6.4: The fundamental Theorem of Calculus

The fundamental Theorem of Calculus :

PART I If $f(x)$ is continuous on $[a,b]$ then $g(x) = \displaystyle\int_a^x f(t)\,\mathrm{d}t$ is continuous on $[a,b]$ and differentiable on (a,b) and $g'(x) = f(x)$.

EXAMPLE 1. *Differentiate* $g(x) = \displaystyle\int_{-4}^{x} e^{2t}\cos^2(1-5t)\,\mathrm{d}t$

EXAMPLE 2. *Let $u(x)$ be a differentiable function and $f(x)$ be a continuous one. Prove that*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^{u(x)} f(t)\,\mathrm{d}t\right) = f(u(x))u'(x).$$

Let $u(x)$ and $v(x)$ be differentiable functions and $f(x)$ be a continuous one. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{v(x)}^{u(x)} f(t)\,\mathrm{d}t\right) = f(u(x))u'(x) - f(v(x))v'(x).$$

EXAMPLE 3. *Differentiate $g(x)$ if*

(a) $g(x) = \displaystyle\int_{-4}^{x^3} e^{2t}\cos^2(1-5t)\,\mathrm{d}t$

(b) $g(x) = \displaystyle\int_{e^{x^2}}^{1} \frac{t+1}{\ln t + 3}\,\mathrm{d}t$

(c) $g(x) = \int_{x^2}^{\sin x} \frac{\cos t}{t}\, \mathrm{d}t$

PART II If $f(x)$ is continuous on $[a, b]$ and $F(x)$ is any antiderivative fort $f(x)$ then

$$\int_a^b f(x)\, \mathrm{d}x = F(x)\Big|_a^b = F(b) - F(a).$$

EXAMPLE 4. *Evaluate*

1. $\int_1^5 \frac{1}{x^2}\, \mathrm{d}x$

2. $\int_{-\pi/2}^{0} (\cos x - 4\sin x)\, \mathrm{d}x$

3. $\displaystyle\int_0^1 (u^3+2)^2\,\mathrm{d}u$

EXAMPLE 5. *Evaluate*

1. $\displaystyle\int_1^2 \frac{2x^5-x+3}{x^2}\,\mathrm{d}x$

2. $\displaystyle\int_0^3 |3t-5|\,\mathrm{d}t$

Applications of the Fundamental Theorem

If a particle is moving along a straight line then application of the Fundamental Theorem to $s'(t) = v(t)$ yields:

$$\int_{t_1}^{t_2} v(t)\,\mathrm{d}t = s(t_2) - s(t_1) = \text{displacement}.$$

Show that

$$\text{total distance traveled} = \int_{t_1}^{t_2} |v(t)|\,\mathrm{d}t.$$

EXAMPLE 6. *A particle moves along a line so that its velocity at time t is $v(t) = t^2 - 2t - 8$. Find the displacement and the distance traveled by the particle during the time period $1 \le t \le 6$.*