p}.
\]

Integrating both sides of

\[
\ln(x-1) + 1 > \ln x + \frac{x-2}{x}.
\]

from \( x = 2 \) to \( p > 2 \), implies that

\[
(p-1)\ln(p-1) > (p-2)\ln p
\]

and, thus,

\[
(p-1)^{p-1} > p^{p-2},
\]

which yields

\[
\alpha_p = p\left(1 - \frac{1}{p}\right)^{p-1} = \frac{(p-1)^{p-1}}{p^{p-2}} > 1
\]

and thus the claim that \( F(0) > 0 \).

Secondly we claim that \( F(1) > 0 \). Indeed,

\[
F(1) = -(\frac{(p-1)^{p-2}}{p^p}) + (p-1)^p = (p-1)^{p-2}\left[(p-1)^2 - \frac{1}{p^2}\right] > 0,
\]

Thirdly, we compute the first and second derivative of \( F \) and get

\[
F'(s) = -\alpha_pp + p(1-s)^{p-1} + (p-1)p^{p-2}, \text{ and } F''(s) = -p(p-1)(1-s)^{p-2} + (p-1)^{p+1}p^{p-2}
\]

and deduce that \( F'(\frac{1}{p}) = F'(\frac{1}{p}) = 0, F''(\frac{1}{p}) > 0 \) and that \( F''(s) \) vanishes for exactly one value of \( s \) (because it is the difference of an increasing functions and a decreasing function). Thus, \( F(s) \) cannot have more points at which it vanishes and it follows that \( F(s) \geq 0 \) for all \( s \in [0,1] \) and we deduce (5.19).

Finally we need to show (5.22). We can (by density argument) assume that \( x \) and \( a \) as well as \( y \) and \( b \) are linear independent as two-dimensional
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Vectors over $\mathbb{R}$. This implies that $|x + t\alpha|$ and $|y + t\beta|$ can never vanish, and, thus, that the function

$$G : \mathbb{R} \to \mathbb{R}, \quad t \mapsto u(x + t\alpha, y + t\beta),$$

is infinitely often differentiable.

We compute the second derivative of $G$ at $0$, getting

$$G''(0) = \alpha_p \left[ -p(p - 1) \left( |a|^2 - |b|^2 \right) \left( |x| + |y| \right)^{p-2} \right. $$

$$- p(p - 2) \left( |b|^2 - \Re(\frac{y}{|y|}, b) \right) |y|^{-1} \left( |x| + |y| \right)^{p-1} $$

$$- p(p - 1)(p - 2) \left( \Re(\frac{x}{|x|}, a) + \Re(\frac{y}{|y|}, b) \right)^2 |x| \left( |x| + |y| \right)^{p-3} \right].$$

Inspecting each term we deduce (recall that $|a| \geq |b|$) from the Cauchy inequality that $G''(0) < 0$. Since for $t \neq 0$ it follows that $G''(t) = G''(0)$ where

$$\tilde{G}(s) : \mathbb{R} \to \mathbb{R}, \quad s \mapsto u(x + t\alpha + sa, y + t\beta + sb),$$

we deduce that $G''(t) \leq 0$ for all $t \in \mathbb{R}$. Thus, $G$ is a concave function which yields

$$\frac{1}{2} [u(x + a, y + b) + u(x - a, y - b)] = \frac{1}{2} [G(1) + G(-1)] \leq G(0) = u(x, y),$$

which proves (5.22).

\[\square\]

Proof of Theorem 5.5.1. Assume that $\tilde{h}_n$ is normalized in $L_\infty$ so that $h_n = \tilde{h}_n/\|\tilde{h}_n\|_\infty$ is a linear reordering of $(h_t^{(p)})_t \in T$ which is compatible with the order on $T$. For $n \in \mathbb{N}$ let $f_n = \sum_{i=1}^n a_i \tilde{h}_i$ and $g_n = \sum_{i=1}^n b_i \tilde{h}_i$, where $(a_i)_{i=1}^n, (b_i)_{i=1}^n$ in $\mathbb{R}$, with $|a_j| \leq |b_j|$, for $j = 1, 2, \ldots n$, we need to show that $\|g_n\|_p \leq (1 - p^*)\|f_n\|$. The fact that we are considering the normalization in $L_\infty[0, 1]$ instead of the normalization in $L_p[0, 1]$ (i.e. $\tilde{h}_n$ instead of $h_n$) will not effect the outcome. We deduce from (5.19) that

$$\|g_n\|_p^p - (p - 1)^p \|f_n\|_p^p = \int_0^1 v(f_n(t), g_n(t)) \, dt \leq \int_0^1 u(f_n(t), g_n(t)) \, dt.$$

Let $A = \text{supp}(\tilde{h}_n), A^+ = A \cap \{\tilde{h}_n > 0\}$ and $A^- = A \cap \{\tilde{h}_n < 0\}$. Since $f_{n-1}$ and $g_{n-1}$ are constant on $A$ we deduce

$$\int_0^1 u(f_n(t), g_n(t)) \, dt$$
\[
\begin{align*}
= & \int_{[0,1] \setminus A} u(f_{n-1}(t), g_{n-1}(t)) \, dt \\
& + \int_{A^+} u(f_{n-1}(t) + a_n, g_{n-1}(t) + b_n) \, dt \\
& + \int_{A^-} u(f_{n-1}(t) - a_n, g_{n-1}(t) - b_n) \, dt \\
= & \int_{[0,1] \setminus A} u(f_{n-1}(t), g_{n-1}(t)) \, dt \\
& + \frac{1}{2} \int_A u(f_{n-1}(t) + a_n, g_{n-1}(t) + b_n) + u(f_{n-1}(t) - a_n, g_{n-1}(t) - b_n) \, dt \\
\leq & \int_{[0,1] \setminus A} u(f_{n-1}(t), g_{n-1}(t)) \, dt + \int_A u(f_{n-1}(t), g_{n-1}(t)) \, dt \\
& [\text{By (5.22)}] \\
= & \int_0^1 u(f_{n-1}(t), g_{n-1}(t)) \, dt
\end{align*}
\]

Iterating this argument yields
\[
\int_0^1 u(f_n(t), g_n(t)) \, dt \leq \int_0^1 u(f_1(t), g_1(t)) \, dt = u(a_1, b_1) = \frac{1}{2} (u(a_1, b_1) + u(-a_1, -b_1)) \quad [\text{By (5.20)}] \\
\leq u(0, 0) = 0 \quad [\text{By (5.21) and (5.22)}],
\]

which implies our claim that \( \|g_n\| \leq (p-1)\|f_n\| \).

\[\square\]
Bibliography


[So] Sobczyk, A. Projection of the space $m$ on its subspace $c_0$, Bull. Amer. Math. Soc. 47 (1941), 938 – 947.