2.2: Systems of Linear Equations: Unique Solution

2.3: Systems of Linear Equations: Undetermined and Overdetermined Systems

- **The Gauss-Jordan elimination method** The Gauss-Jordan Method is a suitable technique for solving systems of linear equations of any size.

  A sequence of operations (see below) of the Gauss-Jordan elimination method allows us to obtain at each step an equivalent system - that is, a system having the same solution as the original system.

The operations of the Gauss-Jordan elimination method are

1. Interchange any two equations. $E_i \leftrightarrow E_k$
2. Replace an equation by a nonzero multiple of itself. $E_k \leftrightarrow c\,E_k$
3. Replace an equation by itself plus a nonzero multiple of any other equation.

$E_k \leftrightarrow E_k + c\,E_i$

EXAMPLE 1. *Solve the system of linear equations:* $\begin{cases} 3x - 2y = -3 & E_1 \\ x + 2y = 7 & E_2 \end{cases}$

using Gauss-Jordan elimination method:

$E_1 \leftrightarrow E_2$

$\begin{cases} x + 2y = 7 & E_1 \\ 3x - 2y = -3 & E_2 \end{cases}$

$E_2 \leftrightarrow E_2 + (-3)E_1$

$\begin{cases} x + 2y = 7 & E_1 \\ -8y = -24 & E_2 \end{cases}$

$E_2 \leftrightarrow -\frac{1}{8}E_2$

$\begin{cases} x + 2y = 7 & E_1 \\ y = 3 & E_2 \end{cases}$

$E_1 \leftrightarrow E_1 + (-2)E_2$

$\boxed{\begin{matrix} x = 1 \\ y = 3 \end{matrix}}$

using augmented matrix

$\begin{array}{c} \left[\begin{array}{cc|c} 3 & -2 & -3 \\ 1 & 2 & 7 \end{array}\right] \end{array} \begin{array}{c} R_1 \\ R_2 \end{array}$

$R_1 \leftrightarrow R_2$

$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 3 & -2 & -3 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \end{array}$

$R_2 \leftrightarrow R_2 + (-3)R_1$

$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & -8 & -24 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \end{array}$

$R_2 \leftrightarrow -\frac{1}{8}R_2$

$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 3 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \end{array}$

$R_1 \leftrightarrow R_1 + (-2)R_2$

$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \end{array}\right]$

$AX=B$

- **Augmented Matrices** $[A|B]$

An augmented matrix that is formed by combining the coefficient matrix (A) and the constant matrix (B) as shown in the next example.

EXAMPLE 2. *Find the initial augmented matrix for the system of linear equations below:*

(a) $\begin{cases} 3x_1 + 12x_2 = 20 \\ 2x_2 = x_1 + 7 \end{cases} \Rightarrow \begin{cases} 3x_1 + 12x_2 = 20 \\ -x_1 + 2x_2 = 7 \end{cases}$ $\left[\begin{array}{cc|c} 3 & 12 & 20 \\ -1 & 2 & 7 \end{array}\right]$

(b) $\begin{cases} 3x - 2y + 5z = 100 \\ x + z = 7 \\ y - z = 11 \end{cases}$

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\left[\begin{array}{ccc|c} 3 & -2 & 5 & 100 \\ 1 & 0 & 1 & 7 \\ 0 & 1 & -1 & 11 \end{array}\right]$$

(c) $\begin{cases} x = 1 \\ y = 12 \\ z = 5 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

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The goal of the Gauss-Jordan Elimination Method is to get the augmented matrix into Row-Reduced Echelon Form (rref).

Row-Reduced Echelon Form of Matrix (note that all rules apply to the coefficient part only.):

(I) The 1st nonzero entry in each row is a 1, called the **leading 1**.

(II) The leading 1 in a lower row must lie to the right of the leading 1 in a upper row.

(III) A row made entirely of zeros lies below any other row. (for coeff. matrix)

(IV) A column containing a leading 1 must be a **unit column**, that is, all other entries in the column are zeros.

