2.5 : Multiplication of Matrices

- If A is a row matrix of size $1 \times n$, $A = [a_1 \ a_2 \ \dots \ a_n]$ $1 \times n$

and B is a column matrix of size $n \times 1$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$ $n \times 1$

$[1 \times n] \ [n \times 1] \quad [1 \times 1]$

$A \cdot B = C$

then the **matrix product** of A and B is defined by

$$AB = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} = a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_nb_n$$

EXAMPLE 1. *Let* $A = \begin{bmatrix} 1 & 2 & -3 & 0 & -1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \end{bmatrix}$. *Find* BA^T.

$[1 \times 5] \quad [1 \times 5]$

AB, BA: $[1 \times 5] \cdot [1 \times 5]$ undefined

$BA^T = [1 \ 2 \ 0 \ 1 \ -1] \begin{bmatrix} 1 \\ 2 \\ -3 \\ 0 \\ -1 \end{bmatrix} \Rightarrow$

$[1 \times 5] \cdot [5 \times 1]$

$BA^T = 1 \cdot 1 + 2 \cdot 2 + 0 \cdot (-3) + 1 \cdot 0 + (-1) \cdot (-1) =$

$= 1 + 4 + 1 = 6$

$[m \times p] \cdot [p \times n] \rightarrow [m \times n]$

- If A is an $m \times p$ matrix and matrix B is $p \times n$, then the product AB is an $m \times n$ matrix, and its element in the ith row and jth column is the product of the ith row of A and the jth column of B.

$AB = C$

$C_{ij} = \underbrace{A_i}_{\text{row}} \underbrace{B^j}_{\text{column}}$

- RULE for multiplying matrices:

  The column of the 1st matrix must be the same size as the row of the 2nd matrix.

EXAMPLE 2. *Multiply*

(a) $\begin{bmatrix} 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\cdot 1 + (-2)\cdot 1 = 1$

(b) $\begin{bmatrix} 3 & -2 \end{bmatrix} \begin{bmatrix} -1/3 \\ -1 \end{bmatrix} = 3\left(-\frac{1}{3}\right) + (-2)\cdot(-1) = -1 + 2 = 1$

(c) $\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} YY & YG \\ GY & GG \end{bmatrix} = \begin{bmatrix} -2+2 & 4-4 \\ 2-2 & -4+4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$-1\cdot 2 + (-2)\cdot(-1)$

(d) $\underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 0 \end{bmatrix}}_{2\times 4} \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}}_{2\times 2}$ undefined

(e) $\underbrace{\begin{bmatrix} 1 & 2 & 5 \\ 3 & 2 & -3 \\ 4 & 3 & 9 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} = \begin{bmatrix} YY & YG & YW \\ GY & GG & GW \\ WY & WG & WW \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 & 5 \\ 3 & 2 & -3 \\ 4 & 3 & 9 \end{bmatrix}}_{C}$

(f) $\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{B} = \begin{bmatrix} 1\cdot 1 + 2\cdot 1 & 1\cdot 1 + 2\cdot 1 \\ 3\cdot 1 + 2\cdot 1 & 3\cdot 1 + 2\cdot 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 5 & 5 \end{bmatrix}$

(g) $\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}}_{A} = \begin{bmatrix} 1\cdot 1 + 1\cdot 3 & 1\cdot 2 + 1\cdot 2 \\ 1\cdot 1 + 1\cdot 3 & 1\cdot 2 + 1\cdot 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$

- **Identity Matrix**

  The identity matrix of size n (square) is given by $I_n = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix}$

  $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $I_3 = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$

- FACTS and LAWS FOR MATRIX MULTIPLICATION: If the size requirements are met for matrices A, B and C, then

  – $AB \neq BA$ (NOT always Commutative)
  – $A(B + C) = AB + AC$ (always Distributive)
  – $(AB)C = A(BC)$ (always Associative)
  – $AB = 0$ does not imply that $A = 0$ or $B = 0$.
  – $AB = AC$ does not imply that $B = C$.
  – $I_nA = A$ for every $n \times p$ matrix A.
  – $BI_n = B$ for every $s \times n$ matrix B.
  – $I_nA = AI_n = A$ for any square matrix A of size n.

