6.3: The Definite Integral

DEFINITION 1. *The* **definite integral of** f **from** a **to** b *is*

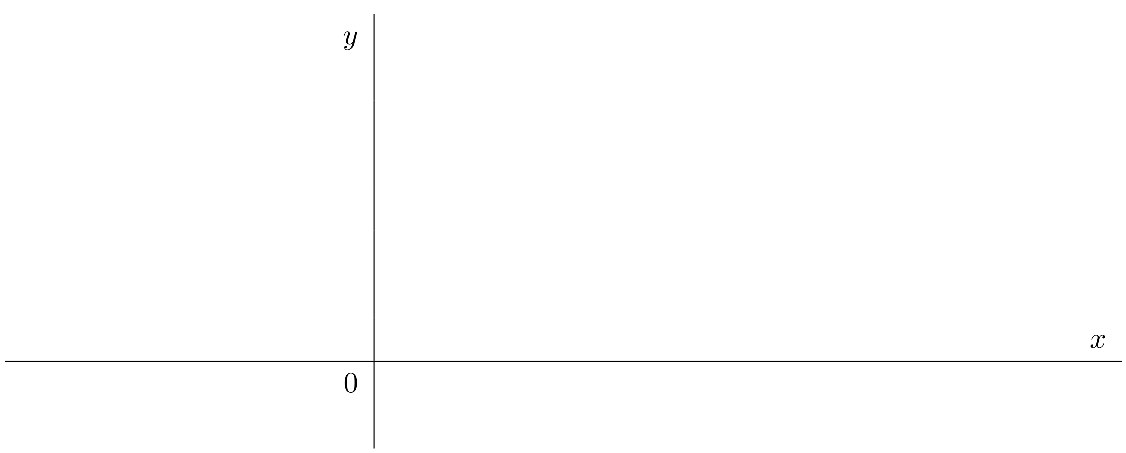
$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

if this limit exists. Here P *is a partition of the interval* $[a,b]$*,* $\Delta x = (b-a)/n$*, and* x_i^* *is any point in the* i*th subinterval. If the limit does exist, then* f *is called* **integrable** *on the interval* $[a,b]$*.*

EXAMPLE 2. *Express the limit as a definite integral:*

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[3\left(1+\frac{2i}{n}\right)^5 - 6\right]\frac{2}{n}$$

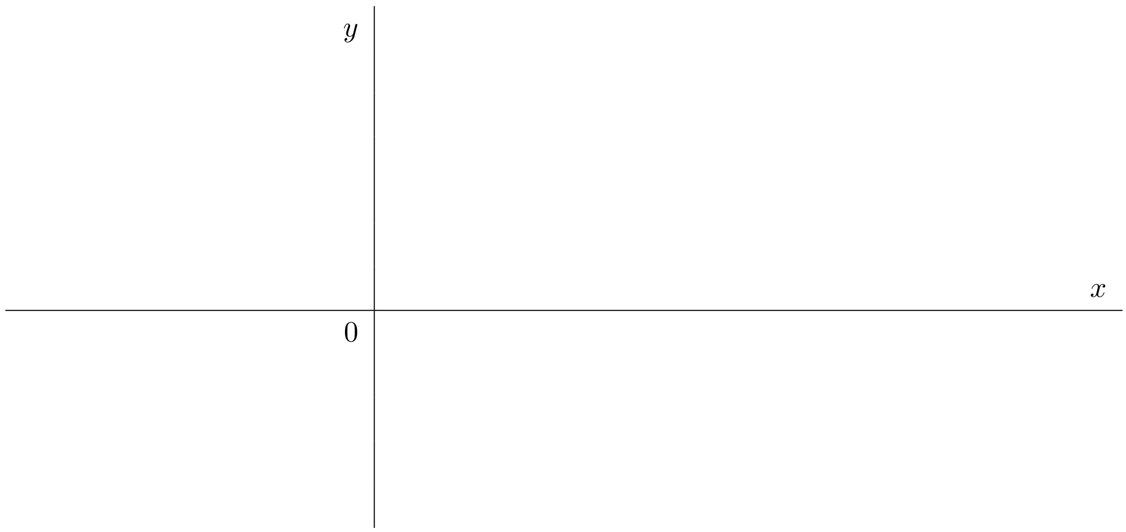
If $f(x) > 0$ on the interval $[a,b]$, then the definite integral is the area bounded by the function f and the lines $y=0$, $x=a$ and $x=b$.



In general, a definite integral can be interpreted as a difference of areas:

$$\int_a^b f(x)\,\mathrm{d}x = A_1 - A_2$$

where A_1 is the area of the region above the x and below the graph of f and A_2 is the area of the region below the x and above the graph of f.



EXAMPLE 3. *Evaluate the following integrals by interpreting each in terms of areas:*

(a) $\displaystyle\int_{-5}^{0} (1+\sqrt{25-x^2})\,\mathrm{d}x$

(b) $\int_{-1}^{3} (2-x)\,\mathrm{d}x$

Properties of Definite Integrals:

- $\int_a^b \mathrm{d}x = b - a$
- $\int_a^b f(x) \pm g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$
- $\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$, where c is any constant
- $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$, where $a \le c \le b$
- $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$
- $\int_a^a f(x)\,\mathrm{d}x = 0$
- If $f(x) \ge 0$ for $a \le x \le b$, then $\int_a^b f(x)\,\mathrm{d}x \ge 0$
- If $f(x) \ge g(x)$ for $a \le x \le b$, then $\int_a^b f(x)\,\mathrm{d}x \ge \int_a^b g(x)\,\mathrm{d}x$
- If $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)\,\mathrm{d}x \le M(b-a)$.

EXAMPLE 4. *Write as a single integral:*

$$\int_3^5 f(x)\,\mathrm{d}x + \int_0^3 f(x)\,\mathrm{d}x - \int_6^5 f(x)\,\mathrm{d}x + \int_5^5 f(x)\,\mathrm{d}x$$

EXAMPLE 5. *Estimate the value of* $\int_0^\pi (4\sin^5 x + 3)\,\mathrm{d}x$