EXAMPLE 3. *Indicate whether the matrix is in row reduced form:*

unit column

(a) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 12 \\ 0 & 1 & 3 & 0 & -5 \\ 0 & 0 & 0 & 0 & 11 \end{array}\right]$ RREF

(b) ? $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & 0 & 11 \\ 0 & 1 & 7 & 0 & 11 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ no RREF (III)

(c) $\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{array}\right]$ RREF

(d) $\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$? no rref (I)

(e) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -11 \end{array}\right]$ RREF

(f) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 1 & 0 & 11 \end{array}\right]$ no RREF (II)

(g) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 15 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -10 \end{array}\right]$ no RREF (2)

(h) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 15 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -10 \end{array}\right]$ RREF

To put a matrix in Row Reduced Form, there are three valid Row Operations:

1. Interchange any two rows ($R_i \leftrightarrow R_j$).
2. Replace any row by a nonzero constant multiple of itself (cR_i). $R_i \leftrightarrow cR_i$
3. Replace any row by the sum of that row and a constant multiple of any other row $R_i \leftrightarrow (R_i + cR_j)$.

   – The process in which a column transforms into a unit column is called pivoting.

   – How to pivot:

     1. Use ROW OPERATIONS to convert the circled pivot element into a 1.
     2. Use ROW OPERATIONS to convert the other elements in the column into 0s.

The Gauss-Jordan method is to use pivot operations to work column by column until the matrix is in reduced form. The solution(s) can then be reached from the corresponding system.

EXAMPLE 4. *Solve:*

$$\begin{aligned} 2x + 4y + 2z &= -4 \\ 3x + 8y - z &= -18 \\ 2x + y + 5z &= 8 \end{aligned}$$

pivot

augmented matrix

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 2 & 4 & 2 & -4 \\ 3 & 8 & -1 & -18 \\ 2 & 1 & 5 & 8 \end{array}\right] \xrightarrow{R_1 \leftrightarrow \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 3 & 8 & -1 & -18 \\ 2 & 1 & 5 & 8 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 0 & 2 & -4 & -12 \\ 2 & 1 & 5 & 8 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_3 - 2R_1}$$

$\boxed{3 \times 4}$

unit column

pivot

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 0 & 2 & -4 & -12 \\ 0 & -3 & 3 & 12 \end{array}\right] \xrightarrow{R_2 \leftrightarrow \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 0 & 1 & -2 & -6 \\ 0 & -3 & 3 & 12 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 10 \\ 0 & 1 & -2 & -6 \\ 0 & -3 & 3 & 12 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_3 + 3R_2}$$

unit column

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 10 \\ 0 & 1 & -2 & -6 \\ 0 & 0 & -3 & -6 \end{array}\right] \xrightarrow{R_3 \leftrightarrow -\frac{1}{3}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 10 \\ 0 & 1 & -2 & -6 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_1 - 5R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -6 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_2 + 2R_3}$$

pivot

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} x = 0 \\ y = -2 \\ z = 2 \end{array} \longrightarrow \boxed{(x, y, z) = (0, -2, 2)}$$

unique solution

To put an augmented matrix into row-reduced form we can use calculator:

1. Enter the augmented matrix into your calculator.
2. Go to your home screen.
3. Hit $\boxed{\text{2nd}}$ $\boxed{x^{-1}}$, cursor right to MATH, and select B:rref.
4. Call the matrix you want to reduce and hit $\boxed{\text{ENTER}}$.

Now you can simply read off the solutions to your original system of equations.

EXAMPLE 5. *Solve the system of Example 4 using your calculator.*

EXAMPLE 6. *Solve the system of linear equations:*

(a)
$$\begin{aligned} 2x &+ 11y &= 70 \\ 5x &- 2y &= 25 \end{aligned}$$

$$\begin{array}{cc} x & y \\ \end{array}$$
$$\left[\begin{array}{cc|c} 2 & 11 & 70 \\ 5 & -2 & 25 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 415/59 \\ 0 & 1 & 300/59 \end{array}\right]$$

$\boxed{2\times 3}$

$$\boxed{x = \frac{415}{59}, \quad y = \frac{300}{59}}$$

(b)
$$\begin{aligned} x &- 4y &= 9 \\ 3y &- 2x &= 8 \\ 2x &+ 4y &= 6 \end{aligned}$$

$$\rightarrow \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 1 & -4 & 9 \\ -2 & 3 & 8 \\ 2 & 4 & 6 \end{array}\right] \end{array} \xrightarrow{\text{rref}} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \end{array}$$

3×3

no solution

$x = 0$
$y = 0$
$0 = 1$ impossible

(c)
$$\begin{aligned} x &+ y &+ z &= 1 \\ x &+ 5y &+ 5z &= -1 \\ 3x &- y &- z &= 4 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 5 & 5 & -1 \\ 3 & -1 & -1 & 4 \end{array}\right] \xrightarrow{\text{rref}} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$3x \rightarrow$

no solution.

