8.2: Expected Value $=$ Characteristic of random variable

- MEAN

The **average**, or **mean**, of a set of n numbers

$$x_1, x_2, x_3 \ldots, x_n$$

is denoted by $\overline{x}$, where

$$\overline{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

EXAMPLE 1. *In a class of* 10*, the grades on the last test were:*

$\overline{x} = (44 + 53 + 73 + 76 + 76 + 76 + 80 + 85 + 95 + 95)/10$

What was the class average for this test?

$\overline{x} = 75.3$

DEFINITION 2. *Let X denote the random variable with values of $x_1, x_2, x_3 \ldots, x_n$ and the probabilities of $p_1, p_2, p_3, \ldots, p_n$ respectively. Then the* ***expected value*** *of X is:*

$$\boxed{E(X) = x_1p_1 + x_2p_2 + x_3p_3 + \ldots + x_np_n.}$$

REMARK 3. A game is said to be *fair* when $E(X) = 0$.

EXAMPLE 4. *The chef at the local Italian restaurant recorded the weekly consumption of pasta over 16 week period.*

	x_1	x_2	x_3	x_4	x_5	
x Qts Pasta	100	120	140	160	180	
Frequency Weeks	2	3	4	4	3	$=16$
$P(X=x)$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	
	p_1	p_2	p_3	p_4	p_5	

(a) *Find the average weekly consumption of pasta at this restaurant.*

$$\bar{X} = \frac{2\cdot 100 + 3\cdot 120 + 4\cdot 140 + 4\cdot 160 + 3\cdot 180}{16} = \boxed{143.75}$$

Qts pasta

(b) *Let X denote the number of quarts of ~~salsa~~ pasta in a week at the restaurant. Compute the expected value of X.* same

$$E(X) = \frac{2}{16}\cdot 100 + \frac{3}{16}\cdot 120 + \frac{4}{16}\cdot 140 + \frac{4}{16}\cdot 160 + \frac{3}{16}\cdot 180 = \boxed{143.75}$$

EXAMPLE 5. *A sample of 7 DVD's is selected from a lot of 25 DVD's, of which 2 are defective. What is the expected number of defective DVD's.*

X = # of defective DVD's = $\{0, 1, 2\}$ Find domain of X

x	0	1	2
$P(X=x)$	$\frac{C(2,0)C(23,7)}{C(25,7)}$	$\frac{C(2,1)C(23,6)}{C(25,7)}$	0.07
	$= 0.51$	$= 0.42$	$= 1-0.51-0.42$

$$E(X) = 0\cdot 0.51 + 1\cdot 0.42 + 2\cdot 0.07 = 0.56$$

EXAMPLE 6. *The Island Club is holding a fund-raising raffle. Ten thousand tickets have been sold for $2 each. There will be a first prize of $3000, 3 second prizes of $1000 each, 5 third prizes of $500 each , and 20 consolation prizes of $100 each. Letting X denote the net winnings (that is, winnings less the cost of the ticket) associated with the tickets, find E(X). Interpret your results.*

10,000

winnings, $	0	100	500	1000	3000
− ticket cost	2	2	2	2	2
X: Net winnings x	−2	98	498	998	2998
Frequency	9971	20	5	3	1
$P(X=x)$	0.9971	0.002	0.0005	0.0003	0.0001

Frequency total $= 10,000$

$$E(x) = 0.9971\cdot(-2) + 0.002\cdot 98 + 0.0005\cdot 498 + 0.0003\cdot 998 + 0.0001\cdot 2998 = \boxed{-0.95}$$

Answer: − 95 cent

Conclusion: The game is unfair

EXAMPLE 7. *Bill, Laura and Amy play a card game with a standard deck of 52 cards. Bill selects a card from a well-shuffled deck and receives \$A from Laura and \$A from Amy if the card selected is a diamond; otherwise Bill pays Laura and Amy 3 dollars each. Determine the value of A if the game is to be fair.*

Define random variable:

X=random variable associated with Bill's winnings

Scenario	◇	$◇^c$	
x	$\overset{A+A}{2A}$	-6^{3+3}	
Frequency	13	39 (52−13)	Total = 52

Find A such that E(X)=0

$$E(X) = \bar{X}$$

$$\frac{13\cdot 2A + 39\cdot(-6)}{52} = 0$$

$$26A - 39\cdot 6 = 0$$

$$26A = 39\cdot 6$$

$$A = \frac{\cancel{39}^{3}\cdot\cancel{6}^{3}}{\cancel{26}_{\cancel{2}_1}} = 9$$

Answer: $9

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EXAMPLE 8. *A man purchased a* \$10,000 *1-yr term life insurance policy for* \$150. *Assuming that the probability that he will live another year is* 0.993, *find the company's expected gain.*

Define random variable:
X= company's gain. Then find E(X).
Probability distribution:

scenario	will live	NOT
x	150	150 − 10,000
$P(X=x)$	0.993	1 − 0.993 = 0.007

$$E(X) = 0.993 \cdot 150 + 0.007 \cdot (150 - 10{,}000) = 80$$

$$\boxed{\$80}$$

EXAMPLE 9. *Bob wishes to purchase a 5-yr term life-insurance policy that will pay the beneficiary* \$25,000 *in the event he dies in the next* 5 *years. The probability Bob lives the next* 5 *years is* 0.99. *What is the minimum amount Bob can expect to pay for this policy? (NOTE: the minimum premium occurs when the insurance company's expected profit is zero.)*

A

X - company's gain
Find A such that $E(X)=0$

Scenario	dies	not
x	A − 25,000	A
$P(X=x)$	0.01	0.99

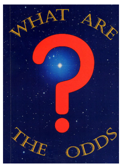
1 − 0.99

$$E(X) = 0.01(A - 25{,}000) + 0.99 \cdot A = 0$$

$$0.01A - 0.01 \cdot 25{,}000 + 0.99A = 0$$

$$A - 250 = 0$$

$$A = \$250$$



• ODDS

The odds that an event E will occur, odds in favor of E, are given as a to b or $a : b$ where a and b are integers and the fraction $\frac{a}{b}$ is in reduced form.

