Chapter 1

The Threshold Algorithm

1.1 Greedy and Quasi Greedy Bases

We start with the Threshold Algorithm:

Definition 1.1.1. Let $X$ be a separable Banach space with a normalized $M$-basis $((e_i, e_i^*) : i \in \mathbb{N})$; we mean by that $\|e_i\| = 1$, for $i \in \mathbb{N}$) For $n \in \mathbb{N}$ and $x \in X$ let $\Lambda_n \subset \mathbb{N}$ so that

$$\min_{i \in \Lambda_n} |e_i^*(x)| \geq \max_{i \in \mathbb{N} \setminus \Lambda_n} |e_i^*(x)|,$$

i.e. we are reordering $(e_i^*(x))$ into $(e_{\sigma_{(i)}}^*(x))$, so that

$$|e_{\sigma_1}^*(x)| \geq |e_{\sigma_2}^*(x)| \geq |e_{\sigma_3}^*(x)| \geq \ldots,$$

and put for $n \in \mathbb{N}$

$$\Lambda_n = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}.$$

Then define for $n \in \mathbb{N}$

$$G^T_n(x) = \sum_{i \in \Lambda_n} e_i^*(x)e_i.$$

$(G^T_n)$ is called the Threshold Algorithm.

Definition 1.1.2. A normalized $M$-basis $(e_i)$ is called Quasi-Greedy, if for all $x$

(QG) \[ x = \lim_{n \to \infty} G^T_n(x). \]

A basis is called greedy if there is a constant $C$ so that

(G) \[ \|x - G_T(x)\| \leq C \sigma_n(x), \]

where we define

$$\sigma_n(x) = \sigma_n(x, (e_j)) = \inf_{\Lambda \subset \mathbb{N}, \#\Lambda = n} \inf_{z \in \text{span}(e_j: j \in \Lambda)} \|z - x\|.$$
In that case we say that \((e_i)\) is \(C\)-greedy. We call the smallest constant \(C\) for which (G) holds the greedy constant of \((e_n)\) and denote it by \(C_g\).

**Remarks.** Let \(((e_i, e_i^*) : i \in \mathbb{N})\) be a normalized \(M\) basis.

1. From the property that \((e_n)\) is fundamental we obtain that for every \(x \in X\)
   \[
   \sigma_n(x) \to_{n \to \infty} 0,
   \]
   it follows therefore that every greedy basis is quasi greedy.

2. If \((e_j)\) is an unconditional basis of \(X\), and \(x = \sum_{i=1}^{\infty} a_i \in X\), then
   \[
   x = \lim_{n \to \infty} \sum_{j=1}^{n} a_{\pi(j)} e_{\pi(j)},
   \]
   for any permutation \(\pi : \mathbb{N} \to \mathbb{N}\) and thus, in particular, also for a greedy permutation, i.e. a permutation, so that
   \[
   |a_{\pi(1)}| \geq |a_{\pi(2)}| \geq |a_{\pi(3)}| \ldots.
   \]
   Thus, an unconditional basis is always quasi-greedy.

3. Schauder bases have a special order and might be reordered so that the cease to be basis. But
   - unconditional bases,
   - \(M\) bases,
   - quasi greedy \(M\)-bases,
   - greedy bases
   keep their properties under any permutation, and can therefore be indexed by any countable set.

4. In order to obtain a quasi greedy \(M\)-Basis which is not a Schauder basis, one could take quasi greedy Schauder basis, which is not unconditional (its existence will be shown later), but admits a suitable reordering under which is not a Schauder basis anymore. Nevertheless, by the observations in (3), it will still be a quasi greedy \(M\)-basis. But it seems unknown whether or not there is a quasi greedy \(M\)-basis which cannot be reordered into a Schauder basis.

**Examples 1.1.3.** 1. If \(1 \leq p < \infty\), then the unit vector basis \((e_i)\) of \(\ell_p\) is \(1\)-greedy.
2. The unit vector basis \((e_i)\) in \(c_0\) is 1-greedy.

