2.5 : Multiplication of Matrices

- If A is a row matrix of size $1 \times n$,

  and B is a column matrix of size $n \times 1$,

  then the **matrix product** of A and B is defined by

$$AB = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} = a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_nb_n$$

EXAMPLE 1. *Let* $A = \begin{bmatrix} 1 & 2 & -3 & 0 & -1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \end{bmatrix}$. *Find* BA^T.

- If A is an $m \times p$ matrix and matrix B is $p \times n$, then the product AB is an $m \times n$ matrix, and its element in the ith row and jth column is the product of the ith row of A and the jth column of B.

- RULE for multiplying matrices:

  The column of the 1st matrix must be the same size as the row of the 2nd matrix.

EXAMPLE 2. *Multiply*

(a) $\begin{bmatrix} 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -2 \end{bmatrix} \begin{bmatrix} -1/3 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 2 & -3 \\ 4 & 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

- **Identity Matrix**

The identity matrix of size n is given by $I_n = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix}$

- FACTS and LAWS FOR MATRIX MULTIPLICATION: If the size requirements are met for matrices A, B and C, then
  - $AB \neq BA$ (NOT always Commutative)
  - $A(B + C) = AB + AC$ (always Distributive)
  - $(AB)C = A(BC)$ (always Associative)
  - $AB = 0$ does not imply that $A = 0$ or $B = 0$.
  - $AB = AC$ does not imply that $B = C$.
  - $I_nA = A$ for every $n \times p$ matrix A.
  - $BI_n = B$ for every $s \times n$ matrix B.
  - $I_nA = AI_n = A$ for any square matrix A of size n.

  NOTE: Since the multiplication of matrices is NOT commutative, you MUST multiply left to right.

  EXAMPLE 3. *The sizes of matrices A, B, C, D are given in the table below:*

A	B	C	D
2×3	3×5	3×3	5×5

Find the size of the following matrices whenever they are defined.

(a) AB

(b) BA

(c) ABD

(d) ABB^T

(e) CBD

(f) $BD + CB$

(g) $D^2 + I_5$

(h) BI_3

EXAMPLE 4. *Given* $J = \begin{bmatrix} a & -2 \\ b & 2c \end{bmatrix}$, $K = \begin{bmatrix} 3 & 0 \\ 4 & -5 \end{bmatrix}$ *and* $P = \begin{bmatrix} 1 & 10 \\ -14 & 20 \end{bmatrix}$

Find the values of a, b, c *s.t.* $JK = P$

- A system of linear equations can be written as a matrix equation $AX = B$.

EXAMPLE 5. *Express the following system of linear equations in matrix form:*

(a)
$$\begin{array}{rcrcrcl} 2x & + & 4y & - & 7z & = & 6 \\ -x & - & 3y & + & 11z & = & 0 \\ & - & y & + & z & = & 1 \end{array}$$

(b)
$$\begin{array}{rcrcl} x_1 & + & 5x_2 & = & 16 \\ 4x_1 & - & 3x_2 & = & 11 \end{array}$$

- Matrix product is used for application problems:

  EXAMPLE 6. *Matrix A shows the number of calories from fat, protein and carbohydrates per unit of each food. Matrix B shows the number of units of each food eaten by each person. Explain the meaning of entries of BA and AB.*

$$A = \begin{array}{c} \\ \text{Food A} \\ \text{Food B} \\ \text{Food C} \end{array} \begin{array}{c} \begin{array}{ccc} \text{fat} & \text{carbs} & \text{protein} \end{array} \\ \begin{bmatrix} 25 & 8 & 12 \\ 31 & 25 & 19 \\ 30 & 12 & 18 \end{bmatrix} \end{array}, \quad B = \begin{array}{c} \\ \text{fat} \\ \text{carbs} \\ \text{protein} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Tom} & \text{Bob} & \text{Jill} \end{array} \\ \begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & 6 \\ 2 & 2 & 0 \end{bmatrix} \end{array}$$

Solution:

$$BA = \begin{array}{c} \\ \text{fat} \\ \text{carbs} \\ \text{protein} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Tom} & \text{Bob} & \text{Jill} \end{array} \\ \begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & 6 \\ 2 & 2 & 0 \end{bmatrix} \end{array} * \begin{array}{c} \\ \text{Food A} \\ \text{Food B} \\ \text{Food C} \end{array} \begin{array}{c} \begin{array}{ccc} \text{fat} & \text{carbs} & \text{protein} \end{array} \\ \begin{bmatrix} 25 & 8 & 12 \\ 31 & 25 & 19 \\ 30 & 12 & 18 \end{bmatrix} \end{array} = \begin{bmatrix} 166 & 73 & 91 \\ 335 & 197 & 203 \\ 112 & 66 & 62 \end{bmatrix}$$

$$AB = \begin{array}{c} \\ \text{Food A} \\ \text{Food B} \\ \text{Food C} \end{array} \begin{array}{c} \begin{array}{ccc} \text{fat} & \text{carbs} & \text{protein} \end{array} \\ \begin{bmatrix} 25 & 8 & 12 \\ 31 & 25 & 19 \\ 30 & 12 & 18 \end{bmatrix} \end{array} * \begin{array}{c} \\ \text{fat} \\ \text{carbs} \\ \text{protein} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Tom} & \text{Bob} & \text{Jill} \end{array} \\ \begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & 6 \\ 2 & 2 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} 99 & 89 & 98 \\ 131 & 194 & 212 \\ 126 & 126 & 132 \end{bmatrix}$$

EXAMPLE 7. *Matrix A below gives the percentage of eligible voters in a city, classified according to party affiliation and age group:*

$$A = \begin{bmatrix} 0.50 & 0.30 & 0.20 \\ 0.45 & 0.38 & 0.17 \\ 0.38 & 0.52 & 0.10 \end{bmatrix}$$

where rows $1, 2\&3$ *represent Under 35, 35-55 and over 55, respectively. Columns* $1, 2\&3$ *represent Democrat, Republican and Independent, respectively. The city currently has* $28,000$ *eligible voters under the age of* 35. *They have* $44,000$ *eligible voters between 35-55 years of age, and* $18,000$ *eligible voters over 55 years old. Find the matrix B representing the population of eligible voters. Then use it to find a matrix giving the total number of eligible voters in the city who vote Independent.*