

1.3: Linear Functions and Mathematical Models

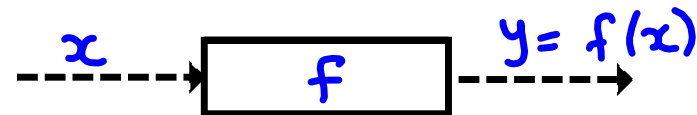
1.4: Intersection of Straight Lines

- Linear Function

DEFINITION 1. A function, f , is a rule that assigns to each value of x one and only one value of y :

$$y = f(x).$$

The variable x is referred to as the independent variable (the "input"). The variable y is referred to as the dependent variable (the "output").



$$\checkmark y = 2x - 4$$

$$\checkmark y = 3x$$

$$\checkmark y = 0$$

$$\checkmark y = 5$$

$$\cancel{y} = x^2 + x$$

$$\cancel{y} = \frac{5}{x} - 16$$

$$\cancel{y} = 4 - \sqrt{x}$$

$$\checkmark y = x + 1$$

$$\cancel{y} = x + 1 - x^2$$

We will use linear functions (those that can be represented graphically by a line) to create linear models.

DEFINITION 2. *The function f defined by*

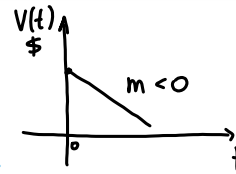
$$y = f(x) = mx + b$$

where m and b are constants, is called a linear function.

• Linear Depreciation:

- An item has an initial value and a final value.
- Assume the value decreases linearly with time:

$$V(t) = mt + b.$$

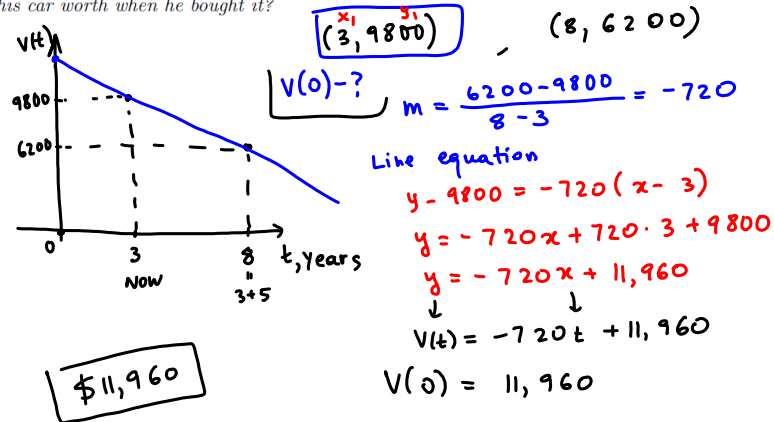


To solve linear depreciation problems:

find points → find equation of line → answer question.

$(t, V(t))$
 ↑ ↑
 time \$

EXAMPLE 3. Bob bought a car three years ago that is currently valued at \$9,800. Five years from now it will be worth \$6200. Assuming it is linearly depreciating, how much was his car worth when he bought it?



• **Linear Cost, Revenue, and Profit Functions**

x = # items
 c = item cost
 s = item selling price
 F = fixed costs.

– The total cost function is

$C(x)$ = total cost of producing x units of the product.

$C(x) = cx + F$, where

- * slope c represents the cost of making one additional unit of the product;
- * variable costs, cx , depend on the number of units produced;
- * F represents the fixed costs that are independent of the number of units produced.

– The revenue function is

$R(x)$ = total revenue (all the money brought in) from the sale of x units of the product.

$R(x) = sx$, where s is the selling price per unit.

– The profit function is

$P(x)$ = total profit realized from manufacturing and selling x units of the product.

$P(x) = R(x) - C(x)$ = Revenue - Cost = $(s - c)x - F$

EXAMPLE 4. A manufacturer has a monthly fixed cost of \$100,000 and a production cost of \$14 for each unit produced. The product sells for \$20/unit.