3. The summing basis \(s_n\) of \(c_0\) \((s_n = \sum_{j=1}^{n} e_j)\) is not quasi greedy.

4. The unit bias of \((\ell_p \oplus \ell_q)_1\) is not greedy (but 1-unconditional and thus quasi greedy).

**Proof.** To prove (1) let \(x = \sum_{j=1}^{\infty} x_j e_j \in \ell_p\), and let \(\Lambda_n \subset \mathbb{N}\) be of cardinality \(n\) so that

\[
\min\{|x_j| : j \in \Lambda_n\} \geq \max\{|x_j| : j \in \mathbb{N} \setminus \Lambda_n\}
\]

and \(\Lambda \subset \mathbb{N}\) be any subset of cardinality \(n\) and \(z = \sum z_i e_i \in \ell_p\) with

\[
\text{supp}(z) = \{i \in \mathbb{N} : |z_i| \neq 0\} \subset \Lambda.
\]

Then

\[
\|x - z\|_p = \sum_{j \in \Lambda} |x_j - z_j|^p + \sum_{j \notin \Lambda} |x_j|^p
\geq \sum_{j \in \Lambda} |x_j - z_j|^p + \sum_{j \notin \Lambda \setminus \Lambda_n} |x_j|^p
\geq \sum_{j \notin \mathbb{N} \setminus \Lambda_n} |x_j|^p = \|G^T(x) - x\|_p.
\]

Thus

\[
\sigma_n(x) = \inf\{|z - x|_p : \text{#supp}(z) \leq n\} = \|G^T(x) - x\|_p.
\]

(2) can be shown in the same way as (1).

In order to show (3) we choose sequences \((\varepsilon_j) \subset (0, 1)\), \((n_j) \subset \mathbb{N}\) as follows:

\[
\varepsilon_{2j} = 2^{-j} \quad \text{and} \quad \varepsilon_{2j-1} = 2^{-j} \left(1 + \frac{1}{j^3}\right), \quad \text{for} \ j \in \mathbb{N}
\]

and

\[
n_j = j 2^j \quad \text{and} \quad N_j = \sum_{i=1}^{n} n_i \quad \text{for} \ i \in \mathbb{N}_0.
\]

Note that the series

\[
x = \sum_{j=1}^{\infty} \sum_{i=N_{j-1}+1}^{N_j} (\varepsilon_{2j-1} s_{2i-1} - \varepsilon_{2j} s_{2i})
= \sum_{j=1}^{\infty} \sum_{i=N_{j-1}+1}^{N_j} \left((\varepsilon_{2j-1} - \varepsilon_{2j}) s_{2i-1} - \varepsilon_{2j} e_{2i}\right)
\]

\[
\begin{align*}
\end{align*}
\]
converges, because
\[
\sum_{j=1}^{\infty} \sum_{i=N_{j-1}+1}^{N_j} \varepsilon_{2j} e_{2i} \in c_0
\]
and
\[
\sum_{j=1}^{\infty} \sum_{i=N_{j-1}+1}^{N_j} \| (\varepsilon_{2j-1} - \varepsilon_{2j}) s_{2i-1} \| = \sum_{j=1}^{\infty} n_j (\varepsilon_{2j-1} - \varepsilon_{2j}) = \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty.
\]

Now we compute for \( l \in \mathbb{N}_0 \) the vector \( x - G_{2N_l+n_i+1}^T(x) \):
\[
x - G_{2N_l+n_i+1}^T(x) = - \sum_{i=N_l+1}^{N_{l+1}} \varepsilon_{2i+2} s_{2i} + \sum_{j=l+2}^{\infty} \sum_{i=N_{j-1}+1}^{N_j} (\varepsilon_{2j-1} s_{2i-1} - \varepsilon_{2j} s_{2i}).
\]
From the monotonicity of \((s_i)\) we deduce that
\[
\left| \sum_{j=l+2}^{\infty} \sum_{i=N_{j-1}+1}^{N_j} \varepsilon_{2j-1} s_{2i-1} - \varepsilon_{2j} s_{2i} \right| \leq \|x\|.
\]
However,
\[
\sum_{i=N_l+1}^{N_{l+1}} \varepsilon_{2i+2} = \sum_{i=N_l+1}^{N_{l+1}} \varepsilon_{2i+2} = l + 1 \rightarrow \infty \quad \text{as} \quad l \rightarrow \infty,
\]
which implies that \( G_{n}^T(x) \) is not convergent.