$a : b$

⇒ Computing ODDS from probability (Given $a : b$)

If $P(E)$ is the probability of an event, E, occurring, then

- the **odds in favor** of E occurring are

$$\frac{a}{b} = \frac{P(E)}{1-P(E)} = \frac{P(E)}{P(E^c)}, \quad P(E) \neq 1.$$

- the **odds against** E occurring are

$$\frac{b}{a} = \frac{1-P(E)}{P(E)} = \frac{P(E^c)}{P(E)}, \quad P(E) \neq 0.$$

EXAMPLE 10. *Find the odds in favor of drawing a face card from a deck of 52.*

E

$$P(E) = \frac{12}{52} = \frac{3}{13}$$

$$P(E^c) = 1 - \frac{3}{13} = \frac{10}{3}$$

$$\Rightarrow \frac{a}{b} = \frac{P(E)}{P(E^c)} = \frac{3/13}{10/13} = \frac{3}{10}$$

$$\boxed{3:10}$$

EXAMPLE 11. *If the probability David will win his tennis match tomorrow is* 0.75. *What are the odds against his winning his match tomorrow?* $b : a$

E

$$P(E) = 0.75$$

$$P(E^c) = 1 - 0.75 = 0.25$$

$$\Rightarrow \frac{b}{a} = \frac{P(E^c)}{P(E)} = \frac{0.25}{0.75} = \frac{1}{3}$$

$$\boxed{1:3}$$

If the odds in favor of an event, E, occurring are a to b, then the probability of E occurring is

$$P(E) = \frac{a}{a+b}.$$

derived from

$$\frac{a}{b} = \frac{P(E)}{1-P(E)}$$

EXAMPLE 12. *The odds in favor of the event E are 5 to 11.*

$a=5$
$b=11$

(a) *Find the odds against E.* $\boxed{11:5}$

(b) *Find $P(E)$ and $P(E^c)$.*

$$P(E) = \frac{a}{a+b} = \frac{5}{5+11} = \boxed{\frac{5}{16}}$$

$$P(E^c) = 1 - \frac{5}{16} = \boxed{\frac{11}{16}}$$

EXAMPLE 13. *Roger feels that the odds ~~in favor~~ of a business deal going through are 9 to 5. What is the (subjective) probability that this deal will* not *materialize?*

$a=9$, $b=5$

$E = \{$this deal will materialized$\}$

Find $P(E^c)$.

$$P(E) = \frac{a}{a+b} = \frac{9}{9+5} = \frac{9}{14}$$

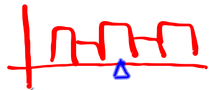
$$P(E^c) = 1 - \frac{9}{14} = \boxed{\frac{5}{14}}$$

• MEDIAN and MODE

odd

The **median** of a data set is the middle value of a set of data that is arranged in increasing or decreasing order. If there are 2 such values, the median is the mean of those 2 numbers. even

Median is the place where the area of the histogram is cut in half



The **mode** of a data set are the values that have the largest frequencies (occur most often).

most frequent

Mode corresponds to the TALLEST rectangle of the histogram.

EXAMPLE 14. *For the data set:*

50 46 43 42 45 49 48 40 52

(a) *What is the median?*

x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9

40 42 43 45 46 48 49 50 52

odd

9 . 2 = 4.5 median 46

(b) *What is the mode?*

All numbers have the same frequency = 1

$\Rightarrow$ $\boxed{\text{no mode}}$

EXAMPLE 15. *For the data set:*

2 3 4 4 4 5 5 5 11 11

10 numbers

10 : 2 = 5

(a) *What is the median?*

median $\dfrac{4+5}{2} = 4.5$

(b) *What is the mode?*

x	2	3	4	5	11
frequency	1	1	3	3	2

mode: $\boxed{\text{4 and 5}}$

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EXAMPLE 16. *The frequency distribution of the hourly wage rates (in dollars) among blue-collar workers in a certain factory is given in the following table. Find the mean (or average) wage rate, the mode, and the median wage rate of these workers.*

most frequent

x Wage Rate	10.70	10.80	10.90	11.00	11.10	11.20
Frequency	60	90	75	120	60	45

$= 450$

225 225

$$\bar{x} = \frac{60 \cdot 10.7 + 90 \cdot 10.8 + 75 \cdot 10.9 + 120 \cdot 11.00 + 60 \cdot 11.10}{450}$$

mean $\bar{x} \approx \$10.94$

mode $= \$11.00$

$450 : 2 = 225$

median:

$$\frac{x_{225} + x_{226}}{2} = \frac{10.9 + 11.00}{2} = \boxed{\$10.95}$$

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