(a) What is the cost function?

$$C(x) = cx + F = 14x + 100,000$$

$$C(x) = 14x + 100,000$$

$$F = 100,000$$

$$c = 14$$

$$s = 20$$

(b) What is the revenue function?

$$R(x) = sx = 20x$$

(c) What is the profit function?

$$P(x) = R - C = 20x - (14x + 100,000)$$

$$P(x) = 6x - 100,000$$

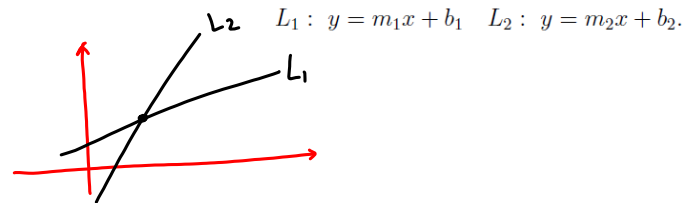
(d) Compute the profit (loss) corresponding to production levels of $\overbrace{12,000}$ and $\overbrace{20,000}$ units.

$$P(12,000) = 6 \cdot 12,000 - 100,000 = -28,000 < 0 \text{ loss}$$

$$P(20,000) = 6 \cdot 20,000 - 100,000 = 20,000 > 0 \text{ makes a profit}$$

• Finding the point of intersection:

Given two lines:



The point of intersection of two lines is the point at which those two lines are equal to each other. To find this point

- algebraically - solve the system of equations composed of the two equations.
- graphically - solve both equations for y. Put into your calculator under Y1 and Y2. Press 2nd TRACE and #5 is INTERSECT.

EXAMPLE 5. Find the point of intersection of $y = \frac{2}{3}x - 4$ and $y = -\frac{1}{3}x - 1$ both algebraically and graphically.

$$\begin{cases} y = \frac{2}{3}x - 4 \\ y = -\frac{1}{3}x - 1 \end{cases} \Rightarrow \frac{2}{3}x - 4 = -\frac{1}{3}x - 1$$

$\frac{2}{3}x + \frac{1}{3}x = -1 + 4$
 $x = 3$

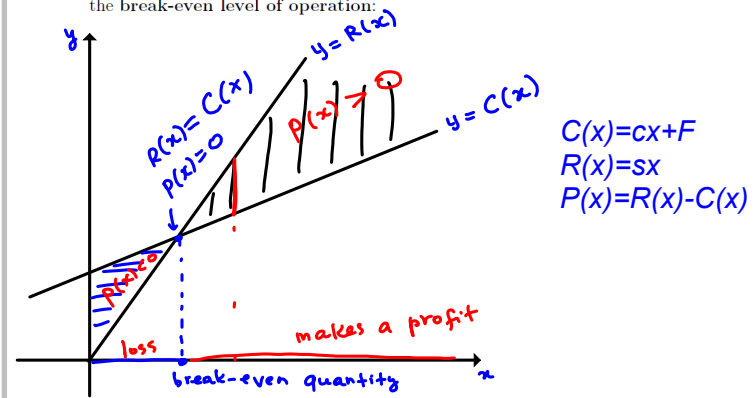
$y = -\frac{1}{3} \cdot 3 - 1 = -1 - 1 = -2$

(3, -2)



- Break-Even Analysis:

The level of production at which the firm neither makes a profit nor sustains a loss is called the break-even level of operation:



EXAMPLE 6. A company makes a wine cooler with capacity of 24 bottles. Each wine cooler sells for \$245. The monthly fixed costs incurred by the company are \$393,600, and the variable cost of producing each wine cooler is \$85.

(a) Find the Cost, Revenue and Profit functions.

$$C(x) = 85x + 393,600$$

$$R(x) = 245x$$

$$P(x) = 245x - 85x - 393,600$$

$$P(x) = 160x - 393,600$$

$$S = 245$$

$$F = 393,600$$

$$C = 85$$

$$x = \# \text{ coolers}$$

Using Calculator

$$Y1 = 85x + 393600$$

$$Y2 = 245x$$

(b) What is the break-even point for this company?

$$0 = P(x) = 160x - 393,600$$

$$160x = 393,600 \Rightarrow x = \frac{393,600}{160} = 2460$$

$$R(2460) = 245 \cdot 2460 = 602,700$$

$$\left(\underbrace{2460}_{\text{coolers}} ; \underbrace{602,700}_{\$} \right)$$

(c) How many coolers should be sold when the company breaks even?