  NOTE: Since the multiplication of matrices is NOT commutative, you MUST multiply left to right.

EXAMPLE 3. *The sizes of matrices A, B, C, D are given in the table below:*

A	B	C	D
2×3	3×5	3×3	5×5

Find the size of the following matrices whenever they are defined.

(a) AB

[2×3]·[3×5]

[2×5]

(b) BA

[3×5]·[2×3]

undefined

(c) ABD

[2×5][5×5]

[2×5]

(d) ABB^T

[2×5][5×3]

[2×3]

(e) CBD

[3×3][3×5][5×5]

[3×5][5×5] → [3×5]

(f) $BD + CB$

[3×5][5×5] + [3×3][3×5]

[3×5] + [3×5]

[3×5]

(g) $D^2 + I_5$ = D·D + I_5

[5×5][5×5] + [5×5]

[5×5]

(h) BI_3

[3×5][3×3]

undefined

EXAMPLE 4. *Given* $J = \begin{bmatrix} a & -2 \\ b & 2c \end{bmatrix}$, $K = \begin{bmatrix} 3 & 0 \\ 4 & -5 \end{bmatrix}$ *and* $P = \begin{bmatrix} 1 & 10 \\ -14 & 20 \end{bmatrix}$

Find the values of a, b, c *s.t.* $JK = P$

$$J \quad K = P$$

$$\begin{bmatrix} a & -2 \\ b & 2c \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 10 \\ -14 & 20 \end{bmatrix}$$

$$\begin{bmatrix} 3a + (-2)\cdot 4 & a\cdot 0 + (-2)(-5) \\ 3b + 2c\cdot 4 & b\cdot 0 + 2c\cdot(-5) \end{bmatrix} = \begin{bmatrix} 1 & 10 \\ -14 & 20 \end{bmatrix}$$

$$\begin{bmatrix} 3a - 8 & 10 \\ 3b + 8c & -10c \end{bmatrix} = \begin{bmatrix} 1 & 10 \\ -14 & 20 \end{bmatrix}$$

$3a - 8 = 1 \Rightarrow 3a = 1 + 8 \Rightarrow a = \frac{9}{3} \Rightarrow \boxed{a = 3}$

$10 = 10$

$3b + 8c = -14$

$-10c = 20 \Rightarrow c = \frac{20}{-10} \Rightarrow \boxed{c = -2}$

$3b + 8\cdot(-2) = -14$

$3b - 16 = -14 \Rightarrow 3b = -14 + 16 \Rightarrow$

$\Rightarrow 3b = 2 \Rightarrow \boxed{b = \frac{2}{3}}$

- A system of linear equations can be written as a matrix equation $AX = B$.

EXAMPLE 5. *Express the following system of linear equations in matrix form:*

(a)
$$\begin{array}{rcrcrcl} 2x & + & 4y & - & 7z & = & 6 \\ -x & - & 3y & + & 11z & = & 0 \\ 0\cdot x & - & y & + & z & = & 1 \end{array}$$

$$\underbrace{\begin{bmatrix} 2 & 4 & -7 \\ -1 & -3 & 11 \\ 0 & -1 & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix}}_{B}$$

coefficient matrix

(b)
$$\begin{aligned} 1 \cdot x_1 &+ 5x_2 = 16 \\ 4x_1 &- 3x_2 = 11 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 1 & 5 \\ 4 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 16 \\ 11 \end{bmatrix}}_{B}$$