To show (4) assume w.l.o.g. \( p < q \), and denote the unit vector basis of \( \ell_p \) by \((e_i)\) and the unit vector basis of \( \ell_q \) by \((f_j)\) for \( n \in \mathbb{N} \) and we put
\[
x(n) = \sum_{j=1}^{n} \frac{1}{2} e_j + \sum_{j=1}^{n} f_j.
\]
Thus
\[
G_{n}^T(x(n)) = \sum_{j=1}^{n} f_j, \text{ and thus } \| G_{n}^T(x(n)) - x(n) \| = \frac{1}{2} n^{1/p}.
\]
Nevertheless
\[
\| x - \sum_{j=1}^{n} \frac{1}{2} e_j \| = n^{1/q},
\]
and since \( \frac{1}{2} n^{1/p}/n^{1/q} \not\to \infty \), for \( n \not\to \infty \), the basis \( \{e_j : j \in \mathbb{N}\} \cup \{f_j : j \in \mathbb{N}\} \) cannot be greedy.
Remarks. With the arguments used in (4) Examples 1.1.3 one can show that the usual bases of \((\oplus_{n=1}^{\infty} \ell_p^n)_{\ell_p}\) and \(\ell_p(\ell_q) = (\oplus_{n=1}^{\infty} \ell_q^n)_{\ell_p}\) are also not greedy but of course unconditional.

Now in [BCLT] it was shown that \((\oplus_{n=1}^{\infty} \ell_p^n)_{\ell_p}\) has up to permutation and up to isomorphic equivalence a unique unconditional basis, namely the one indicated above. Since, as it will be shown later, every greedy basis must be unconditional, the space does not have any greedy basis.

Due to a result in [FDOS] however \((\oplus_{n=1}^{\infty} \ell_p^n)_{\ell_p}\) has a greedy bases if \(1 < p, q < \infty\). More precisely, the following was shown:

Let \(1 < p, q < \infty\).

a) If \(1 < q < \infty\) then the Banach space \((\oplus_{n=1}^{\infty} \ell_p^n)_{\ell_q}\) has a greedy basis.

b) If \(q = 1\) or \(q = \infty\), and \(p \neq q\), then \((\oplus_{n=1}^{\infty} \ell_p^n)_{\ell_q}\) has not a greedy basis. Here we take \(c_0\)-sum if \(q = \infty\).

The question whether or not \(\ell_p(\ell_q)\) has a greedy basis is open and quite an interesting question.

The following result by Wojtaszczyk can be seen the analogue of the characterization of Schauder bases by the uniform boundedness of the canonical projections for quasi-greedy bases.

**Theorem 1.1.4.** [Wo2] A bounded M-basis \((e_i, e_i^*)\), with \(|e_i| = 1\), \(i \in \mathbb{N}\), of a Banach space \(X\) is quasi greedy if and only if there is a constant \(C\) so that for any \(x \in X\) and any \(m \in \mathbb{N}\) it follows that

\[
\|G_m^T(x)\| \leq C\|x\|
\]

We call the smallest constant so that (1.1) is satisfied the Greedy Projection Constant.

**Remark.** Theorem 1.1.4 is basically a uniform boundedness result. Nevertheless, since the \(G_m^T\) are nonlinear projections we need a direct proof.

