

8.4: The Binomial Distribution (corresponds to finite discrete random variable)

Binomial Experiments have the following properties:

1. The number of trials in the experiment is fixed. n
2. There are 2 possible outcomes in each trial: success and failure. $q = 1 - p$
3. The probability of success in each trial is the same.
4. Trials are independent of each other.

EXAMPLE 1. Determine whether each of the following experiments is binomial or not:

(a) ✓ Draw a single card from a deck of 52 and its color is observed.

$n=1$
fixed
Outcomes: Black or Red
2 outcomes
 $p = \frac{1}{2}$
YES

(b) ✗ Draw a single card from a deck of 52 and record the suit.

$n=1$
fixed
Outcomes: ♠ ♥ ♦ ♣
4 outcomes $\neq 2$
NO

(c) ✗ Draw a single card from a deck of 52 and its color is observed. A second card is drawn (without replacement), and its color observed.

$n=2$
fixed
Outcomes: B or R
2
 $p = \frac{1}{2}$ $n=1$
 $p \neq \frac{1}{2}$ $n=2$
NO

(d) ✗ Roll a fair 6-sided die 10 times and record the number falling uppermost.

$n=10$
fixed
Outcomes: 1, 2, 3, 4, 5, 6
 $6 \neq 2$
NO

(e) ✓ Roll a fair 6-sided die 100 times and record if the number is even or odd.

$n=100$
fixed
Outcomes: Even or Odd
2
 $p = \text{the same}$
trials independent
YES

(f) ✗ Roll a fair 6-sided die until a 5 appears.

n is not fixed
NO

(g) ✓ Analyze the composition of a 5-child family in which each child was born at a different time (no twins, triplets, etc.).

n fixed
Outcomes B or G
2
YES

$p = \frac{1}{2}$
indep.

Calculating a Binomial Probability:

n, p, r

1. Determine if the experiment is binomial.
2. Determine the number of trials (n).
3. Define success in the experiment and find the probability of that success occurring (p).
(the probability of failure (q); note $q = 1 - p$)
4. Determine the number of successes desired (r). *exactly r , at least r , at most r*
5. Calculate the desired probability:

(a) Analytically, the probability is found by doing the following calculation:

$$P(X = r) = C(n, r)p^r(1 - p)^{n-r} = C(n, r)p^r q^{n-r}$$

EXAMPLE 2. Find the probability of exactly 4 successes in 6 trials of a binomial experiment in which the probability of success is $1/4$.

$$\begin{aligned} n &= 6 \\ r &= 4 \\ p &= \frac{1}{4} \end{aligned}$$

$$P(X=4) = \text{binompdf}(6, \frac{1}{4}, 4) = 0.033$$

(b) By the calculator: use functions binompdf or binomcdf:

• When you want to compute the probability of exact number of successes, use binompdf(n,p,r) command:



- Press 2nd VARS and select option A:binompdf
- Enter in the following format: binompdf(n,p,r), where
 - * n = number of trials
 - * p = probability of success in a single trial
 - * r = the exact # of successes you want.

• When you want to compute the probability of at least or at most a certain number of successes, use binomcdf(n,p,r) command: $P(0 \leq X \leq r)$

- Press 2nd VARS and select option B:binomcdf (this will give you the probability from 0 to r.)
- Enter in the following format: binomcdf(n,p,m), where
 - * n = number of trials
 - * p = probability of success in a single trial
 - * m = the maximum # of successes you want.

The binomcdf(n,p,m) command computes $P(0 \leq X \leq m)$, where X represents the number of successes.

EXAMPLE 3. Redo Example 2 using calculator.

EXAMPLE 4. Consider a 10-child family in which all children were born at different times (no multiples: twins, triplets, etc.).

Binomial $n=10$, outcomes Boy or Girl
 trials independent $\frac{1}{2}$

(a) What is the probability exactly 3 of the children are boys?

Success "boy"
 $p = \frac{1}{2}$
 $n = 10$
 $r = 3$

$$P(X=3) = \text{binompdf}(10, \frac{1}{2}, 3) = 0.117$$

(b) What is the probability at most 6 of the children are boys?

$0 \leq r \leq 6$

$$P(0 \leq X \leq 6) = \text{binomcdf}(10, \frac{1}{2}, 6) =$$

$$= 0.828$$

(c) What is the probability at least 6 of the children are girls?

Success "girl"
 $p = \frac{1}{2}$

$$P(X \geq 6) = 1 - P(0 \leq X \leq 5) = 1 - \text{binomcdf}(10, \frac{1}{2}, 5) =$$

$$= 1 - 0.623 = 0.377$$

(d) What is the probability at least 3, but no more than 6 of the children are girls?

