6.4: Permutation and Combinations

EXAMPLE 1. *How many ways can you arrange 10 books on a shelf together?* $10!$

Given a set of distinct objects, a permutation of a set is an arrangement of these objects in a definite order. Therefore, *the number of permutations of n distinct objects taken n at a time is* $n!$.

DEFINITION 2. *The number of permutations, $P(n,r)$, of n distinct items of which r objects are chosen to be placed in an ordered setting (i.e. row, list,...) is given by*

$$P(n,r) = \frac{n!}{(n-r)!}$$

$N_1 \cdot N_2 = P(n,r) = C(n,r) \cdot r!$

On your calculator: $\boxed{\text{MATH}}$ → PRB → #2nPr.

T_1 : Choose r objects from n | $N_1 = C(n,r)$

T_2 : arrange r objects | $N_2 = r!$

EXAMPLE 3. *Compute*

$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$ $\quad P(n,1) = \frac{n!}{(n-1)!} = \frac{n\cancel{(n-1)!}}{\cancel{(n-1)!}} = n$ $\quad P(7,4) = \boxed{840}$

cf. Ex.10 sec. 6.3

EXAMPLE 4. *Find the number of ways a chairman, a vice-chairman, and a secretary can be chosen from a committee of eleven members.*

$n = 11$ $\quad T_1$: Choose 3 from 11

$r = 3$ $\quad T_2$: arrange 3

$\Big\}\ P(11,3) = \boxed{990}$

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DEFINITION 5. *The number of combinations, $C(n,r)$, of n distinct items of which r objects are chosen to be placed in an unordered setting is given by*

$$C(n,r) = \frac{n!}{(n-r)!r!}, \quad \textit{where} \quad r \le n.$$

$= P(n,r)$

$C(n,r) = \frac{P(n,r)}{r!}$

$\boxed{P(n,r) = C(n,r)\cdot r!}$

On your calculator: $\boxed{\text{MATH}}$ $\to$ PRB $\to$ #3nCr.

EXAMPLE 6. *A subcommittee of three members is to be selected from a committee of eleven members. Determine the number of ways this can be done.*

$n = 11$

$r = 3$

$N = C(11,3) = 165$

EXAMPLE 7. *How many ways can 2 subcommittees be formed where one has 4 people and the other has 3 people from a committee of eleven members.*

OR

The same person can be in both subcomm.		The same person can only be in one subcomm.	
T_1 choose 4 from 11	$N_1 = C(11,4)$	T_1 choose 4 from 11	$N_1 = C(11,4)$
T_2 choose 3 from 11	$N_2 = C(11,3)$	T_2 choose 3 from 7 ($11-4$)	$N_2 = C(7,3)$

$N = C(11,4)\cdot C(11,3) = 54{,}450$

$N = C(11,4)\cdot C(7,3) = 11{,}550$

Total # $= 54{,}450 + 11{,}550 = \boxed{66{,}000}$

EXAMPLE 8. **(a)** *How many different ways can 4 books be selected from a pile of 10 different books and arranged on a shelf?*

$n=10$
$r=4$

$$P(n,r) = P(10,4) = \boxed{5{,}040}$$

$$= C(10,4)\cdot 4!$$

(b) *How many ways can you select 4 books to read next week from a pile of 10 different books?*

$n=10$
$r=4$

$$C(n,r) = C(10,4) = 210$$

EXAMPLE 9. *Six friends go to the movies and all sit in the first row, which has 10 seats available. How many different seating arrangements of these six friends are possible in these 10 seats?*

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T_1: choose 6 seats from 10

or

and

T_2: arrange 6 friends on those 6 seats

$$\# \text{ways} = P(10,6) = \boxed{151{,}200}$$

$$\text{or } C(10,6)\cdot 6!$$

EXAMPLE 10. *Lauren has a bucket of sidewalk chalk. In the bucket there are 2 green, 8 red, 2 yellow, 4 blue and 4 pieces of white chalk. If she randomly pulls out 6 pieces of chalk, in how many ways can she pull out exactly 2 red chalks and 1 white chalk?*

20 chalks.

