3.3: Graphical solution of Linear Programming Problems

PROBLEM: *Find the* **optimal** *(maximum/minimum) solution for a linear programming problem with a feasible region FR and* linear *objective function $P = ax + by$.*

- *FUNDAMENTAL THEOREM OF LINEAR PROGRAMMING:*

  If the optimal max/min value of the objective function $P=ax+by$ exists, it must occur at one (or more) of the corner points of the feasible region.

- *EXISTENCE OF A SOLUTION:*

  For a linear programming problem with a feasible region FR and objective function $P = ax + by$,

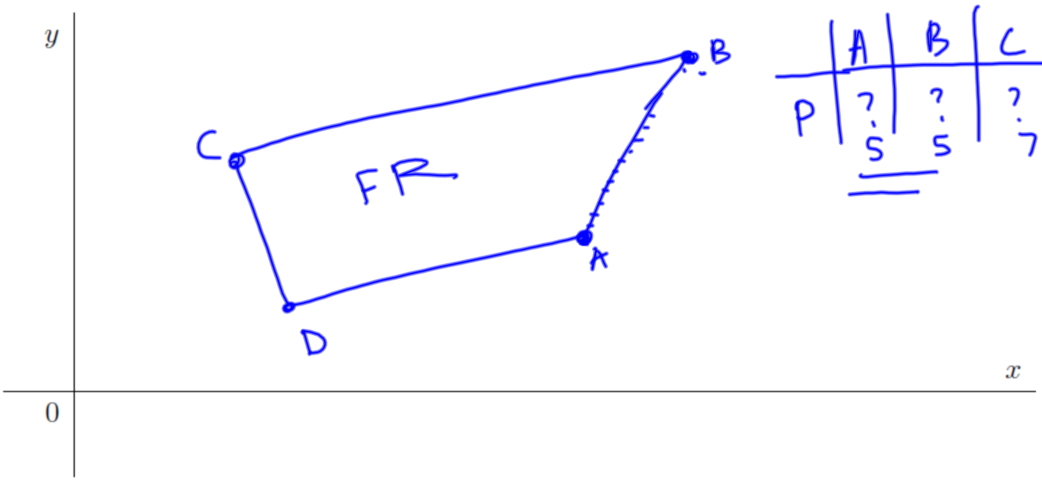
  (a) If FR is bounded, then P has both a maximum and a minimum value on FR.

  (b) If FR is unbounded you might have a max or min (or neither). In particular, if FR is unbounded and $a, b > 0$, then P has a minimum value on FR provided that $x, y \geq 0$ are part of the constraints. P has no maximum value then.

  (c) If FR is the empty set, then P has no maximum or minimum value.

- *THE METHOD OF CORNERS:*

  1. Graph the feasible region.
  2. Find the coordinates of all the corner points (vertices).
  3. Evaluate the objective function at each corner point.
  4. Use the existence of solutions to determine the optimal solutions. If two adjacent corner points give the same optimal objective function value, all the points on the line segment joining the two corner points will be the optimal solutions.



	A	B	C	D
P	? 5	? 5	? 7	? 10

EXAMPLE 1. (a) *Set up the following linear programming problem:*

The water-supply manager for a Midwest city needs to supply the city with at least $\geq$ *10 million gal of potable (drinkable) water per day.*

The supply may be drawn from the local reservoir or from a pipeline to an adjacent town. The local reservoir has a maximum daily yield of $\leq$ *5 million gal of potable water, and the pipeline has a maximum daily yield of* $\leq$ *10 million gallons.*

By contract, the pipeline is required to supply a minimum of $\geq$ *6 million gal/day.*

If the cost for 1 million gallons of reservoir water is \$300 and that for pipeline water is \$500, how much water (million of gallons) *should the manager get from each source to minimize daily water cost for the city. What is the minimum daily cost?*

Solution:

x - the number of gallons of water in millions obtained from the local reservoir per day;

y - the number of gallons of water in millions obtained from the pipeline.