$3 \leq r \leq 6$

$$P(3 \leq X \leq 6) = P(0 \leq X \leq 6) - P(0 \leq X \leq 2) =$$

$$= \text{binomcdf}(10, \frac{1}{2}, 6) - \text{binomcdf}(10, \frac{1}{2}, 2) =$$

$$= 0.828 - 0.055 = 0.773$$

EXAMPLE 5. There are 20 multiple choice questions on an exam in which each question has 5 answers. If Max knows the answer to 12 of them and randomly guesses at the remaining 8, what

is the probability Max will answer all 20 questions correctly?

Max will try to guess 8 questions with probability

of success $p = \frac{1}{5}$

$n = 8$
 $r = 8$

Binomial

$$P(X=8) = \text{binompdf}(8, \frac{1}{5}, 8) =$$

$$= 2.56 \cdot 10^{-6}$$

DEFINITION 6. For a binomial random variable X , we have the following:

$$\begin{array}{l}
 \text{mean} = E(X) = np \\
 \text{Var}(X) = npq = np(1-p) \\
 \sigma = \sqrt{npq} = \sqrt{np(1-p)}
 \end{array}
 \left. \begin{array}{l}
 q=1-p \\
 \text{Var}(X) = \sigma^2
 \end{array} \right\}$$

EXAMPLE 7. A certain university estimates that 89% of the freshman class will graduate within four years. From the incoming class of 70 students,

(a) find the expected number of ~~students~~ ^{students} who will graduate within 4 years. Success

$n=70$ $p=0.89$

$$E(X) = np = 70 \cdot 0.89 = 62.3 \approx \boxed{62 \text{ students}}$$

(b) compute the standard deviation of the number of students who will graduate within 4 years.

$$\sigma = \sqrt{npq} = \sqrt{70 \cdot 0.89 \cdot (1-0.89)} \approx 2.6 \approx \boxed{3 \text{ students}}$$

(c) what is the probability at least 63 will graduate within four years?

$$\begin{aligned}
 P(X \geq 63) &= 1 - P(0 \leq X \leq 62) = \\
 &= 1 - \text{binomcdf}(70, 0.89, 62) = 1 - 0.57 = \boxed{0.4}
 \end{aligned}$$

EXAMPLE 8. A fair die is rolled 2 times. If a 3 or a 6 lands uppermost in a trial, then the throw is considered a success. Otherwise, the throw is considered a failure.

(a) Find the probability of obtaining exactly 0, 1, or 2 successes. $r=0, 1, \text{ or } 2$

$$n=2 \quad p = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$X = \# \text{ success}$
 $X = \{0, 1, 2\}$

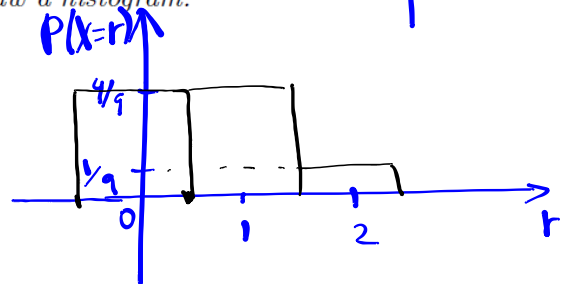
$$P(X=0) = \text{binompdf}(2, \frac{1}{3}, 0) = \frac{4}{9}$$

$$P(X=1) = \text{binompdf}(2, \frac{1}{3}, 1) = \frac{4}{9}$$

$$P(X=2) = \text{binompdf}(2, \frac{1}{3}, 2) = \frac{1}{9}$$

(b) Construct a binomial distribution and draw a histogram.

r	0	1	2
$P(X=r)$	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$



(c) What is the probability of obtaining 0 or 1 success in the experiment?

$$P(X=0 \cup X=1) = P(X=0) + P(X=1) = \frac{4}{9} + \frac{4}{9} = \frac{8}{9}$$

(d) What is the probability of obtaining at least one success in the experiment?

$$P(X \geq 1) = P(X=1) + P(X=2) = \frac{4}{9} + \frac{1}{9} = \frac{5}{9}$$

(e) Compute the mean, variance, and standard deviation of X , binomial random variable representing # of successes.

$$\text{mean} = E(X) = np = 2 \cdot \frac{1}{3} = \frac{2}{3}$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{\frac{2}{3} \cdot \frac{2}{3}} = \frac{2}{3}$$

$$\text{Var}(X) = \sigma^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$