$x = 0$
$y + z = 0$
$0 = 1$
impossible

$$\begin{array}{rcrcrcr} x & + & 2y & + & z & = & -2 \\ 2x & + & 3y & + & z & = & -1 \\ 3x & + & 5y & + & 2z & = & -3 \end{array}$$

EXAMPLE 7. (a) *Solve* the system above.

basic variables

check it!

parameter (free variable)

Solution: The resulting rref matrix is

$$\begin{array}{ccc|c} x & y & z & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

There is no unique solution. We have a situation with infinite solutions.

How can we express the solutions in a helpful way?

First, translate the rows back into equations:

$$\begin{cases} x - z = 4 \\ y + z = -3 \end{cases}$$

Then solve for all other variables in terms of one of them in order to define a "pattern" for the infinite solutions.

– We can let the parameter t represent the z value (then x and y represent so called *basic variables*)

$z = t$ any number

$x - t = 4 \Longrightarrow x = 4 + t$

$y + t = -3 \Longrightarrow y = -3 - t$

and get the pattern:

$$\boxed{(x, y, z) = (4 + t, -3 - t, t)}$$

infinitely many solutions

general solution (pattern)

(b) Use the general pattern for the solutions, (4+t,-3-t,t), to find the missing portions of these specific *solutions:*

(1) (14, -13, 10) (2) (4, -3, 0) (3) (55, -54, 51)

$z = 10 = t$
$x = 4 + t = 4 + 10 = 14$
$y = -3 - t = -3 - 10 = -13$

$z = t = 0$
$x = 4 + 0 = 4$
$y = -3 - 0 = -3$

$x = 4 + t = 55$
$t = 55 - 4 = 51$
$y = -3 - t = -3 - 51 = -54$
$z = t = 51$

EXAMPLE 8. *Solve:*

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & - & x_3 & - & 3x_4 & = & 7 \\ 2x_1 & + & 4x_2 & - & & & 2x_4 & = & 10 \end{array}$$

Basic variables   Free variables (parameters)

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -1 & -3 & 7 \\ 2 & 4 & 0 & -2 & 10 \end{array}\right] \xrightarrow{\text{rref}} \begin{array}{c} \begin{array}{cccc} \boxed{x_1 \quad x_2} & \boxed{x_3 \quad x_4} \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ 0 & 1 & -1 & -2 & 2 \end{array}\right] \end{array}$$

2×5

TO SYSTEM

$$\begin{cases} x_1 + 2x_3 + 3x_4 = 1 \\ x_2 - x_3 - 2x_4 = 2 \\ x_3 = t \\ x_4 = s \end{cases} \longrightarrow \begin{cases} x_1 + 2t + 3s = 1 \\ x_2 - t - 2s = 2 \\ x_3 = t \\ x_4 = s \end{cases} \longrightarrow$$

$$\longrightarrow \boxed{\begin{cases} x_1 = 1 - 2t - 3s \\ x_2 = 2 + t + 2s \\ x_3 = t \\ x_4 = s \end{cases}}$$

OR $(x_1, x_2, x_3, x_4) = \underbrace{(1-2t-3s,\ 2+t+2s,\ t,\ s)}_{\text{general solution}}$

(b) Find one specific solution

$$(\underline{1-3},\ \underline{2+2},\ 0,\ 1) \to \boxed{(-2, 4, 0, 1)}$$

TOTAL #

EXAMPLE 9. *A chemical manufacturer wants to lease a fleet of 24 railroad tank cars with a combined carrying capacity of 520,000 gallons. Tank cars with three different carrying capacities are available: 8,000 gallons, 16,000 gallons, and 24,000 gallons. How many of each type of tank car should be leased?*

TOTAL

Define variables

		Capacity
x	= # tank cars with capacity 8,000 g.	8000x
y	= # " 16,000 g.	16,000y
z	= # " 24,000 g	24,000z
TOTAL	24	520,000

$$\begin{cases} x+y+z=24 \\ 8\cancel{000}x + 16\cancel{000}y + 24\cancel{000}z = 520,\cancel{000} \end{cases}$$

Solve for x, y, z

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 24 \\ 8 & 16 & 24 & 520 \end{array}\right] \xrightarrow{\text{rref}} \begin{array}{cccc} x & y & z & \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & -17 \\ 0 & 1 & 2 & 41 \end{array}\right] \end{array}$$

