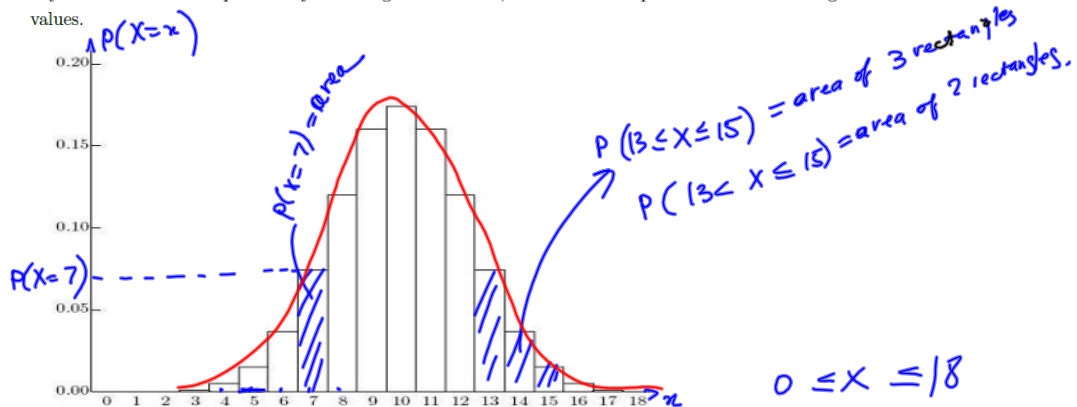# 8.5&8.6: The Normal Distribution and Applications

RECALL: To find a probability distribution with a *discrete finite* random variable can be represented graphically by a histogram. Characteristics of a histogram:

- Each rectangle is centered around x.

- Each rectangle has a base of width 1.

- The height of each rectangle represents the probability for that particular value of X, $P(X = x)$.

- The area of any given rectangle (width x height) is equal to the probability of that random variable.

- If you want to find the probability of a range of X values, we would add up the ares over the range of X values.

$$P(13 \leq X \leq 15) = \text{area of 3 rectangles}$$
$$P(13 < X \leq 15) = \text{area of 2 rectangles}.$$

$$0 \leq X \leq 18$$

Continuous Probability Distributions are probability distributions associated with conti random variables. For a continuous random variable, the graph of the probability distril becomes a smooth curve. To find probability with a *continuous* random variable,X, we **probability density function, $f(x)$.** This function has the properties:

1. $f(x) \geq 0$ for all values of X;

2. The area under the graph of $f(x)$ is equal to 1 (on the values of X).

area = 1
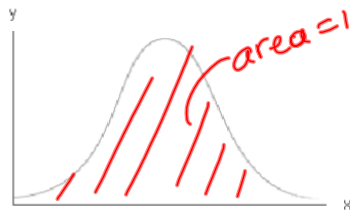
$P(a < x < b) =$ area under the probability distribution from $x = a$ to $x = b$.
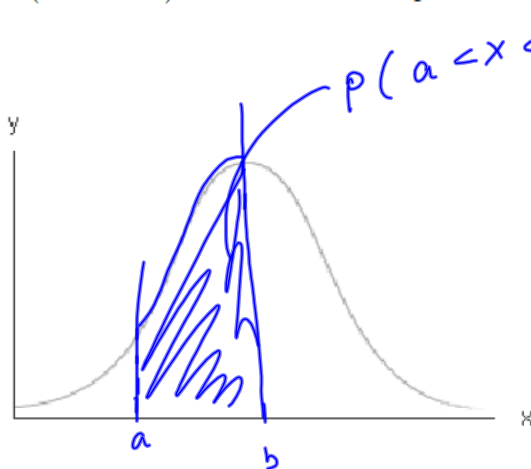
$$P(a < x < b)$$

REMARK 1. Because the area under one point of the graph of f is equal to zero, we see that:

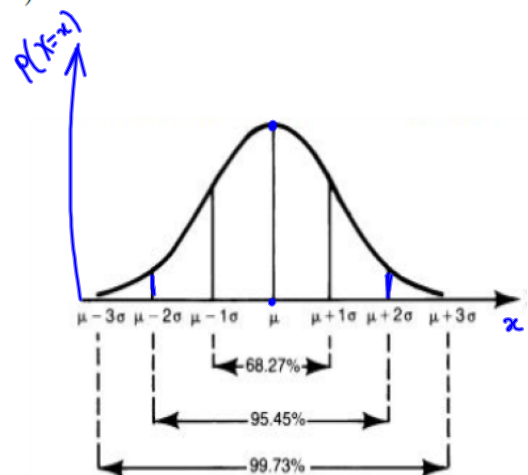$$\boxed{P(a < x < b) = P(a < x \leq b) = P(a \leq x < b) = P(a \leq x \leq b)}$$

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We concentrate on a special class of continuous probability distributions known as **normal distributions**.

Each normal distribution is defined by μ (mean) *average* and σ (standard deviation; σ determines the sharpness or flatness of the curve).

Characteristics of the Normal Distribution Curve:

1. The curve has a peak at $x = \mu$.

2. The curve is symmetric about a vertical line passing through the mean, $x = \mu$.

3. The curve always lies above the x-axis but approaches the x-axis as x extends indefinitely in either direction. (The curve never crosses the x-axis.)

4. The area under the curve is always 1.

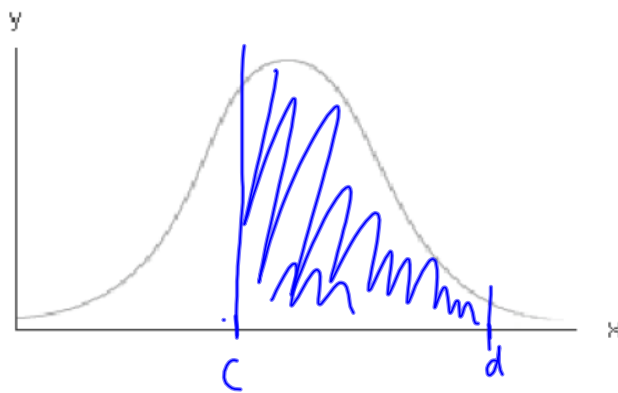5. Regardless of the value of μ and σ:

   - 68.27% of the area lies within 1 σ from the mean.

   - 95.45% of the area lies within 2 σ from the mean.
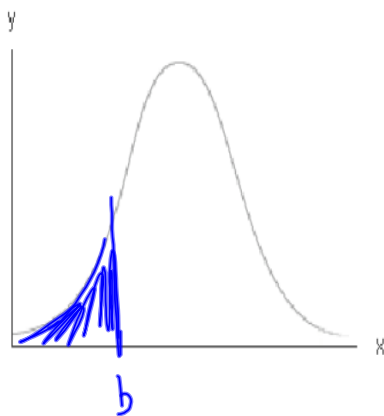
   - 99.73% of the area lies within 3 σ from the mean.

$P(X=x)$

$\mu-3\sigma$ $\mu-2\sigma$ $\mu-1\sigma$ μ $\mu+1\sigma$ $\mu+2\sigma$ $\mu+3\sigma$ x

68.27%

95.45%

99.73%

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EXAMPLE 2. *Shade the area under the normal curve that represents these probabilities:*
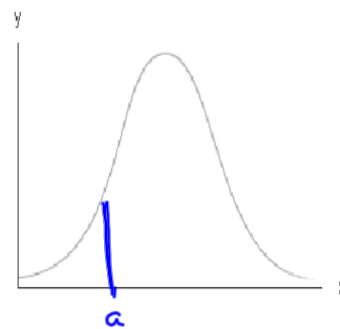
(a) $P(X \geq a)$
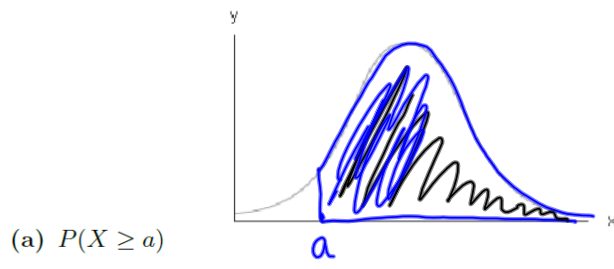
(b) $P(c < X \leq d)$

(c) $P(X \leq b)$

(d) $P(X = a) = 0$

To find the probability of a normal distribution, we will use the built-in feature of your calculator: Go to DIST which is found by pressing 2nd VARS.

normalcdf(lower,upper,μ, σ) computes the probability that a continuous random variable X is between the lower bound and the upper bound, $P(\text{lower} < X < \text{upper})$.