2460 coolers

EXAMPLE 7. Mark manages a heaters store which has a monthly rent \$900. If Mark needs to sell 45 heaters each month to break even and he sells the heaters for \$30, what does Mark pay for each heater?

$$F = 900$$

$$s = 30$$

$$P(45) = 0$$

$$c = ?$$

$$\text{or } \begin{cases} C(45) = R(45) \\ C(x) = cx + 900 \\ R(x) = 30x \end{cases} \Rightarrow \begin{matrix} C(45) = 45c + 900 \\ R(45) = 30 \cdot 45 \end{matrix}$$

$$45c + 900 = 30 \cdot 45 \quad \left(\times \frac{1}{45} \right)$$

$$c + 20 = 30$$

$$c = 30 - 20 = 10$$

$\boxed{\$10}$

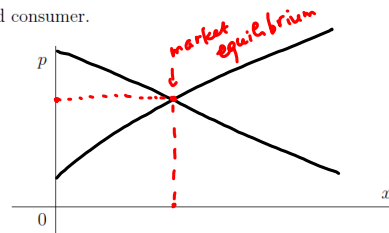
- **Linear Demand and Supply Curves** The demand function models the relationship between the price p per item and the number of items x purchased (demanded) by consumers:

$$D(x) : p = mx + b$$

The supply function models the relationship between the price p per item and the number of items x supplied to the market"

$$S(x) : p = mx + b.$$

Market equilibrium is represented by the point of intersection on the graph of the supply and demand equations. It shows the price and number of items that satisfy both the producer and consumer.



Market Equilibrium Point where the x-value is the equilibrium quantity and the y-value is the equilibrium price

$(x, p) \rightarrow$ price
#items

EXAMPLE 8. When a digital voice recorder is priced at \$40, then 200 sell. If the price increases by \$10, then 150 sell. The producer is willing to provide 240 recorders when the price is \$160 and is willing to provide 120 recorders when the price is \$88. Find

(a) the supply equation;

(x, p)
 $(240, 160)$
 $(120, 88)$

$$m = \frac{88 - 160}{120 - 240} = 0.6$$

$$p - 160 = 0.6(x - 240)$$

$$p = 160 = 0.6x - 144 \quad \boxed{p = 0.6x + 16}$$

$$p = 0.6x - 144 + 160$$

(b) the demand equation;

(x, p)
 $(200, 40)$
 $(150, 50)$

$$m = \frac{50 - 40}{150 - 200} = -0.2$$

$$p - 40 = -0.2(x - 200) \quad \boxed{p = -0.2x + 80}$$

$$p = -0.2x + 40 + 40$$

(c) the equilibrium point.

Calculator

$$Y_1 = 0.6x + 16$$

$$Y_2 = -0.2x + 80$$

$$0.6x + 16 = -0.2x + 80$$

$$0.6x + 0.2x = 80 - 16$$

$$0.8x = 64$$

$$x = 80$$

$$p = -0.2 \cdot 80 + 80 = 64$$

$(x, p) = \boxed{(80, 64)}$

EXAMPLE 9. A company will not market any gadget if the price per item is \$25 or less. If the price is \$35 per item, they will make 100 gadgets available per month.

(a) Find the supply function.

$$(x, p) \quad m = \frac{35 - 25}{100 - 0} = 0.1$$

$$\begin{array}{l} (0, 25) \\ (100, 35) \end{array} \quad \underline{p = 0.1x + 25}$$

(b) The monthly demand function for the gadgets is given by $x + 20p = 740$. What is the equilibrium quantity and what is the equilibrium price?

$$x + 20p = 740 \Rightarrow 20p = 740 - x \Rightarrow p = \frac{740}{20} - \frac{x}{20}$$
$$\boxed{\begin{array}{l} \text{y1: } p = 0.1x + 25 \\ \text{y2: } p = 37 - \frac{x}{20} \end{array}} \Rightarrow (x, p) = (80, 33)$$

eq. quantity
↓
equil.
price