EXAMPLE 6. *Matrix A shows the number of calories from fat, protein and carbohydrates per unit of each food. Matrix B shows the number of units of each food eaten by each person. Explain the meaning of entries of BA and AB.*

$$A = \begin{array}{c|ccc} & \text{fat} & \text{carbs} & \text{protein} \\ \hline \text{Food A} & 25 & 8 & 12 \\ \text{Food B} & 31 & 25 & 19 \\ \text{Food C} & 30 & 12 & 18 \end{array}, \quad B = \begin{array}{c|ccc} & \text{Tom} & \text{Bob} & \text{Jill} \\ \hline \text{fat} & 3 & 1 & 2 \\ \text{carbs} & 0 & 5 & 6 \\ \text{protein} & 2 & 2 & 0 \end{array}$$

[3x3] [3x3]

A → [Food x nutrients] B → [nutrients x person]

Solution:

$$BA = \begin{array}{c|ccc} & \text{Tom} & \text{Bob} & \text{Jill} \\ \hline \text{fat} & 3 & 1 & 2 \\ \text{carbs} & 0 & 5 & 6 \\ \text{protein} & 2 & 2 & 0 \end{array} * \begin{array}{c|ccc} & \text{fat} & \text{carbs} & \text{protein} \\ \hline \text{Food A} & 25 & 8 & 12 \\ \text{Food B} & 31 & 25 & 19 \\ \text{Food C} & 30 & 12 & 18 \end{array} = \begin{bmatrix} 166 & 73 & 91 \\ 335 & 197 & 203 \\ 112 & 66 & 62 \end{bmatrix}$$

[nutrients x person] [Food x nutrients] → ? no meaning

$3 \cdot 25 + 1 \cdot 31 + 2 \cdot 30 = 166$

$$AB = \begin{array}{c|ccc} & \text{fat} & \text{carbs} & \text{protein} \\ \hline \text{Food A} & 25 & 8 & 12 \\ \text{Food B} & 31 & 25 & 19 \\ \text{Food C} & 30 & 12 & 18 \end{array} * \begin{array}{c|ccc} & \text{Tom} & \text{Bob} & \text{Jill} \\ \hline \text{fat} & 3 & 1 & 2 \\ \text{carbs} & 0 & 5 & 6 \\ \text{protein} & 2 & 2 & 0 \end{array} = \begin{array}{ccc|c} \text{Tom} & \text{Bob} & \text{Jill} & \\ \hline 99 & 89 & 98 & \text{Food A} \\ 131 & 194 & 212 & \text{Food B} \\ 126 & 126 & 132 & \text{Food C} \end{array}$$

AB

[Food x nutrients] x [nutrients x person] → [food x person]

Each entry in AB represents number of calories from each food for each person.

EXAMPLE 7. *Matrix A below gives the percentage of eligible voters in a city, classified according to party affiliation and age group:*

[Age × party affiliation]

$$A = \begin{bmatrix} 0.50 & 0.30 & 0.20 \\ 0.45 & 0.38 & 0.17 \\ 0.38 & 0.52 & 0.10 \end{bmatrix}$$

Columns: D R I; Rows: 35−, 35-55, 55+ [3 × 3]

where rows 1, 2&3 represent Under 35, 35-55 and over 55, respectively. Columns 1, 2&3 represent Democrat, Republican and Independent, respectively. The city currently has 28,000 eligible voters under the age of 35. They have 44,000 eligible voters between 35-55 years of age, and 18,000 eligible voters over 55 years old. Find the matrix B representing the population of eligible voters. Then use it to find a matrix giving the total number of eligible voters in the city who vote Independent.

\# voters

$$B = \begin{bmatrix} 28\,000 \\ 44\,000 \\ 18\,000 \end{bmatrix} \begin{matrix} 35- \\ 35-55 \\ 55+ \end{matrix}$$

[3 × 1]

[age × # voters]

B A undefined because [3×1] [3×3] impossible

A B [3×3] [3×1] → [3×1]
[age × party aff] [age × # voters] — ?

