## 7.5: Conditional Probability and Independent Events

EXAMPLE 1. *Two cards are drawn from a deck of 52 without replacement.* experiment $n(S)=C(52,2)$

(a) *What is the probability of that the first card drawn is a King?*

Experiment: draw 1 card of 52 $n(S_1)=C(52,1)=52$

Event E: draw 1K of 4K $n(E)=C(4,1)=4$

$$P(E)=\frac{n(E)}{n(S_1)}=\frac{4}{52}=\frac{1}{13}$$

Conditional probabilities

(b) *What is the probability that the second card drawn is a King giving that the first card drawn was a King?*

Experiment: draw 1 card of $52-1=51$ | $n(S_2)=51$

Event F: draw 1K of 3K | $n(F)=3$

$$P(F)=P(\text{2nd King}\mid\text{1st King})=P(F\mid E)=$$

$$=\frac{n(F)}{n(S_2)}=\frac{3}{51}=\frac{1}{17}$$

(c) *What is the probability the second card drawn is a King giving that the first card drawn was not a King?* $n(S_2)=51$

Event G: draw 1K of 4K $\Rightarrow n(G)=4$

$$P(G)=\frac{n(G)}{n(S_2)}=\frac{4}{51}$$

$$\| $$

$$P(F\mid E^c)$$

DEFINITION 2. *If $A$ and $B$ are events in an experiment and $P(A) \neq 0$, then the* ***conditional probability*** *that the event $B$ will occur, given that the event $A$ has already occurred, is*

$$\boxed{P(B|A) = \frac{P(A \cap B)}{P(A)}.} = \frac{n(A \cap B)}{n(A)}$$

EXAMPLE 3. *The data below were obtained from the financial aid office of a certain university:*

| | | $B$ | $B^c$ | |
|---|---|---|---|---|
| | | *Receiving Fin. Aid* | *Not Receiving Fin. Aid* | *TOTAL* |
| $A =$ | *Undergraduates* | 4,222 $= n(A \cap B)$ | 3,898 | 8,120 $= n(A)$ |
| $A^c =$ | *Graduates* | 1,879 $= n(B \cap A^c)$ | 731 | 2,610 $= n(A^c)$ |
| | *TOTAL* | 6,101 $= n(B)$ | 4,629 $= n(B^c)$ | 10,730 $= n(S)$ |

*Let $A$ be the event that a student selected at random from this university is an undergraduate student, and let $B$ be the event that a student selected at random is receiving financial aid. Find each of the following probabilities:*

1. $P(A) = \frac{n(A)}{n(S)} = \frac{8120}{10730} \approx 0.757$

2. $P(B) = \frac{n(B)}{n(S)} = \frac{6101}{10730} \approx 0.569$

3. $P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{4222}{10730} \approx 0.393$

4. $P(B|A) = \frac{n(B \cap A)}{n(A)} = \frac{4222}{8120} \approx 0.520$

5. $P(B|A^c) = \frac{n(B \cap A^c)}{n(A^c)} = \frac{1879}{2610} \approx 0.72$

EXAMPLE 4. *Given the probability distribution*

| $S$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $p$ | 0.1 | 0.2 | 0.3 | 0.15 | 0.08 | 0.17 |

probability distribution

*and the events* $A = \{a, c, d, e\}$, $B = \{b, d, f\}$, $C = \{a, b, d\}$.
*Find each of the following probabilities:*

1. $P(A|C) = \dfrac{P(A\cap C)}{P(C)} = \dfrac{P(\{a\}) + P(\{d\})}{P(\{a\}) + P(\{b\}) + P(\{d\})} = \dfrac{0.1+0.15}{0.1+0.2+0.15} \approx 0.55$

$A\cap C = \{a,c,d,e\}\cap\{a,b,d\} = \{a,d\}$

2. $P(C|B) = \dfrac{P(C\cap B)}{P(B)} = \dfrac{0.2+0.15}{0.2+0.15+0.17} \approx 0.673$

