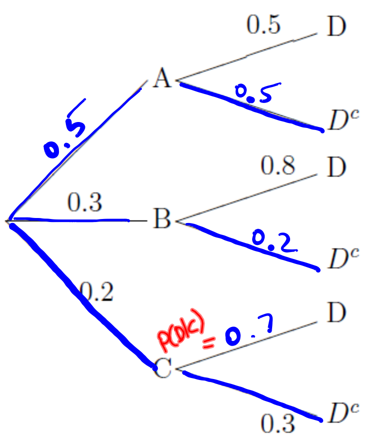
7.6: Bayes Theorem

In Section 7.5 weve been calculating a *priori probabilities* - the likelihood an event will occur. In this section we are going to look at a *posteriori probabilities* - calculating probability after an outcome has been observed.

Bayes' Theorem: $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$.

EXAMPLE 1. *Using the tree diagram below, find:*



(a) $P(D^c) = 0.5\cdot 0.5 + 0.3\cdot 0.2 + 0.2\cdot 0.3 = \boxed{0.37}$

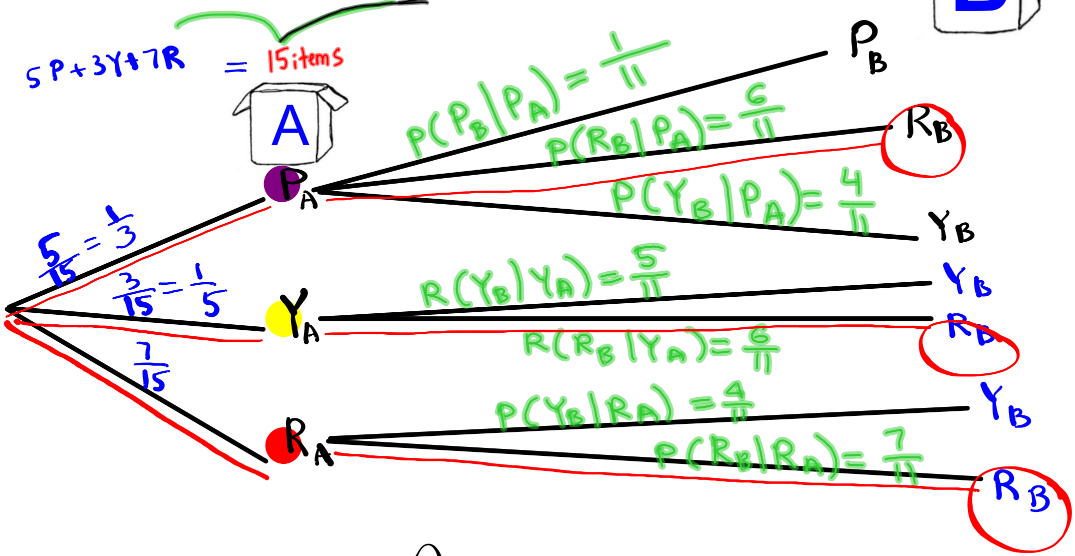
(b) $P(D) = 1 - P(D^c) = 1 - 0.37 = \boxed{0.63}$

(c) $P(D|C) = 0.7$

(d) $P(C|D) = \dfrac{P(C \cap D)}{P(D)} = \dfrac{0.2\cdot 0.7}{0.63} \approx \boxed{0.222}$

EXAMPLE 2. *Box A has 5 purple, 3 yellow, and 7 red items in it. Box B has 4 yellow and 6 red items in it. An item is drawn from Box A and transferred to Box B. An item is then drawn from Box B. The color of item drawn from each box is recorded.*

(a) *Draw a tree diagram representing this experiment.*



(b) *What is the probability that both items are purple?*

$$P(P_A \cap P_B) = P(P_A)\cdot P(P_B|P_A) = \frac{1}{3}\cdot\frac{1}{11} = \boxed{\frac{1}{33}}$$

(c) *What is the probability that the item drawn from the box A was yellow or the item drawn from the Box B was red?*

$$P(Y_A \cup R_B) = P(Y_A) + P(R_B) - P(Y_A \cap R_B) =$$

$$= \frac{1}{5} + \left(\frac{1}{3}\cdot\frac{6}{11} + \cancel{\frac{1}{5}\cdot\frac{6}{11}} + \frac{7}{15}\cdot\frac{7}{11}\right) - \cancel{\frac{1}{5}\cdot\frac{6}{11}} = \frac{112}{165}$$

(d) *What is the probability that the item drawn from the box B is purple if the item drawn from the Box A was yellow?*

$$P(P_B|Y_A) = 0$$

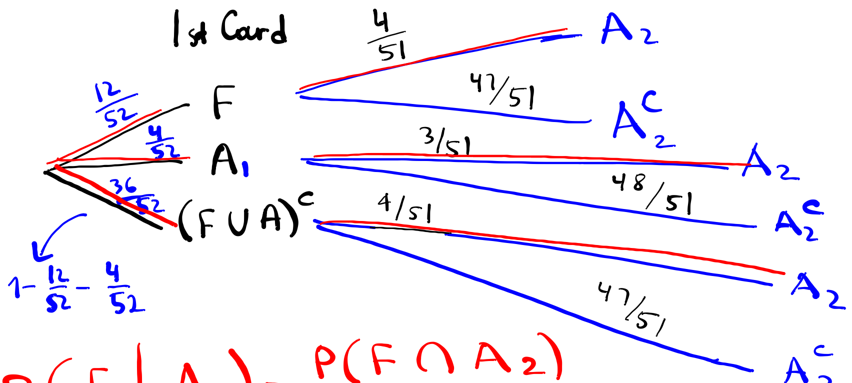
impossible event $= \varnothing$

(e) *What is the probability that the transferred item was red if a yellow item was selected from the Box B?*

$$P(R_A|Y_B) = \frac{P(R_A \cap Y_B)}{P(Y_B)} =$$

$$= \frac{\frac{7}{15}\cdot\frac{4}{11}}{\frac{1}{3}\cdot\frac{4}{11} + \frac{1}{5}\cdot\frac{5}{11} + \frac{7}{15}\cdot\frac{4}{11}} = \boxed{\frac{28}{63}}$$