We need first the following Lemma:

**Lemma 1.1.5.** Assume there is no positive number \(C\) so that \(\|G_m^T(x)\| \leq C\|x\|\) for all \(x \in X\) and all \(m \in \mathbb{N}\). Then the following holds:

For all finite \(A \subset \mathbb{N}\) all \(K > 0\) there is a finite \(B \subset \mathbb{N}\), which is disjoint from \(A\) and a vector \(x\), with \(x = \sum_{j \in B} x_j e_j\), such that \(\|x\| = 1\) and \(\|G_m^T(x)\| \geq K\), for some \(m \in \mathbb{N}\).

**Proof.** For a finite set \(F \subset \mathbb{N}\), define \(P_F\) to be the coordinate projection onto \(\text{span}(e_i : i \in F)\), generated by the \((e_i^*)\), i.e.

\[
P_F : X \rightarrow \text{span}(e_i : i \in F), \quad x \mapsto P_F(x) = \sum_{j \in F} e_j^*(x)e_j.
\]
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Since there are only finitely many subsets of $A$ we can put

$$M = \max_{F \subseteq A} \|P_F\| = \max_{F \subseteq A} \sup_{x \in B_X} \left\| \sum_{j \in F} e_j(x)e_j \right\| \leq \sum_{j \in A} \|e_j\| \cdot \|e_j\| < \infty.$$  

Let $K_1 > 1$ so that $(K_1 - M)/(M+1) > K$, and choose $x_1 \in S_X \cap \text{span}(e_j : j \in \mathbb{N})$ and $k \in \mathbb{N}$ so that $\|G_k^T(x_1)\| \geq K_1$. We assume without loss of generality (after suitable small perturbation) that all the non zero numbers $|e^*_n(x_1)|$ are different from each other.

Then let $x_2 = x_1 - P_A(x_1)$, and note that $\|x_2\| \leq M + 1$ and $G_k^T(x_1) = G_k^T(x_2) + P_F(x_1)$ for some $m \leq k$ and $F \subseteq A$. Thus $\|G_k^T(x_2)\| \geq K_1 - M$, and if we define $x_3 = x_2/\|x_2\|$, we have $\|G_k^T(x_3)\| \geq (K_1 - M)/(M+1) > K$.

It follows that the support $B$ of $x = x_3$ is disjoint from $A$ and that $\|G_k^T(x)\| > K$.

Proof of Theorem 1.1.4. Let $b = \sup_i \|e^*_i\|$.  

"\Rightarrow" Assume there is no positive number $C$ so that $\|G_m^T(x)\| \leq C \|x\|$ for all $x \in X$ and all $m \in \mathbb{N}$.

Applying Lemma 1.1.5 we can choose recursively vectors $y_1, y_2, \ldots$ in $S_X \cap \text{span}(e_j : j \in \mathbb{N})$ and numbers $m_n \in \mathbb{N}$, so that the supports of the $y_n$, which we denote by $B_n$, are pairwise disjoint, (Recall that for $z = \sum_{i=1}^\infty z_i e_i$, we call $\text{supp}(z) = \{i \in \mathbb{N} : e^*_i(z) \neq 0\}$, the support of $z$) and so that

(1.2) \[ \|G_{m_n}^T(y_n)\| \geq 2^n b^n \prod_{j=1}^{n-1} \varepsilon_j^{-1}, \]

where $\varepsilon_j = \min\{2^{-j}, \min\{|e^*_i(y_j)| : i \in B_j\}\}$.

Then we let

$$x = \sum_{n=1}^\infty \left( \prod_{j=1}^{n-1} (\varepsilon_j/b) \right) y_n,$$

(which clearly converges) and write $x$ as

$$x = \sum_{j=1}^\infty x_j e_j.$$

Since $|e^*_i(y_j)| \leq b$, for $i, j \in \mathbb{N}$

$$\min\left\{|x_i| : i \in \bigcup_{j=1}^n B_j\right\} \geq \prod_{j=1}^{n-1} \frac{\varepsilon_j}{b} \varepsilon_n = \prod_{j=1}^n \frac{\varepsilon_j}{b} \geq \max\left\{|x_i| : i \in \mathbb{N} \setminus \bigcup_{j=1}^n B_j\right\}.$$
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We may assume w.l.o.g. that $m_n \leq \#B_n$, for $n \in \mathbb{N}$. Letting $k_j = m_j + \sum_{i=1}^{j-1} \#B_i$, it follows that