Compute	Lower	Upper	
$P(X < a)$	-1E99	a	normalcdf(-1EE99,a,μ, σ)
$P(a < X < b)$	a	b	normalcdf(a,b,μ, σ)
$P(X > b)$	b	1E99	normalcdf(b,1EE99,μ, σ)

REMARK 3. To enter 1E99 press 1 EE 9 9. The EE represents scientific notation and the EE is found by pressing 2nd ,.
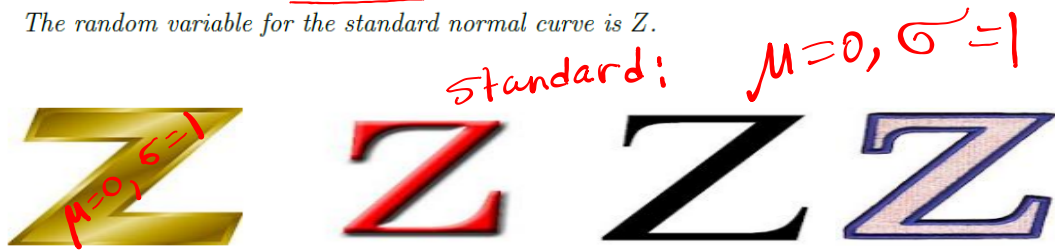
EXAMPLE 4. *Suppose X is a normal random variable with $\mu = 70$ and $\sigma = 4$. each of the following probabilities:*

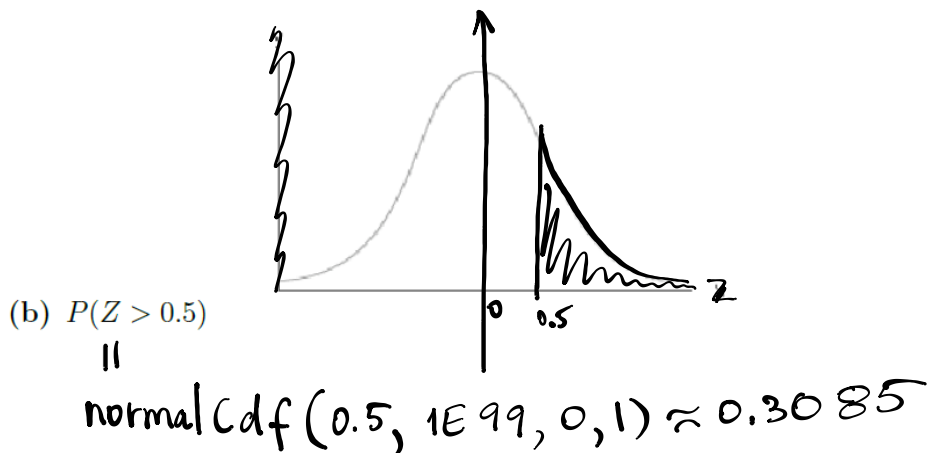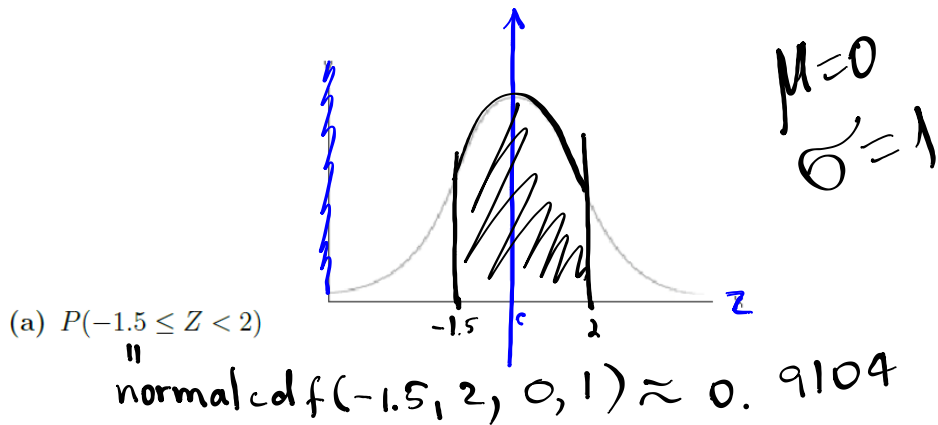(a) $P(66 \leq X < 74)$ = normalcdf $(66, 74, 70, 4) \approx 0.6827$

