2.2 Annihilators, complemented subspaces

Definition 2.2.1. [Annihilators, Pre-Annihilators]
Assume $X$ is a Banach space. Let $M \subset X$ and $N \subset X^*$. We call

$$M^\perp = \{ x^* \in X^* : \forall x \in M \langle x^*, x \rangle = 0 \} \subset X^*,$$

the annihilator of $M$ and

$$N_\perp = \{ x \in X : \forall x^* \in N \langle x^*, x \rangle = 0 \} \subset X,$$

the pre-annihilator of $N$.

Proposition 2.2.2. Let $X$ be a Banach space, and assume $M \subset X$ and $N \subset X^*$.

a) $M^\perp$ is a closed subspace of $X^*$, $M^\perp = (\text{span}(M))^\perp$, and $(M^\perp)_\perp = \text{span}(M)$,

b) $N_\perp$ is a closed subspace of $X$, $N_\perp = (\text{span}(N))_\perp$, and $\text{span}(N) \subset (N_\perp)^\perp$.

c) $\text{span}(M) = X \iff M^\perp = \{ 0 \}$.

Proposition 2.2.3. If $X$ is Banach space an $Y \subset X$ a closed subspace then $(X/Y)^*$ is isometrically isomorphic to $Y^\perp$ via the operator

$$\Phi : (X/Y)^* \to Y^\perp,$$

(recall $\pi := x + Y \in X/Y$ for $x \in X$).

Proof. Let $Q : X \to X/Y$ be the quotient map

For $z^* \in (X/Y)^*$, $\Phi(z^*)$, as defined above, can be written as $\Phi(z^*) = z^* \circ Q$ thus $\Phi(z^*) \in X^*$ since $Q(Y) = \{ 0 \}$ it follows that $\Phi(z^*) \in Y^\perp$. For $z^* \in (X/Y)^*$ we have

$$\|\Phi(z^*)\| = \sup_{x \in B_X} \langle z^*, Q(x) \rangle = \sup_{\pi \in B_{X/Y}} \langle z^*, \pi \rangle = \|z^*\|_{(X/Y)^*},$$

where the second equality follows on the one hand from the fact that $\|Q(x)\| \leq \|x\|$ for $x \in X$, and on the other hand, from the fact that for any $\pi = x + Y \in X/Y$ there is a sequence $(y_n) \subset Y$ so that $\limsup_{n \to \infty} \|x + y_n\| \leq 1$.

Thus $\Phi$ is an isometric embedding. If $x^* \in Y^\perp \subset X^*$, we define

$$z^* : X/Y \to \mathbb{K}, \quad x + Y \mapsto \langle x^*, x \rangle.$$
2.2. ANNIHILATORS, COMPLEMENTED SUBSPACES

First note that this map is well defined (since \( \langle x^*, x + y \rangle = \langle x^*, x \rangle \) for \( y_1, y_2 \in Y \)). Since \( x^* \) is linear, \( z^* \) is also linear, and \( |\langle z^*, \bar{x} \rangle| = |\langle x^*, x \rangle| \), for all \( x \in X \), and thus \( \|z^*\|_{(X/Y)^*} = \|x^*\| \). Finally, since \( \langle \Phi(z^*), x \rangle = \langle z^*, Q(x) \rangle = \langle x^*, x \rangle \), it follows that \( \Phi(x^*) = x^* \), and thus that \( \Phi \) is surjective.

**Proposition 2.2.4.** Assume \( X \) and \( Y \) are Banach spaces and \( T \in L(X, Y) \).

Then

\[
T(X) \perp = \mathcal{N}(T^*) \quad \text{and} \quad T^*(Y^*) \subset \mathcal{N}(T) \perp
\]

\[
\overline{T(X)} = \mathcal{N}(T^*)^\perp \quad \text{and} \quad T^*(Y^*)^\perp \subset \mathcal{N}(T).
\]

**Proof.** We only prove (2.1). The verification of (2.1) is similar. For \( y^* \in Y^* \)

\[
y^* \in T(X) \perp \iff \forall x \in X \quad \langle y^*, T(x) \rangle = 0
\]

\[
\iff \forall x \in X \quad \langle T^*(y^*), x \rangle = 0
\]

\[
\iff T^*(y^*) = 0 \iff y^* \in \mathcal{N}(T^*),
\]

which proves the first part of (2.1), and for \( y^* \in Y^* \) and all \( x \in \mathcal{N}(T) \), it follows that \( \langle T^*(y^*), x \rangle = \langle y^*, T(x) \rangle = 0 \), which implies that \( T^*(Y^*) \subset \mathcal{N}(T) \perp \), and, thus, \( T^*(X^*) \subset \mathcal{N}(T) \).