2 G
8 R
2 Y
4 B
4 W

T_1: pull out 2 R from 8R

and

T_2: pull out 1W from 4W

and

T_3: pull out 3 (not white, not red) from 8 (not W, R)

$N_1 = C(8,2)$

$N_2 = C(4,1) = 4$

$N_3 = C(8,3)$

$$N = N_1 \cdot N_2 \cdot N_3 = C(8,2)\cdot 4 \cdot C(8,3) = 6{,}272$$

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EXAMPLE 11. *A box contains 4 lime, 8 cherry and 10 blue raspberry Jolly Ranchers. If Jessica randomly selects 7 Jolly Ranchers from the box, in how many ways could she select exactly 5 of the same color?*

Total 22

4 L
8 C
10 B

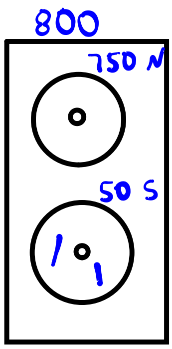
	T_1 select 5C from 8C	$C(8,5)$
and	T_2 select 2 from 4+10=14 other color	$C(14,2)$

OR

	T_1 select 5B from 10 B	$C(10,5)$
and	T_2 select 2 from 4+8=12	$C(12,2)$

$C(8,5)C(14,2) + C(10,5)C(12,2) = \boxed{21,728}$

EXAMPLE 12. *A box contains 800 DVD's of which 50 are scratched. In how many ways can you randomly select 25 DVD's such that at least 2 are scratched?*



I way
2 S & 23 N — $C(50,2)\,C(750,23)$
OR 3 S & 22 N — $+\,C(50,3)\cdot C(750,22)$
OR 4 S & 21 N — $+\,C(50,4)\cdot C(750,21)$
OR ⋮ — $+$ ⋮
25 S & 0 N — $C(50,25)\cdot C(750,0)$

difficult to calculate (TIME CONSUMING)

II WAY Use "compliment" task:

Randomly select 25 DVD's such that at most 1 is scratched.

0 S & 25 N OR 1 S & 24 N

$\underbrace{C(50,0)\cdot C(750,25) + C(50,1)\cdot C(750,24)}_{\#\text{ ways the "compliment" task can be performed}}$

$C(800,25) = \#$ ways to select 25 DVD's from 800

Final answer $= C(800,25) - \big(C(50,0)C(750,25) + C(50,1)C(750,24)\big)$

$= \ldots$

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EXAMPLE 13. *Six* [6] *cards are randomly selected from a standard deck of 52 playing cards. How many 6-card hands are possible*

(a) *if there are no restrictions?*

$$C(52,6) = 20,358,520$$

(b) *that have exactly 4 hearts or exactly 3 spades cards?*

4H OR 3S

4H from 13H & 2 from 52−13 = 39
OR
3S from 13S & 3 from 39

$$\# = C(13,4)\cdot C(39,2) + C(13,3)\cdot C(39,3) = \ldots$$

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identical

Distinct rearrangements (or, permutations of n objects, Not All Distinct):

EXAMPLE 14. *How many distinguishable ways can you rearrange the letters in the word BEAR? What about the word BEER?*

BEAR:

BEAR	*BERA*	*BARE*	*BAER*	*BREA*	*BRAE*
EBAR	*EBRA*	*EABR*	*EARB*	*ERBA*	*ERAB*
ABRE	*ABER*	*ARBE*	*AREB*	*AEBR*	*AERB*
RBEA	*RBAE*	*REBA*	*REAB*	*RABE*	*RAEB*

$\# = 4! = 24$

identical

BEER:

BEER	*BERE*	~~*BERE*~~	~~*BEER*~~	~~*BREE*~~	*BREE*
EBER	*EBRE*	*EEBR*	*EERB*	*ERBE*	*EREB*
~~*EBRE*~~	~~*EBER*~~	~~*ERBE*~~	~~*EREB*~~	~~*EEBR*~~	~~*EERB*~~
RBEE	~~*RBEE*~~	*REBE*	*REEB*	~~*REBE*~~	~~*REEB*~~

$$\# = \frac{4!}{2!} = \frac{24}{2} = 12$$

We do not need to rearrange E,E. Thus we have to drop this task. But that task can be performed in 2! ways, hence we have to divide by 2!.

Example: How many ways can you rearrange 3 identical triangles, 5 identical squares and 4 distinct rectangles?

$$\frac{12!}{3!\ 5!} = \ldots$$

• *If we have n objects in which n_1 of the objects are alike (same), then the number of permutations of these n objects taken n at a time would be $\frac{n!}{n_1!}$.*

EXAMPLE 15. *How many different arrangements can be made from the letters of MASSACHUSETTS?*

M A S C H U E T
A S T
S
S

$1+2+4+1+1+1+1+2 = 13$

$$\frac{13!}{2!\cdot 4!\cdot 2!} = 64{,}864{,}800$$

Appendix: **Standard Deck of Cards:** A deck of cards has 4 suits: *diamonds, hearts, clubs, and spades*. The suits of diamonds and hearts are both *red* and the suits of clubs and spades are both **black**. Each suit has the following denominations: Ace, 2, 3, 4, 5, 6, 7, 8, 9,10, Jack, Queen, and King. The Jacks, Queens and Kings are also called *face cards*.