$$G_{k_j}^T(x) = \sum_{i=1}^{j-1} \left( \prod_{s=1}^{i-1} (\varepsilon_s/b) \right) y_i + G_{m_j}^T \left( \prod_{i=1}^{j-1} (\varepsilon_i/b) \right) y_j,$$

and thus by (1.2)

$$\|G_{k_j}^T(x)\| \geq \|G_{m_j}^T \left( \prod_{i=1}^{j-1} (\varepsilon_i/b) \right) y_j\| - \sum_{i=1}^{j-1} \left( \prod_{s=1}^{i-1} (\varepsilon_s/b) \right) \|y_i\| \geq 2^j b,$$

which implies that $G_{k_j}^T$ does not converge.

“⇐” Let $C > 0$ such that $\|G_m^T(x)\| \leq C\|x\|$ for all $m \in \mathbb{N}$ and all $x \in X$. Let $x \in X$ and assume w.l.o.g. that supp$(x)$ is infinite. For $\varepsilon > 0$ choose $x_0$ with finite support $A$ so that $\|x - x_0\| < \varepsilon$. Using small perturbations we can assume that $A \subset$ supp$(x)$ and that $A \subset$ supp$(x - x_0)$. We can therefore choose $m \in \mathbb{N}$ large enough so that $G_m^T(x)$ and $G_m^T(x - x_0)$ are of the form

$$G_m^T(x) = \sum_{j \in B} e_j^*(x)e_i \quad \text{and} \quad G_m^T(x - x_0) = \sum_{j \in B} e_j^*(x - x_0)e_i$$

with $B \subset \mathbb{N}$ such that $A \subset B$. It follows therefore that

$$\|x - G_m^T(x)\| \leq \|x - x_0\| + \|x_0 - G_m^T(x)\| = \|x - x_0\| + \|G_m^T(x_0 - x)\| \leq (1 + C)\varepsilon,$$

which implies our claim by choosing $\varepsilon > 0$ to be arbitrarily small.

**Definition 1.1.6.** An $M$ basis $(e_j, e_j^*)$ is called **unconditional for constant coefficients** if there is a positive constant $C$ so that for all finite sets $A \subset \mathbb{N}$ and all $(\sigma_n : n \in A) \subset \{\pm 1\}$ we have

$$\frac{1}{C} \left\| \sum_{n \in A} e_n \right\| \leq \left\| \sum_{n \in A} \sigma_n e_n \right\| \leq C \left\| \sum_{n \in A} e_n \right\|.$$  

**Proposition 1.1.7.** A quasi-greedy $M$ basis $(e_n, e_n^*)$ is unconditional for constant coefficients. Actually the constant in Definition 1.1.6 can be chosen to be equal to twice the projection constant in Theorem 1.1.4.

**Remark.** We will show later that there are quasi-greedy bases which are not unconditional. Actually there are Banach spaces which do not contain any unconditional basic sequence, but in which every normalized weakly null sequence contains a quasi-greedy subsequence.
Proof of Proposition 1.1.7. Let $A \subset \mathbb{N}$ be finite and $(\sigma_n : n \in A) \subset \{\pm 1\}$. Then if we let $\delta \in (0,1)$ and put $m = \# \{j \in A : \sigma_j = +1\}$ we obtain
\[
\left\| \sum_{n \in A, \sigma_n = +1} e_n \right\| = \left\| \sum_{n \in A, \sigma_n = +1} e_n + \sum_{n \in A, \sigma_n = -1} (1 - \delta)e_n \right\| 
\leq C \left\| \sum_{n \in A, \sigma_n = +1} e_n + \sum_{n \in A, \sigma_n = -1} (1 - \delta)e_n \right\|.
\]
By taking $\delta > 0$ to be arbitrarily small, we obtain that
\[
\left\| \sum_{n \in A, \sigma_n = +1} e_n \right\| \leq C \left\| \sum_{n \in A} e_n \right\|.
\]
Similarly we have
\[
\left\| \sum_{n \in A, \sigma_n = -1} e_n \right\| \leq C \left\| \sum_{n \in A} e_n \right\|,
\]
and thus,
\[
\left\| \sum_{n \in A} \sigma_ne_n \right\| \leq 2C \left\| \sum_{n \in A} e_n \right\|.
\]

We now present a characterization of greedy bases obtained by Konyagin and Temliakov. We need the following notation.