$A^T B$ [3×3] [3×1] → [3×1]
[party aff × age] [age × # voters]
↓
[party aff × # voters]

Use calculator

$$A^T B = \begin{bmatrix} 40\,640 \\ 34\,980 \\ 14\,880 \end{bmatrix} \begin{matrix} D \\ R \\ I \end{matrix}$$

\# voters

14880 eligible voters who vote Independent

2.6: The Inverse of a Square Matrix

Let A be a square matrix of size n. A square matrix, A^{-1}, of size n, such that $AA^{-1} = I_n$ (or equivalently, $A^{-1}A = I_n$) is called an **inverse matrix**.

EXAMPLE 1. *Are these matrices inverses?*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.5 & 0.5 \\ 0.75 & -0.25 \end{bmatrix}$$

Check if $AB = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Yes (use calculator)

Conclusion $A = B^{-1}$ and $B = A^{-1}$

We shall use the calculator to find A^{-1} if it exists.

EXAMPLE 2. *If possible, find the inverse of the following matrices and express it with exa values (fractions).*

(a) $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ -4 & -2 & 7 \end{bmatrix}$.

$$A^{-1} = \begin{bmatrix} -\frac{13}{15} & \frac{4}{5} & -\frac{7}{15} \\ \frac{4}{5} & -\frac{1}{5} & \frac{1}{5} \\ -\frac{4}{15} & \frac{2}{5} & -\frac{1}{15} \end{bmatrix}$$

(b) $B = \begin{bmatrix} 0 & 1 & 3 \\ -4 & -2 & 7 \end{bmatrix}$ DNE, because B is not square

(c) $C = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 3 & 0 \\ -4 & -2 & 7 & 0 \\ 1 & 2 & -1 & 1 \end{bmatrix}$. not possible, because C is singular

DEFINITION 3. *A matrix that does NOT have an inverse is called* **singular**.

EXAMPLE 4. *Solve the matrix equation for* X. *Assume all matrices are square and all inverses are possible.*

(a) $XA - 4B = D$

$$XAA^{-1} = (D + 4B)A^{-1}$$

$$XI = (D + 4B)A^{-1}$$

$$\boxed{X = (D + 4B)A^{-1}}$$

(b) $X + AX = B$

$$IX + AX = B$$

$$(I+A)^{-1}(I+A)X = (I+A)^{-1}B$$

$$IX = (I+A)^{-1}B \Rightarrow \boxed{X = (I+A)^{-1}B}$$

$AX = B$

EXAMPLE 5. *Suppose we have the following system of linear equations:*

$$
\begin{aligned}
2x + y + 2z &= -1 \\
3x + 2y + z &= 2 \\
2x + y + z &= 1
\end{aligned}
$$

(a) *write a matrix equation that is equivalent to the system of linear equations.*

$$\underbrace{\begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}_{B}$$

(b) *solve the system of equations by using the inverse of the coefficient matrix.* A

$AX = B$

$\underbrace{A^{-1}A}_{I}X = A^{-1}B \Rightarrow \boxed{X = A^{-1}B}$

use calculator

$A \Rightarrow [3 \times 3]$

$B \rightarrow [3 \times 1]$

$$X = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

$\boxed{(x, y, z) = (2, -1, -2)}$

OR

$x = 2$
$y = -1$
$z = -2$

EXAMPLE 6. *Solve the following system of linear equations:*

$$\begin{array}{rcrcrcr} 2x & + & y & + & 2z & = & -1 \\ 3x & + & 2y & + & z & = & 2 \\ x & & & + & 3z & = & -4 \end{array}$$

→ $1 \cdot x + 0 \cdot y + 3z = -4$

$$\underbrace{\begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ -4 \end{bmatrix}}_{B} \Rightarrow X = A^{-1}B$$

singular

A^{-1} DNE

⇓

the system has no unique solution

infinitely many solutions ? no solution

use another method to solve (see Sections 2.2–2.3)