(b) $P(X > 70)$ = $P(70 < X < \infty)$ = normalcdf $(70, 1E99, 70, 4) \approx 0.5$

(c) $P(X \leq 72)$ = $P(-\infty < X \leq 72)$ = normalcdf $(-1E99, 72, 70, 4) \approx 0.6915$

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DEFINITION 5. The **standard normal curve** is the normal curve with $\mu = 0$ and $\sigma = 1$. The random variable for the standard normal curve is Z.

standard: $\mu = 0, \sigma = 1$

$\mu = 0, \sigma = 1$

EXAMPLE 6. Let Z be the standard normal variable. Make a sketch of the appropriate r under the standard normal curve, and then find the values of

$\mu = 0$
$\sigma = 1$

(a) $P(-1.5 \leq Z < 2)$

$=$

normalcdf$(-1.5, 2, 0, 1) \approx 0.9104$

(b) $P(Z > 0.5)$

$=$

normalcdf$(0.5, 1E99, 0, 1) \approx 0.3085$

EXAMPLE 7. Find the area under the standard normal curve to the left of $z = 1.27$.

area $= P(Z < 1.27) =$
$=$ normalcdf$(-1E99, 1.27, 0, 1)$
$=$
0.8980

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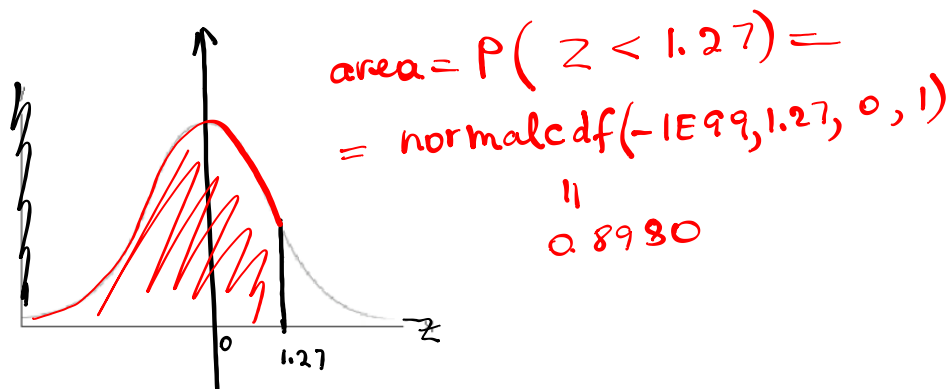
To find the cutoff, when given a probability:

Go to DIST which is found by pressing 2nd VARS. Your 3rd choice is invnorm":

$$\text{invnorm}(\text{probability}, \mu, \sigma).$$

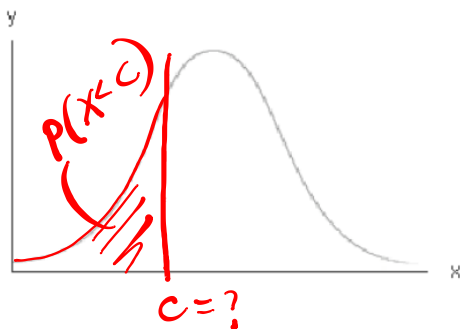Here the probability is given and is sometimes referred to as "area".

REMARK 8. In order for this feature on your calculator to work, you must always enter in the probability LESS THAN the cutoff you are looking for, i.e.

we use invnorm(d, μ, σ) to find c in $P(X < c) = d$, given d.