**Definition 2.2.5.** Let \( X \) be a Banach space and let \( U \) and \( V \) be two closed subspaces of \( X \). We say that \( X \) is the complemented sum of \( U \) and \( V \) and we write \( X = U \oplus V \), if for every \( x \in X \) there are \( u \in U \) and \( v \in V \), so that \( x = u + v \) and so that this representation of \( x \) as sum of an element of \( U \) and an element of \( V \) is unique.

We say that a closed subspace \( Y \) of \( X \) is complemented in \( X \) if there is a closed subspace \( Z \) of \( X \) so that \( X = Y \oplus Z \).

**Remark.** Assume that the Banach space \( X \) is the complemented sum of the two closed subspaces \( U \) and \( V \). We note that this implies that \( U \cap V = \{0\} \).

We can define two maps

\[
P : X \to U \quad \text{and} \quad Q : X \to V
\]

where we define \( P(x) \in U \) and \( Q(x) \in V \) by the equation \( x = P(x) + Q(y) \), with \( P(x) \in U \) and \( Q(x) \in V \) (which, by assumption, has a unique solution).

Note that \( P \) and \( Q \) are linear. Indeed if \( P(x_1) = u_1, P(x_2) = u_2, Q(x_1) = v_1, Q(x_2) = v_2 \), then for \( \lambda, \mu \in \mathbb{K} \) we have \( \lambda x_1 + \mu x_2 = \lambda u_1 + \mu u_2 + \lambda v_1 + \mu v_2 \), and thus, by uniqueness \( P(\lambda x_1 + \mu x_2) = \lambda u_1 + \mu u_2 \), and \( Q(\lambda x_1 + \mu x_2) = \lambda v_1 + \mu v_2 \).

Secondly it follows that \( P \circ P = P \), and \( Q \circ Q = Q \). Indeed, for any \( x \in X \) we write \( P(x) = P(x) + 0 \in U + V \), and since this representation
of $P(x)$ is unique it follows that $P(P(x)) = P(x)$. The argument for $Q$ is the same.

Finally it follows that, again using the uniqueness argument, that $P$ is the identity on $U$ and $Q$ is the identity on $V$.

We therefore proved that

a) $P$ is linear,

b) the image of $P$ is $U$

c) $P$ is idempotent, i.e. $P^2 = P$

We say in that case that $P$ is a linear projection onto $U$. Similarly $Q$ is a linear projection onto $V$, and $P$ and $Q$ are complementary to each other., meaning that $P(X) \cap Q(X) = \{0\}$ and $P+Q = Id$. A linear map $P : X \to X$ with the properties (a) and (c) is called projection.

The next Proposition will show that $P$ and $Q$ as defined in above remark are actually bounded.

**Lemma 2.2.6.** Assume that $X$ is the complemented sum of two closed subspaces $U$ and $V$. Then the projections $P$ and $Q$ as defined in above remark are bounded.

**Proof.** Consider the norm $\| \cdot \|$ on $X$ defined by

$$\|x\| = \|P(x)\| + \|Q(x)\|, \text{ for } x \in X.$$  

We claim that $(X, \|\cdot\|)$ is also a Banach space. Indeed if $(x_n) \subset X$ with

$$\sum_{n=1}^{\infty} \|x_n\| = \sum_{n=1}^{\infty} \|P(x_n)\| + \sum_{n=1}^{\infty} \|Q(x_n)\| < \infty.$$  

Then $u = \sum_{n=1}^{\infty} P(x_n) \in U$, $v = \sum_{n=1}^{\infty} Q(x_n) \in V$ ($U$ and $V$ are assumed to be closed) converge in $U$ and $V$, respectively, and since $\| \cdot \| \leq \| \cdot \|$ also $x = \sum_{n=1}^{\infty} x_n$ converges and

$$x = \sum_{n=1}^{\infty} x_n = \lim_{n \to \infty} \sum_{j=1}^{n} P(x_n) + Q(x_n) = \lim_{n \to \infty} \sum_{j=1}^{n} P(x_n) + \lim_{n \to \infty} \sum_{j=1}^{n} Q(x_n) = u + v,$$

and

$$\|x - \sum_{j=1}^{n} x_n\| = \|u - \sum_{j=1}^{n} P(x_n) + v - \sum_{j=1}^{n} Q(x_n)\|.$$
2.2. ANNIHILATORS, COMPLEMENTED SUBSPACES

\[ = \left\| u - \sum_{j=1}^{n} P(x_n) \right\| + \left\| v - \sum_{j=1}^{n} Q(x_n) \right\| \rightarrow_{n \to \infty} 0, \]

which proves that \((X, \| \cdot \|)\) is complete.