$\boxed{2\times 4}$

basic variables

free variable (parameter)

infinitely many solutions

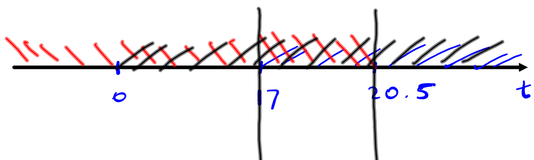
$$\begin{cases} x \quad - z = -17 \\ y + 2z = 41 \\ z = t \end{cases} \rightarrow \begin{array}{l} x - t = -17 \\ y + 2t = 41 \\ z = t \end{array} \rightarrow \boxed{\begin{array}{l} x = -17 + t \\ y = 41 - 2t \\ z = t \end{array}}$$

pattern

general solution of the system

According to the problem, the variables x,y,z must be nonnegative integers, i.e.

$$\begin{array}{l} -17 + t \geqslant 0 \\ 41 - 2t \geqslant 0 \\ t \geqslant 0 \end{array} \Rightarrow \begin{array}{l} t \geqslant 17 \\ 41 \geqslant 2t \\ t \geqslant 0 \end{array} \rightarrow \begin{cases} t \geqslant 17 \\ t \leq \frac{41}{2} = 20.5 \\ t \geqslant 0 \end{cases}$$



t is integer

$17 \leq t \leq 20.5$

$$\underbrace{\qquad\qquad}_{t = 17,\ 18,\ 19,\ 20}$$

One specific solution: $t = 18$

$(x, y, z) = ($ 1 , 5 , $18)$

1 tank car with capacity 8,000 g, 5 tank cars with capacity 16,000 g and 18 tank cars with capacity 24,000 g

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EXAMPLE 10. *A company wants to purchase* TOTAL *25 trucks with a combined capacity of* TOTAL CAPACITY *28,000* ft^3. *Three different types of trucks are available: a 10-foot truck with a capacity of 350* ft^3*, a 14-foot truck with a capacity of 700* ft^3*, and a 24-foot truck with a capacity of 1400* ft^3. *How many of each type of truck should the company purchase?*

Define variables	CAPACITY
x - # 10-ft trucks	$350x$
y - # 14-ft trucks	$700y$
z - # 24-ft trucks	$1400z$
TOTAL 25	28,000 ft^3

$$\begin{cases} x+y+z=25 \\ 350x+700y+1400z=28000 \end{cases}$$

augmented matrix

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25 \\ 350 & 700 & 1400 & 28000 \end{array}\right]$$

$\downarrow$ rref

basic: x, y — z parameter

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -30 \\ 0 & 1 & 3 & 55 \end{array}\right]$$

$$\begin{cases} x-2z=-30 \\ y+3z=55 \end{cases}$$

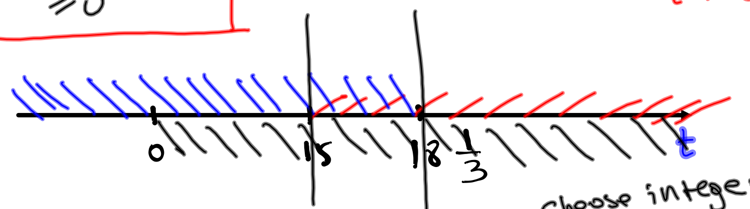
$$\begin{cases} z=t \\ x-2t=-30 \\ y+3t=55 \end{cases}$$

$$\boxed{\begin{array}{l} x=-30+2t \\ y=55-3t \\ z=t \end{array}}$$

general solution for the system

According to the problem, the variables x,y,z must be nonnegative integers, i.e.

$$\boxed{\begin{array}{ll} x= -30+2t \geq 0 & \Rightarrow 2t \geq 30 \Rightarrow t \geq \frac{30}{2} \Rightarrow t \geq 15 \\ y= 55-3t \geq 0 & \Rightarrow -3t \geq -55 \Rightarrow t \leq \frac{55}{3} = 18\frac{1}{3} \\ z= t \geq 0 & t \geq 0 \end{array}}$$



$15 \leq t \leq 18\frac{1}{3}$ $\Rightarrow$ choose integers $t = 15, 16, 17, 18,$

One specific solution (Suggestion)

$t=15 \Rightarrow x=-30+2\cdot 15=0$

$y=55-3\cdot 15=10$

$z=15$

The company may purchase 0 10-ft trucks, 10 14-ft trucks and 15 24-ft trucks.

Useful Remark:

- The system has exactly one solution: $\begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right] \end{array}$, $\begin{array}{c} \begin{array}{ccc} x & y & \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{array}\right] \end{array}$

- No solution: $\left[\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & \underline{0} & a \end{array}\right]$, where a is a non zero constant.

- Infinitely many solutions: $\begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$, $\begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{array}\right] \end{array}$