↳ area to the left of $c = P(X < c)$

⇩

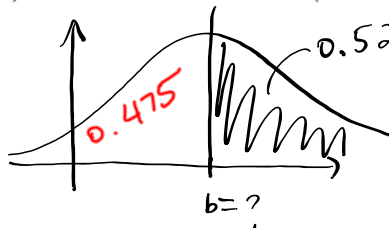$$c = \text{invnorm}(d, \mu, \sigma)$$

$c = ?$

EXAMPLE 9. *Let X be a normally distributed random variable with $\mu = 53$ and $\sigma = 4$.*

(a) *Find c such that $P(X < c) = 0.325$*

$$c = \text{inv norm}(0.325, 53, 4) \approx \boxed{51.185}$$

(b) *Find b such that $P(X \geq b) = 0.525$*

$b = ?$

$$P(X \leq b) = 1 - P(X \geq b) =$$
$$= 1 - 0.525 = 0.475$$
$$b = \text{invnorm}(0.475, 53, 4) \approx \boxed{52.74}$$

EXAMPLE 10. *Find the value of c such that*   Z is standard normal distr.

(a) $P(Z < c) = 0.14$

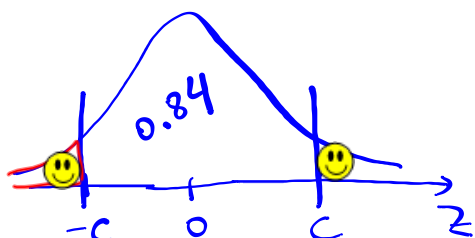$$c = \text{invnorm}(0.14, 0, 1) \approx 1.0803 \qquad \mu = 0, \quad \sigma = 1$$

(b) $P(Z > c) = 0.65$

$$P(Z < c) = 1 - P(Z > c) = 1 - 0.65 = 0.35$$
$$c = \text{invnorm}(0.35, 0, 1) = \boxed{-0.3853}$$

(c) $P(-c < z < c) = 0.84$

☺ $+ 0.84 +$ ☺ $= 1$

$$☺ = \frac{1 - 0.84}{2} = 0.08$$

$$P(Z < c) = 0.08$$

$$-c = \text{invnorm}(0.08, 0, 1) \approx -1.405$$
$$\boxed{c = 1.405}$$

EXAMPLE 11. *A certain company manufactures artificial starter logs for fireplaces. These logs are accepted by the buyer if they fall within the tolerance limits of 0.695 inches and 0.780 inches in length. Assuming that the length of the logs is normally distributed with a mean of 0.72 inches and a standard deviation of 0.03 inches, estimate the percentage of logs that will be rejected by the buyer.*

event $E = \{$logs are accepted$\}$

$X = $ length

$\mu = 0.72$

$\sigma = 0.03$

Find $P(E^c) = ?$

$$P(E) = P(0.695 \le X \le 0.780) =$$
$$= \text{normalcdf}(0.695, 0.780, 0.72, 0.03) =$$
$$= 0.7749$$
$$P(E^c) = 1 - 0.7749 \approx 0.2251$$
$$\boxed{22.51\%}$$

EXAMPLE 12. *At a certain hospital, the weight of infants is normally distributed with a mean of 7.5 lbs and a standard deviation of 1.1 lbs.* $\mu = 7.5 \text{ lbs}$, $\sigma = 1.1 \text{ lbs}$

$X = $ weight of infants

(a) *What is the probability that a randomly selected infant at this hospital weighs more than 8 lbs?*

$P(E) = P(X > 8) = $ normalcdf$(8, 1E99, 7.5, 1.1) = 0.325$

(b) *What is the probability that a randomly selected infant at this hospital weighs exactly 7.5 lbs?*