Definition 1.1.8. We call a a normalized basic sequence democratic if there is a constant $C$ so that for all finite $E, F \subset \mathbb{N}$, with $\#E = \#F$ it follows that
\[
\left\| \sum_{j \in E} e_j \right\| \leq C \left\| \sum_{j \in F} e_j \right\|
\]
In that case we call the smallest constant, so that (1.3) holds, the Constant of Democracy of $(e_i)$ and denote it by $C_d$.

Theorem 1.1.9. [KT1] A normalized basis $(e_n)$ is greedy if and only it is unconditional and democratic. In this case
\[
\max(C_s, C_d) \leq C_g \leq C_d C_s C_u^2 + C_u,
\]
where $C_u$ is the unconditional constant and $C_s$ is the suppression constant.

Remark. The proof will show that the first inequality is sharp. Recently it was shown in [DOSZ] that the second inequality is also sharp.
Proof of Theorem 1.1.9. “⇒” Let \( x = \sum e_i^*(x)e_i \in X \), \( n \in \mathbb{N} \) and let \( \eta > 0 \). Choose \( \tilde{x} = \sum_{i \in \Lambda_n^*} a_ie_i \) so that \( \#\Lambda_n^* = n \) which is up to \( \eta \) the best \( n \) term approximation to \( x \) (since we allow \( a_i \) to be 0, we can assume that \( \#\Lambda \) is exactly \( n \)), i.e.

\[
\|x - \tilde{x}\| \leq \sigma_n(x) + \eta.
\]

Let \( \Lambda_n \) be a set of \( n \) coordinates for which

\[
b := \min_{i \in \Lambda_n} |e_i^*(x)| \geq \max_{i \notin \Lambda_n} |e_i^*(x)| \text{ and } G_n^T(x) = \sum_{i \in \Lambda_n} e_i^*(x)e_i.
\]

We need to show that

\[
\|x - G_n^T(x)\| \leq (C_dC_uC_s^2 + C_u)(\sigma_n(x) + \eta).
\]

Then

\[
x - G_n^T(x) = \sum_{i \notin \Lambda_n \setminus \Lambda_n} e_i^*(x)e_i = \sum_{i \in \Lambda_n^* \setminus \Lambda_n} e_i^*(x)e_i + \sum_{i \notin \Lambda_n^* \cup \Lambda_n} e_i^*(x)e_i.
\]

But we also have

\[
\left\| \sum_{i \in \Lambda_n^* \setminus \Lambda_n} e_i^*(x)e_i \right\| \leq bC_u \left\| \sum_{i \in \Lambda_n^* \setminus \Lambda_n} e_i \right\| \quad \text{(By Proposition 4.1.11)}
\]

\[
\leq bC_uC_d \left\| \sum_{i \in \Lambda_n^* \setminus \Lambda_n} e_i \right\|
\]

[Note that \( \#(\Lambda_n \setminus \Lambda_n^*) = \#(\Lambda_n^* \setminus \Lambda_n) \)]

\[
\leq C_s^2C_d \left\| \sum_{i \in \Lambda_n^* \setminus \Lambda_n} e_i^*(x)e_i \right\|
\]