Since the identity is a bijective linear bounded operator from \((X, \| \cdot \|)\) to \((X, \| \cdot \|)\) it has by Corollary 1.3.6 of the Closed Graph Theorem a continuous inverse and is thus an isomorphy. Since \(\|P(x)\| \leq \|x\|\) and \(\|Q(x)\| \leq \|x\|\) we deduce our claim. \(\square\)

**Proposition 2.2.7.** Assume that \(X\) is a Banach space and that \(P : X \to X\)

is a bounded projection onto a closed subspace of \(X\).

Then \(X = P(X) \oplus \mathcal{N}(P)\).

**Theorem 2.2.8.** There is no linear bounded operator \(T : \ell_\infty \to \ell_\infty\) so that the kernel of \(T\) equals to \(c_0\).

**Corollary 2.2.9.** \(c_0\) is not complemented in \(\ell_\infty\).

**Proof of Theorem 2.2.10.** For \(n \in \mathbb{N}\) we let \(e_n^*\) be the \(n\)-th coordinate functional on \(\ell_\infty\), i.e.

\[ e_n^* : \ell_\infty \to \mathbb{K}, \quad x = (x_j) \mapsto x_n. \]

Step 1. If \(T : \ell_\infty \to \ell_\infty\) is bounded and linear, then

\[ \mathcal{N}(T) = \bigcap_{n=1}^{\infty} \mathcal{N}(e_n^* \circ T). \]

Indeed, note that

\[ x \in \mathcal{N}(T) \iff \forall n \in \mathbb{N} \quad e_n^*(T(x)) = (e_n^*, T(x)) = 0. \]

In order to prove our claim we will show that \(c_0\) cannot be the intersection of the kernel of countably many functionals in \(\ell_\infty^*\).

Step 2. There is an uncountable family \((N_\alpha : \alpha \in I)\) of infinite subsets of \(\mathbb{N}\) for which \(N_\alpha \cap N_\beta\) is finite whenever \(\alpha \neq \beta\) are in \(I\).

Write the rationales \(Q\) as a sequence \((q_j : j \in \mathbb{N})\), and choose for each \(r \in \mathbb{R}\) a sequence \((n_k(r) : k \in \mathbb{N})\), so that \((g_{n_k(r)} : k \in \mathbb{N})\) converges to \(r\). Then, for \(r \in \mathbb{R}\) let \(N_r = \{n_k(r) : k \in \mathbb{N}\}\).

Step 3. For \(i \in I\), put \(x_\alpha = 1_{N_\alpha} \in \ell_\infty\), i.e.

\[ x_\alpha = (\xi^{(\alpha)}_k : k \in \mathbb{N}) \text{ with } \xi_k^{(\alpha)} = \begin{cases} 1 & \text{if } k \in N_\alpha, \\ 0 & \text{if } k \not\in N_\alpha. \end{cases} \]
CHAPTER 2. WEAK TOPOLOGIES, REFLEXIVITY, ADJOINT OPERATORS

If $f \in \ell_\infty^*$ and $c_0 \subset \mathcal{N}(f)$ then $\{ \alpha \in I : f(x_\alpha) \neq 0 \}$ is countable.

In order to verify Step 3 let $A_n = \{ \alpha : |f(x_\alpha)| \geq 1/n \}$, for $n \in \mathbb{N}$. It is enough to show that for $n \in \mathbb{N}$ the set $A_n$ is finite. To do so, let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be distinct elements of $A_n$ and put $x = \sum_{j=1}^k \text{sign}(f(x_{\alpha_j})) x_{\alpha_j}$ (for $a \in \mathbb{C}$ we put sign($a$) = $a/|a|$) and deduce that $f(x) \geq k/n$. Now consider $M_j = N_{\alpha_j} \setminus \bigcup_{i \neq j} N_{\alpha_i}$. Then $N_{\alpha_j} \setminus M_j$ is finite, and thus it follows for

$$\tilde{x} = \sum_{j=1}^k \text{sign}(f(x_{\alpha_j})) 1_{M_j}$$

that $f(x) = f(\tilde{x})$ (since $x - \tilde{x} \in c_0$). Since the $M_j$, $j = 1, 2 \ldots k$ are pairwise disjoint, it follows that $\|\tilde{x}\|_{\infty} = 1$, and thus

$$\frac{k}{n} \leq f(x) = f(\tilde{x}) \leq \|f\|.$$  

Which implies that that $A_n$ can have at most $n\|f\|$ elements.

Step 4. If $c_0 \subset \bigcap_{n=1}^{\infty} \mathcal{N}(f_n)$, for a sequence $(f_n) \subset \ell_\infty^*$, then there is an $\alpha \in I$ so that $x_\alpha \in \bigcap_{n=1}^{\infty} \mathcal{N}(f_n)$. In particular this implies that $c_0 \neq \bigcap_{n \in \mathbb{N}} \mathcal{N}(f_n)$.