$C\cap B = \{b,d\}$

EXAMPLE 5. *During lunch period at a certain middle school 25% of the students buy vanilla ice-cream, 50% buy chocolate ice-cream, and 5% buy both products. Find the probability that*

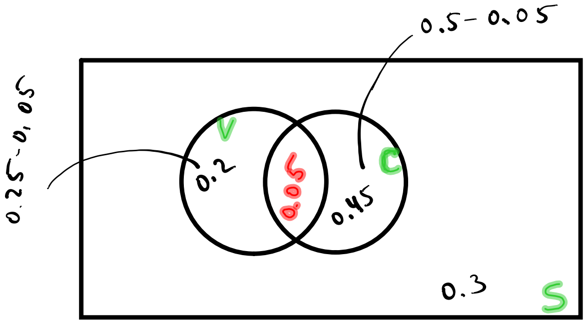
(a) *a student selected at random will buy vanilla ice-cream if he already bought chocolate ice cream.*

condition

$P(V) = \frac{25}{100} = 0.25$

$P(C) = \frac{50}{100} = 0.5$

$P(V\cap C) = \frac{5}{100} = 0.05$




$$P(V|C) = \frac{P(V\cap C)}{P(C)} = \frac{0.05}{0.5} = \boxed{0.1}$$

(b) *a student selected at random will not buy chocolate ice-cream if he already bought vanilla ice cream.*

condition

$$P(C^c|V) = \frac{P(C^c\cap V)}{P(V)} = \frac{0.2}{0.25} = 0.8$$

only vanilla

EXAMPLE 6. *A red die and a blue die are rolled at the same time (they are both fair die). What is the probability the sum of the numbers falling uppermost is 6, if it is known that the number falling uppermost on the red die is 4.*

E condition

| | | | | | |
|---|---|---|---|---|---|
| (1,1) | (1,2) | (1,3) | (1,4) | (1,5) | (1,6) |
| (2,1) | (2,2) | (2,3) | (2,4) | (2,5) | (2,6) |
| (3,1) | (3,2) | (3,3) | (3,4) | (3,5) | (3,6) |
| (4,1) | (4,2) | (4,3) | (4,4) | (4,5) | (4,6) |
| (5,1) | (5,2) | (5,3) | (5,4) | (5,5) | (5,6) |
| (6,1) | (6,2) | (6,3) | (6,4) | (6,5) | (6,6) |

$$P(E|F) = \frac{n(E\cap F)}{n(F)} = \frac{1}{6}$$

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- **INDEPENDENT EVENTS**

Two events $A$ and $B$ are independent if the outcome of one does not affect the outcome of the other.

DEFINITION 7. *If $A$ and $B$ are* ***independent events*** *then*

$$P(A|B) = P(A), \text{ and } P(B|A) = P(B).$$

| |
|---|
| *TEST for the Independence of Two Events: Two events, $A$ and $B$ are independent iff* $$P(A \cap B) = P(A) \cdot P(B)$$ |

EXAMPLE 8. *Are the events $A$ and $B$ from Example 3 independent events?*

$P(A \cap B) = 0.393$

$\neq \qquad \neq$

$P(A) \cdot P(B) = 0.757 \cdot 0.569$

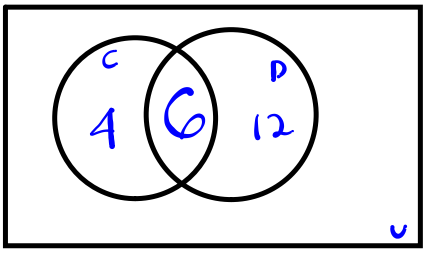
$\Rightarrow$ A & B are not independent.

REMARK 9. Do not confuse *independent* events with *mutually exclusive* events.