	# millions gal	Yield	Contract supply	Cost \$
local reservoir	x	≤ 5		$300x$
pipeline	y	≤ 10	≥ 6	$500y$
Total	≥ 10			min

$C = 300x + 500y \longrightarrow \min$

Test (0,0) $0 \geq 10$ (F)

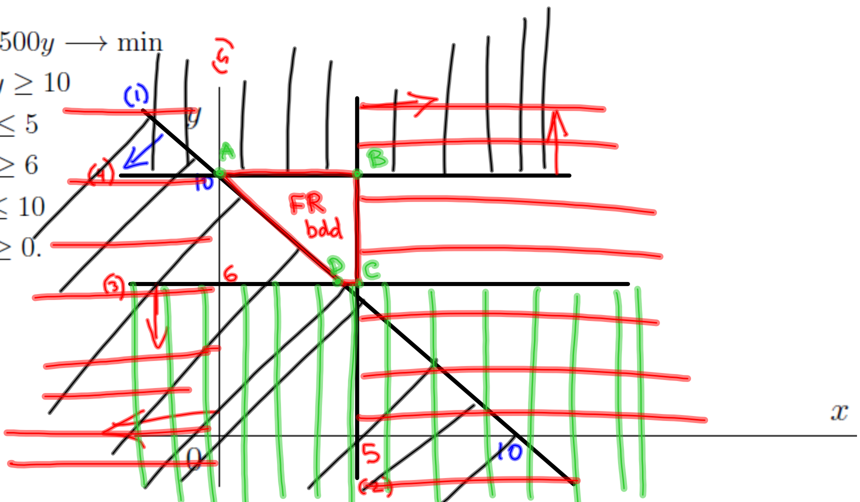
(1) $x + y \geq 10$

(2) $x \leq 5$

(3) $y \geq 6$

(4) $y \leq 10$

(5) $x \geq 0$.



(b) *Find all the corner points and the corresponding values of the objective function* C.

	A	B	C	D
	(0,10)	(5,10)	(5,6)	(1) ∩ (3) (4,6)
$C=300x+500y$	5000	6500	4500	4200

$\begin{cases} x+y=10 \\ y=6 \end{cases} \rightarrow \begin{array}{l} x+6=10 \\ x=4 \end{array}$

(c) *What is the minimum daily cost?*

\$4200 comes from 4 millions gal from the local reservoir and 6 millions gal from the pipeline.

EXAMPLE 2. *Solve the following linear programming problem:*

Maximize $\underline{P = 4x + 2y}$

subject to $x + y \le 8$ (1)

$2x + y \le 10$ (2)

$x \ge 0, y \ge 0$

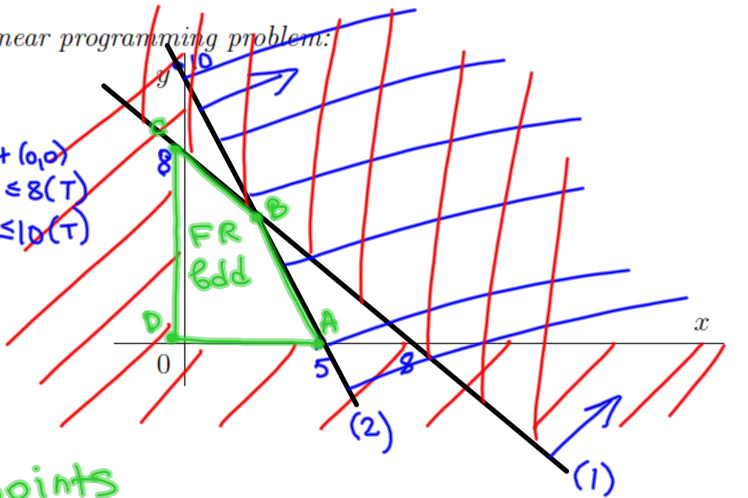
Test (0,0)

$0 \le 8$ (T)

$0 \le 10$ (T)

$2x+y=10$

x	y
0	10
5	0



Find corner points

	A	B	C	D
	(5,0)	(1) ∩ (2) (2,6)	(0,8)	(0,0)
P=4x+2y max	20	20	16	0

$\left.\begin{array}{l} x+y=8 \\ 2x+y=10 \end{array}\right\}$ To find $\underline{B}$

$$\left[\begin{array}{cc|c} 1 & 1 & 8 \\ 2 & 1 & 10 \end{array}\right]$$

$\downarrow$ rref

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 6 \end{array}\right]$$

max P=20 and it occurs at any point of the line segment AB (because A and B are two adjacent corners).