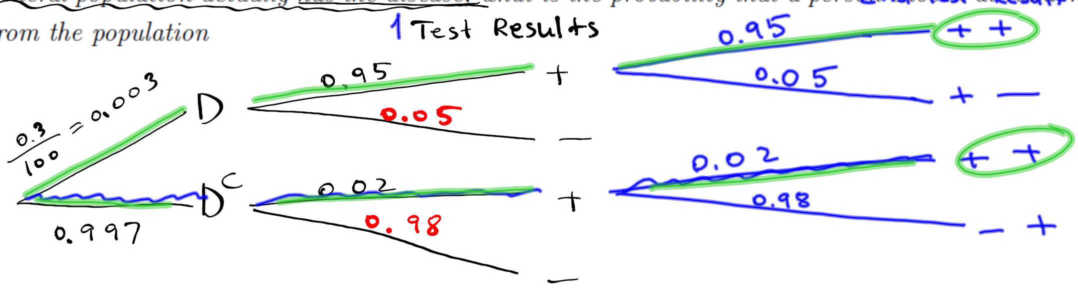
EXAMPLE 3. *Two cards are drawn from a deck of 52. What is the probability the 1st card is a* $\underbrace{\textit{face card}}_{F}$*, given that the 2nd card is an* $\underbrace{\textit{ace}}_{A}$*?*



$$P(F \mid A_2) = \frac{P(F \cap A_2)}{P(A_2)} =$$

$$= \frac{\frac{12}{52} \cdot \frac{4}{51}}{\frac{12}{52} \cdot \frac{4}{51} + \frac{4}{52} \cdot \frac{3}{51} + \frac{36}{52} \cdot \frac{4}{51}} = \frac{4}{17} \approx 0.2353$$

EXAMPLE 4. *If a certain disease is present, then a blood test will reveal it* $95\% = \frac{95}{100}$ *of the time. But the test will also indicate the presence of the disease* $2\% = \frac{2}{100}$ *of the time when in fact the person tested is free of that disease; that is, the test gives a false positive 2% of the time. If* $0.3\% = \frac{0.3}{100}$ *of the general population actually* $\underbrace{\textit{has the disease}}_{D}$*, what is the probability that a person chosen at random from the population*



(a) *has the disease given that he/she tested positive?*

$$P(D \mid +) = \frac{P(D \cap +)}{P(+)} = \frac{0.003 \cdot 0.95}{0.003 \cdot 0.95 + 0.997 \cdot 0.02} \approx$$

$$\approx 0.125$$

(b) *does not have the disease* ~~*given*~~ *that they took the test twice and got positive results both times.*

$$P(D^c \mid ++) = \frac{P(D^c \cap ++)}{P(++)} =$$

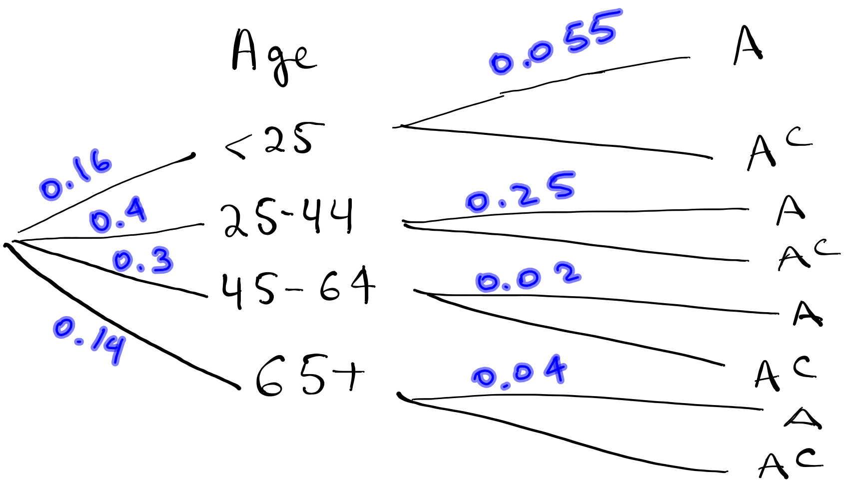
$$= \frac{0.997 \cdot 0.02 \cdot 0.02}{0.003 \cdot 0.95 \cdot 0.95 + 0.997 \cdot 0.02 \cdot 0.02} = 0.128$$

EXAMPLE 5.

A

Age Group	% of Insured Drivers	Accident rate, %
Under 25	16	5.5
25-44	40	2.5
45-64	30	2
65+	14	4

What is the probability that an insured driver selected at random



(a) *will be involved in an accident?*

$$P(A)=0.16\cdot 0.55+0.4\cdot 0.025+0.3\cdot 0.02+0.14\cdot 0.04=\boxed{0.0304}$$

(b) *who is involved in an accident is under 25?*

$$P(<25\mid A)=\frac{P(<25\cap A)}{P(A)}=$$

$$=\frac{0.16\cdot 0.055}{0.0304}\approx\boxed{0.289}$$