Indeed, from Step 2 it follows that

$$C = \{ \alpha \in I : f_n(x_\alpha) \neq 0 \text{ for some } n \in \mathbb{N} \} = \bigcup_{n \in \mathbb{N}} \{ \alpha \in I : f_n(x_\alpha) \neq 0 \},$$

is countable, and thus $I \setminus C$ is not empty. 

\[\square\]

Remark. Assume that $Z$ is any subspace of $\ell_\infty$ which is isomorphic to $c_0$, then $Z$ is not complemented. The proof of that is a bit harder.

Theorem 2.2.10. \[So\] Assume $Y$ is a subspace of a separable Banach space $X$ and $T : Y \to c_0$ is linear and bounded. Then $T$ can be extended to a linear and bounded operator $\hat{T} : X \to c_0$. Moreover, $\hat{T}$ can be chosen so that $\|\hat{T}\| \leq 2\|T\|$.

Corollary 2.2.11. Assume that $X$ is a separable Banach space which contains a subspace $Y$ which is isomorphic to $c_0$. Then $Y$ is complemented in $X$.

Proof. Let $T : Y \to c_0$ be an isomorphism. Then extend $T$ to $\hat{T} : X \to c_0$ and put $P = \hat{T} \circ T^{-1}$.

\[\square\]
2.2. ANNIHILATORS, COMPLEMENTED SUBSPACES

Proof of Theorem 2.2.10. Note an operator $T : Y \to c_0$ is defined by a $\sigma(Y^*, Y)$ null sequence $(y_n^*) \subset Y^*$, i.e.

$$T : Y \to c_0, \quad y \mapsto (\langle y_n^*, y \rangle : n \in \mathbb{N}).$$

We would like to use the Hahn Banach theorem and extend each $y^*n$ to an element $x_n^* \in X_n^*$, with $\|y_n^*\| = \|x_n^*\|$, and define

$$\tilde{T}(x) := (\langle x_n^*, x \rangle : n \in \mathbb{N}), \quad x \in X.$$

But the problem is that $(x_n^*)$ might not be $\sigma(X^*, X)$ convergent to $j$, and thus we can only say that $(\langle x_n^*, x \rangle : n \in \mathbb{N}) \in \ell_\infty$, but not necessarily in $c_0$.

Thus we will need to change the $x_n^*$ somehow so that they are still extensions of the $y_n^*$ but also $\sigma(X^*, X)$ null.

Let $B = \|T\| B_{X^*}$. $B$ is $\sigma(X^*, X)$-compact and metrizable (since $X$ is separable). Denote the metric which generates the $\sigma(X^*, X)$-topology by $d(\cdot, \cdot)$. Put $K = B \cap Y^\perp$. Since $Y^\perp \subset X^*$ is $\sigma(X^*, X)$-closed, $K$ is compact and every $\sigma(X^*, X)$-accumulation point of $(x_n^*)$ lies in $K$. Indeed, this follows from the fact that $x_n^*(y) = y_n^*(y) \to n \to \infty 0$. This implies that $\lim_{n \to \infty} d(x_n^*, K) = 0$, thus we can choose $(z_n^*) \subset K$ so that $\lim_{n \to \infty} d(x_n^*, z_n^*) = 0$, and thus $(x_n^* - z_n^*)$ is $\sigma(X^*, X)$-null and for $y \in Y$ it follows that $\langle x_n^* - z_n^*, y \rangle = \langle x_n^*, y \rangle$, $n \in \mathbb{N}$. Choosing therefore

$$\tilde{T} : \to c_0, \quad x \mapsto ((x_n^* - z_n^*, x) : n \in \mathbb{N}),$$

yields our claim. \qed

Remark. Zippin [Zi] proved the converse of Theorem: if $Z$ is an infinite-dimensional separable Banach space admitting a projection from any separable Banach space $X$ containing it, then $Z$ is isomorphic to $c_0$.

Exercises

1. Prove Proposition 2.2.2.

2. a) Assume that $\ell_\infty$ isomorphic to a subspace $Y$ of some Banach space $X$, then $Y$ is complemented in $X$.

   b) Assume $Z$ is a closed subspace of a Banach space $X$, and $T : Z \to \ell_\infty$ is linear and bounded. Then $T$ can be extended to a linear and bounded operator $\tilde{T} : X \to \ell_\infty$, with $\|\tilde{T}\| = \|T\|$.

3. Show that for a Banach space $X$, the dual space $X^*$ is isometrically isomorphic to complemented subspace of $X^{***}$, via the canonical embedding.
4. Prove Proposition 2.2.7.

5. Prove (2.2) in Proposition 2.2.4.