EXAMPLE 3. *Solve the following linear programming problem:*

(a) *Minimize* $\underline{P = 3x + 4y}$

subject to $x + y \geq 3$ (1)

$x + 2y \geq 4$ (2)

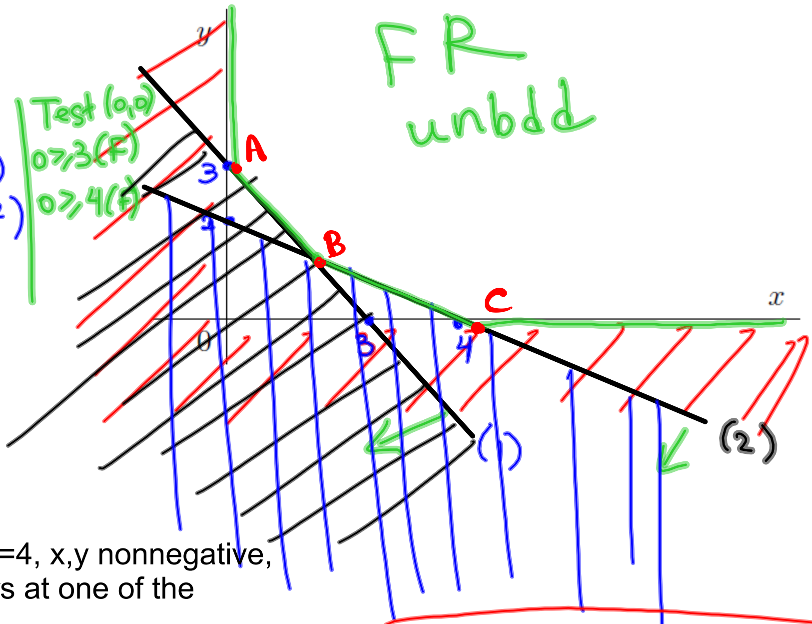
$x \geq 0, y \geq 0$

$x+2y=4$

x	y
0	2
4	0

Test (0,0)
$0 \geq 3$ (F)
$0 \geq 4$ (F)

FR
unbdd



FR is unbounded, but a=3, b=4, x,y nonnegative, thus the minimum of P occurs at one of the corner points.

	A	B	C
	(0,3)	(2,1)	(4,0)
$P=3x+4y$	12	10	12

$B = (1) \cap (2)$

$\begin{cases} x+y=3 \\ x+2y=4 \end{cases}$ $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 2 & 4 \end{array}\right]$

$\Downarrow$ rref

$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$

$\boxed{\text{min } P=10 \text{ and it occurs at } (2,1)}$

(b) *Maximize* $P = 3x + 4y$

subject to $x + y \geq 3$

$x + 2y \geq 4$

$x \geq 0, y \geq 0$

FR is unbounded, but a=3, b=4, x,y nonnegative, thus there is no maximum

EXAMPLE 4. *Solve the following linear programming problem:*

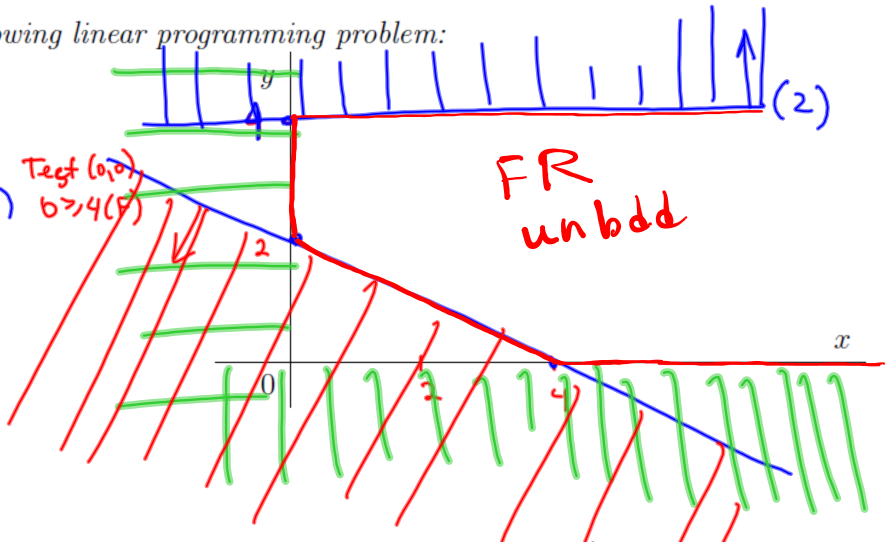
$\overset{a}{4}x+\overset{b}{(-1)}y$

$$\begin{array}{ll} \textit{Maximize} & f = 4x - y \\ \textit{subject to} & x + 2y \geq 4 \quad (1) \\ & y \leq 4 \quad (2) \\ & x \geq 0, y \geq 0 \end{array}$$

Test $(0,0)$: $0 \geq 4$ (F)

$x + 2y = 4$

x	y
0	2
4	0



y is bdd between 0 and 4
$0 \leq x < \infty$, i.e. x increases infinitely $\Rightarrow$

$\Rightarrow f = 4x - y$ is unbdd on FR $\Rightarrow$

$\Rightarrow$ no maximum