$P(X = 7.5) = 0$

(c) *Only 1% of all infants at this hospital weigh less than* ☺ *lbs.*

$P(X < ☺) = 0.01$

$☺ = $ invnorm$(0.01, 7.5, 1.1) = \boxed{4.94 \text{ lbs}}$

(d) *25% of all infants at this hospital weigh more than* ★ *lbs.*

$P(X > ★) = 0.25 \implies P(X < ★) = 1 - 0.25 = 0.75$

$★ = $ invnorm$(0.75, 7.5, 1.1) \approx \boxed{8.24 \text{ lbs}}$

(e) *If you randomly access records of 1000 infants at this hospital, how many of those infants would you expect to weigh more than 9 lbs?*

$n =$

$P(X > 9) = $ normalcdf$(9, 1E99, 7.5, 1.1) = 0.086$

$E(X) = np = 1000 \cdot 0.086 = \boxed{86 \text{ infants}}$

You would expect 86 of the babies to weigh more than 9 lbs.

EXAMPLE 13. *An instructor gave an exam to his class that had an average of 65 and standard deviation of 13. He decided to assign grades as follows: the top 6% and the bottom 6% will receive A's and F's, respectively. The next 16% in either direction will be given B's and D's, and the remaining students will receive C's. Assuming that the grades on the exam are normally distributed, find the cutoffs for each grade level.*

$\mu =$

$\sigma =$

$100 - 6 - 6 - 16 - 16 = 56\%$

16%   0.56   16%

6%   0.16   6%

0.16   0.06

0.06

D   C   B   A
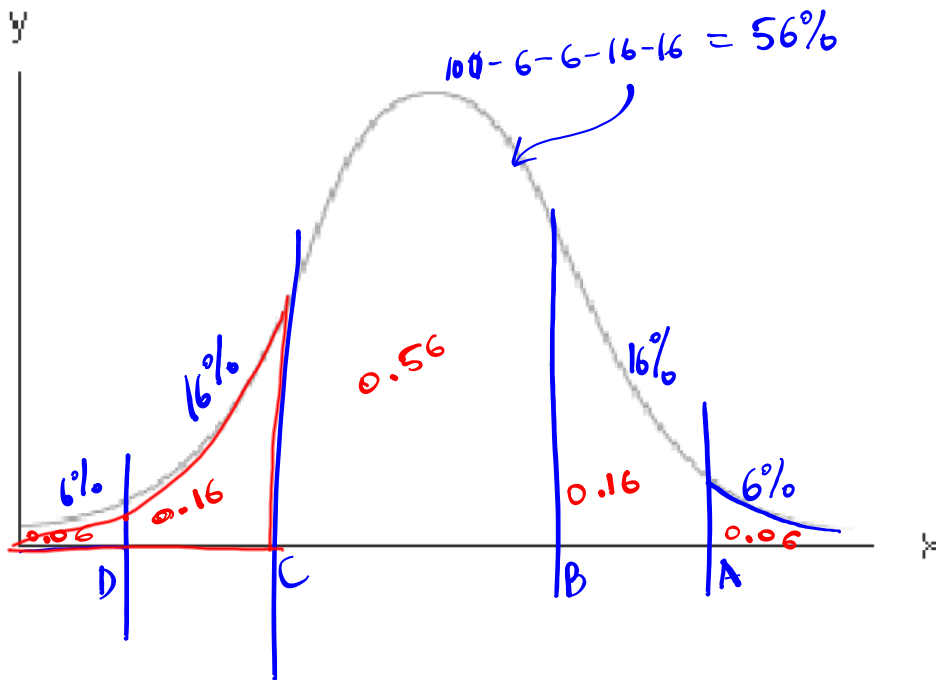
$F = 0$

$P(x < D) = 0.06 \Rightarrow D = \text{invnorm}(0.06, 65, 13) = \boxed{45}$

$P(X < C) = 0.16 + 0.06 = 0.22 \Rightarrow$
$\Rightarrow C = \text{invnorm}(0.22, 65, 13) = \boxed{55}$

$P(X < B) = 0.22 + 0.56 = 0.78$
$B = \text{invnorm}(0.78, 65, 13) = \boxed{75}$

$P(X < A) = 1 - P(X > A) = 1 - 0.06 = 0.94$
$A = \text{invnorm}(0.94, 65, 13) = \boxed{85}$

A	B	C	D	F
85+	75+	55+	45+	0+

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