EXAMPLE 10. *A group of* 30 $= n(S)$ *high school students was asked about the types of pets they have, specifically, whether they have a cat or dog. The survey revealed the following information:*

*4 students have only cat; 12 students have only dog; 6 students have cat and dog.*

*Determine whether the event that the* $\underbrace{\text{student has cat}}_{C}$ *and the event that the* $\underbrace{\text{students has dog}}_{D}$ *are independent?*




C & D are independent if and only if

$$P(C\cap D) = P(C)\cdot P(D)$$

$$\frac{n(C\cap D)}{n(S)} = \frac{n(C)}{n(S)}\cdot\frac{n(D)}{n(S)}$$

$$\frac{6}{\cancel{30}} = \frac{4+6}{\cancel{30}}\cdot\frac{6+12}{30}$$

$$6 = \cancel{10}^{\,1}\cdot\frac{\cancel{18}^{\,6}}{\cancel{30}_{\,\cancel{3}\,1}}$$

$$6 = 6 \quad \checkmark$$

C and D are independent events

EXAMPLE 11. *Let $E, F, G$ are three independent events of an experiment with $P(E) = 0.4$, $P(F) = 0.75$ and $P(G) = 0.3$. Compute each of the following:*

(a) $P(E^c \cap F)$


E & F are independent $\Rightarrow P(E \cap F) = P(E) \cdot P(F) =$
$= 0.4 \cdot 0.75 = 0.3$

$P(E^c \cap F) = 0.75 - 0.3 = \boxed{0.45}$

(b) $P(E^c \cup F) = \underbrace{P(E^c)} + P(F) - P(E^c \cap F) =$

$= 1 - P(E) + P(F) - P(E^c \cap F) =$

$= 1 - 0.4 + 0.75 - 0.45 = \boxed{0.9}$

(c) $P(F|G) = P(F) = 0.75$ because F&G are independent

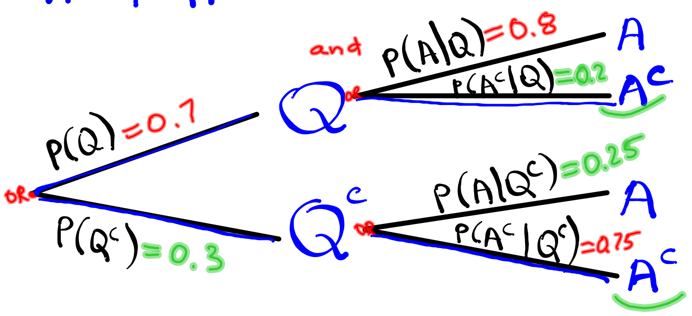
EXAMPLE 12. *A personnel manager at a certain company claims that he approves qualified applicants for a certain job 80% of the time; he rejects an unqualified person 75% of the time. It is known that 70% of applicants for this job are qualified.*

A Q

(a) *Construct the* probability tree *that represents this experiment.*

$Q = \{\text{qualified applicant}\}$

$A = \{\text{approved applicant}\}$




**(b)** *What is the probability that an applicant is not approved?*

$$P(A^c) = P(A^c \cap Q) + P(A^c \cap Q^c) =$$

$$= P(Q) \cdot P(A^c|Q) + P(Q^c)\, P(A^c|Q^c) =$$

$$= 0.7 \cdot 0.2 + 0.3 \cdot 0.75 = 0.365$$

(c) *What is the probability that an applicant is approved for the job if he/she is unqualified?*

$$P(A|Q^c) = 0.25 \quad (\text{use tree})$$

priori

(d) *What is the probability that an applicant is unqualified if he/she is approved for the job?*

$$P(Q^c|A) = \frac{P(Q^c \cap A)}{P(A)} = \frac{0.3 \cdot 0.25}{1 - 0.365} = 0.118$$