[Note that \( e_i^*(x) \geq b \) if \( i \in \Lambda_n \setminus \Lambda_n^* \)]

\[
\leq C_s^2C_d \left\| \sum_{i \in \Lambda_n^*} (e_i^*(x) - a_i)e_i + \sum_{i \notin \Lambda_n^* \cup \Lambda_n} e_i^*(x)e_i \right\|
\]

\[
= C_s^2C_d \|x - \tilde{x}\| \leq C_s^2C_d(\sigma_n(x) + \eta)
\]

and

\[
\left\| \sum_{i \notin \Lambda_n^* \cup \Lambda_n} e_i^*(x)e_i \right\| \leq C_s \left\| \sum_{i \in \Lambda_n^*} (e_i^*(x) - a_i)e_i + \sum_{i \notin \Lambda_n^*} e_i^*(x)e_i \right\|
\]

\[
(1.8) \quad = C_s \|x - \tilde{x}\| \leq C_s(\sigma_n(x) + \eta).
\]

This shows that \((e_i)\) is greedy and, since \( \eta > 0 \) is arbitrary, we deduce that \( C_g \leq C_sC_u^2C_d + C_s \).
“⇒” Assume that \((e_i)\) is greedy. In order to show that \((e_i)\) is democratic let \(\Lambda_1, \Lambda_2 \subset \mathbb{N}\) with \(#\Lambda_1 = #\Lambda_2\). Let \(\eta > 0\) and put \(m = #(\Lambda_2 \setminus \Lambda_1)\) and

\[
x = \sum_{i \in \Lambda_1} e_i + (1+\eta) \sum_{i \in \Lambda_2 \setminus \Lambda_1} e_i.
\]

Then it follows

\[
\left\| \sum_{i \in \Lambda_1} e_i \right\| = \left\| x - G_m^T(x) \right\|
\]

\[
\leq C_g \sigma_m(x) \quad \text{(since \((e_i)\) is } C_g\text{-greedy)}
\]

\[
\leq C_g \left\| x - \sum_{i \in \Lambda_1 \setminus \Lambda_2} e_i \right\| \leq C_g \left\| \sum_{i \in \Lambda_1 \cap \Lambda_2} e_i + (1+\eta) \sum_{i \in \Lambda_2 \setminus \Lambda_1} e_i \right\|.
\]

Since \(\eta > 0\) can be taken arbitrary, we deduce that

\[
\left\| \sum_{i \in \Lambda_1} e_i \right\| \leq C_g \left\| \sum_{i \in \Lambda_2} e_i \right\|.
\]

Thus, it follows that \((e_i)\) is democratic and \(C_d \leq C_g\).

In order to show that \((e_i)\) is unconditional let \(x = \sum e_i^*(x) e_i \in X\) have finite support \(S\). Let \(\Lambda \subset S\) and put

\[
y = \sum_{i \in \Lambda} e_i^*(x) e_i + b \sum_{i \in S \setminus \Lambda} e_i,
\]

with \(b > \max_{i \in S} \left| e_i^*(x) \right|\). For \(n = #(S \setminus \Lambda)\) it follows that

\[
G_n^T(y) = b \sum_{i \in S \setminus \Lambda} e_i,
\]

and since \((e_i)\) is greedy we deduce that (note that \#supp\((y-x)\) = \(n\))

\[
\left\| \sum_{i \in \Lambda} e_i^*(x) e_i \right\| = \left\| y - G_n^T(y) \right\| \leq C_g \sigma_n(y) \leq C_g \left\| y - (y-x) \right\| = C_g \left\| x \right\|,
\]

which implies that \((e_i)\) is unconditional with \(C_s \leq C_g\).

\[\square\]

1.2 The Haar basis is greedy in \(L_p[0,1]\) and \(L_p(\mathbb{R})\)

Theorem 1.2.1. For \(1 < p < \infty\) there are two constants \(c_p \leq C_p\), depending only on \(p\), so that for all \(n \in \mathbb{N}\) and all \(A \subset T\) with \(#A = n\)

\[
c_p n^{1/p} \leq \left\| \sum_{t \in A} h_t^{(p)} \right\| \leq C_p n^{1/p}.
\]

In particular \((h_t^{(p)})_{t \in T}\) is democratic in \(L_p[0,1]\).