2.4: Matrices

- A matrix with m rows and n columns has the *dimension* (aka: *order* or *size*)

  $m \times n$ (ROW× COLUMN).

$$A = \begin{array}{cc} \begin{array}{c} \quad\quad\quad A^2 \quad\quad\quad\quad\quad\quad\quad \end{array} & \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdot & - & - & - & - \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} & \begin{array}{c} \\ A_2 \\ \\ \\ \\ \end{array} \end{array}$$

$$\underbrace{\underset{2 \times 3}{\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}} \quad \underset{3 \times 2}{\begin{bmatrix} 2 & 5 \\ 3 & 6 \\ 4 & 7 \end{bmatrix}}}_{\text{rectangular matrices}} \quad \underbrace{\underset{1 \times 4}{\begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}}}_{\text{Row matrix}} \quad \overbrace{\underset{4 \times 1}{\begin{bmatrix} -1 \\ -2 \\ 3 \\ 5 \end{bmatrix}}}^{\text{Column matrix}} \quad \overset{\text{square matrix}}{\underset{2 \times 2}{\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}}}$$

- If a matrix has the same number of rows and columns ($m = n$), it is called a **square matrix.** An $n \times n$ matrix is a square matrix of order n.

- A matrix consisting of only ONE row is called a **row matrix.**

$$[a_1 \quad a_2 \quad \cdots \quad a_n]$$

- A matrix consisting of only ONE column is called a **column matrix.** $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

- A matrix is usually denoted by a capital letter and its elements by small letters : a_{ij} = entry in the ith row and jth column of A.

$$A = \begin{bmatrix} 2 & 4 & -8 & 1 \\ 0 & 5 & 0 & -6 \\ -1 & 6 & 10 & -7 \\ 3 & 7 & 11 & -3 \end{bmatrix}$$

$a_{23} = 0$

$a_{32} = 6$

$a_{11} = 2$

$a_{33} = 10$

$a_{14} - a_{41} = 1 - 3 = -2$

- A **Zero Matrix** is a matrix consisting of all zeros. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

A matrix is a compact way to organize or display data.

EXAMPLE 1. *Mortality actuarial tables in the U.S. were revised in 2001, the fourth time since 1858. Based on the new life insurance mortality rates,* 1% *of* 60*-yr-old men,*2% *of* 70*-yr-old men,* 7% *of* 80*-yr-old men,* 18.8% *of* 90*-yr-old men and* 36.3% *of* 100*-yr-old men would die within a year. The corresponding rates for women are* 0.8%, 1.8%, 4.4%, 12.2% *and* 27.6%*, respectively. Express this information using a* 2×5 *matrix.*

$$\begin{array}{c|c}
\text{Age} \rightarrow & \begin{array}{ccccc} 60 & 70 & 80 & 90 & 100 \end{array} \\ \hline
\begin{array}{c} M \\ W \end{array} & \begin{bmatrix} 1 & 2 & 7 & 18.8 & 36.3 \\ 0.8 & 1.8 & 4.4 & 12.2 & 27.2 \end{bmatrix}
\end{array}$$

- Two matrices are said to be **equal** if they are the same size and each corresponding entry is equal.

$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 3 \end{bmatrix}$ 2×3 $\quad B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 5 & 3 \end{bmatrix}$ 3×2 $\quad C = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 5 \end{bmatrix}$

$A \neq B$

$B \neq C$

EXAMPLE 2. *Given matrices A and B below*

$$A = \begin{bmatrix} 2x-1 & 3 \\ y+5 & 2z \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & -3u \\ 0 & 1 \\ 0 & uv \end{bmatrix}$$

$3\times 2 \qquad 3\times 2$

Solve the matrix equation $A = B$ for x, y, z, u, v.

$2x-1=5 \Rightarrow 2x = 5+1 \Rightarrow x = \frac{6}{2} \Rightarrow \boxed{x=3}$

$y+5=0 \Rightarrow \boxed{y=-5}$

$0 = 0$

$3 = -3u \Rightarrow u = \frac{3}{-3} \Rightarrow \boxed{u=-1}$

$2z = 1 \Rightarrow \boxed{z = \frac{1}{2}}$

$1 = uv \Rightarrow v = \frac{1}{u} = \frac{1}{-1} \Rightarrow \boxed{v=-1}$

• **ADDITION/SUBTRACTION** of matrices:

Two matrices can be added or subtracted ONLY if they are the SAME SIZE. This is done by adding (or subtracting) each entry with the corresponding entry and resulting in a solution matrix of the same size.

• **Scalar Multiplication**:

If A is a matrix and c ia a real number (or scalar), then the scalar product: cA is the matrix obtained by multiplying each individual entry of A by c. NOTE: Zero is NOT a scalar.

• **LAWS for Matrix Addition**:

If $A, B, andC$ are matrices of the *same* size, then

1. $A + B = B + A$ (Commutative Law)
2. $(A + B) + C = A + (B + C)$ (Associative Law)

EXAMPLE 3. *Given matrices A and B below. Find the matrix X satisfying the matrix equation $\frac{1}{2}X + A = B$.*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 2 & -3 & 0 \\ -3 & 2 & -1 & 0 \end{bmatrix}$$

$\Downarrow$

$\frac{1}{2}X = B - A$

$X = 2(B-A)$

$$B - A = \begin{bmatrix} -1-1 & 2-2 & -3-3 & 0-4 \\ -3-3 & 2-2 & -1-1 & 0-0 \end{bmatrix}$$

$$B - A = \begin{bmatrix} -2 & 0 & -6 & -4 \\ -6 & 0 & -2 & 0 \end{bmatrix}$$

$$\boxed{X = 2(B-A) = \begin{bmatrix} -4 & 0 & -12 & -8 \\ -12 & 0 & -4 & 0 \end{bmatrix}}$$

- **Transpose of a Matrix**

If A is an $m \times n$ matrix with entries a_{ij} , then A^T is the $n \times m$ matrix with entries a_{ji}.
A^T is obtained by interchanging rows and columns of A.

EXAMPLE 4. *Find A^T and $(A^T)^T$ given that*

$$A = \begin{bmatrix} -1 & 2 & 3 \\ -3 & 0 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -1 & -3 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$$

$$(A^T)^T = A$$

$$(A^T)^T = \begin{bmatrix} -1 & 2 & 3 \\ -3 & 0 & 1 \end{bmatrix} = A$$

EXAMPLE 5. *Find the values for $a, b, c,$ and d for the matrix equation below:*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + 3\begin{bmatrix} -2 & 1 \\ 4 & -1 \end{bmatrix}^T = \begin{bmatrix} 6 & 4 \\ 0 & 7 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + 3\begin{bmatrix} -2 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 0 & 7 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -6 & 12 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 0 & 7 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} -6 & 12 \\ 3 & -3 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 6-(-6) & 4-12 \\ 0-3 & 7-(-3) \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 12 & -8 \\ -3 & 10 \end{bmatrix}$$

$$\boxed{a = 12, \quad b = -8, \quad c = -3, \quad d = 10}$$