$P(A) = 1 - P(A^c)$

posteriori

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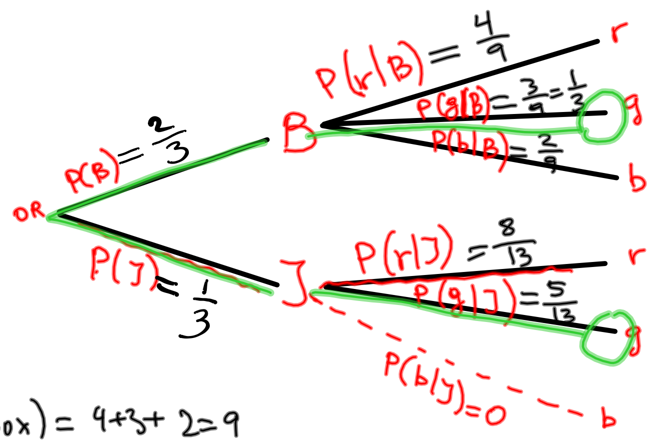
EXAMPLE 13. *Laura has a box (B) and a jar (J). Box contains 4 red, 3 green and 2 black marbles. Jar contains 8 red and 5 green marbles. The experiment is to draw a single marble. It is known that Laura is twice as likely to draw a marble from box as from jar.*

(a) *Construct the* probability tree *that represents this experiment.*




$$\begin{cases} P(B)=2P(J) \\ P(B)+P(J)=1 \end{cases}$$

$2P(J)+P(J)=1$

$3P(J)=1$

$P(J)=\frac{1}{3}$

$P(B)=2P(J)=\frac{2}{3}$

$n(Box)=4+3+2=9$

$n(Jar)=8$

(b) *Compute*

- $P(r|B)=\frac{4}{9}$
- $P(g|J)=\frac{5}{13}$
- $P(b|J)=0$
- $P(J\cap r)=P(J)\cdot P(r|J)=\frac{1}{3}\cdot\frac{8}{13}=\frac{8}{39}$
- $P(B\cap g)=\frac{2}{3}\cdot\frac{1}{3}=\frac{2}{9}$
- $P(g)=\frac{2}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{5}{13}=\frac{41}{117}$

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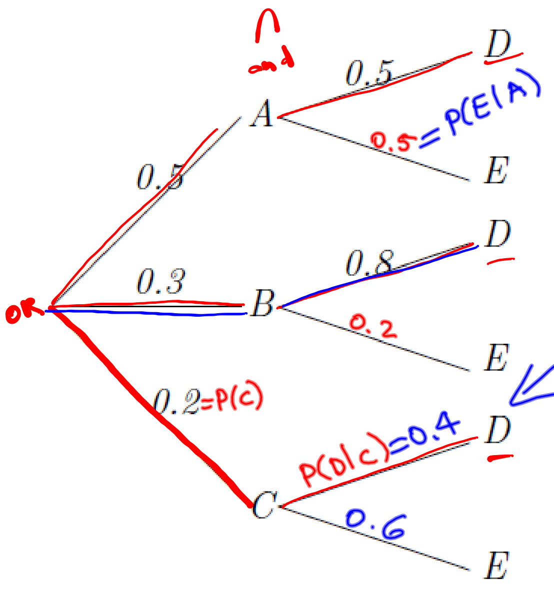
EXAMPLE 14. (a) *Complete the tree diagram below if* $P(C \cap D) = 0.08$.




$P(C \cap D) = P(C) \cdot P(D|C)$

$0.08 = 0.2 \cdot P(D|C)$

$P(D|C) = \dfrac{0.08}{0.2} = 0.4$

(b) *Compute:*

- $P(E|A) = \boxed{0.5}$
- $P(A \cup D) = \underbrace{P(A)}_{=\,0.5} + P(D) - P(A \cap D)$

  $+ 0.57 - 0.5 \cdot 0.5 = \boxed{0.82}$

$P(D) = 0.5 \cdot 0.5 + 0.3 \cdot 0.8 + 0.2 \cdot 0.4 = 0.57$

- $P(B|D) = \dfrac{P(B \cap D)}{P(D)} = \dfrac{0.3 \cdot 0.8}{0.57} \approx 0.421$

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