Section 1.2: The Dot Product

Let's start with two equivalent definitions of dot product.

DEFINITION 1. *The* **dot product** *of two nonzero vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *is the number*

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta,$$

where θ *is the angle between the vectors* $\mathbf{a}$ *and* $\mathbf{b}$, $0 \le \theta \le \pi$. *If either* $\mathbf{a}$ *or* $\mathbf{b}$ *is* $\mathbf{0}$, *then we define* $\mathbf{a}\cdot\mathbf{b} = 0$.

DEFINITION 2. *The* **dot product** *of two given vectors* $\mathbf{a} = \langle a_1, a_2\rangle$ *and* $\mathbf{b} = \langle b_1, b_2\rangle$ *is the number*

$$\mathbf{a}\cdot\mathbf{b} = a_1 b_1 + a_2 b_2.$$

Note that the formula from Definition 1 is often used not to compute a dot product but instead to find the angle between two vectors. Indeed, it implies:

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} =$$

EXAMPLE 3. *Given* $\mathbf{a} = \langle 2, -3\rangle$ *and* $\mathbf{b} = \langle 3, -4\rangle$.

(a) *Compute the dot product of* $\mathbf{a}$ *and* $\mathbf{b}$.

(b) *Determine the angle between* $\mathbf{a}$ *and* $\mathbf{b}$.

Note that

$$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$$

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$$

$$\mathbf{0}\cdot\mathbf{a} = 0$$

The dot product gives us a simple way for determining if two vectors are perpendicular (or orthogonal), namely,

Two nonzero vectors a and b are orthogonal if and only if $\mathbf{a}\cdot\mathbf{b} = 0$. (*Prove it!*)

EXAMPLE 4. *Determine whether the given vectors are orthogonal, parallel, or neither.*

(a) $\langle 3, 4\rangle$, $\langle -8, 6\rangle$

(b) $\langle -7, -4\rangle$, $\langle 28, 16\rangle$

(c) $\langle 1, 1\rangle$, $\langle 2, 3\rangle$

EXAMPLE 5. *What is the dot product of* $12\mathbf{j}$ *and* $11\mathbf{i}$*?*

DEFINITION 6. *The* **work** *done by a force* $\mathbf{F}$ *in moving and object from point* A *to point* B *is given by*

$$W = \mathbf{F} \cdot \mathbf{D}$$

where $\mathbf{D} = \overrightarrow{AB}$ *is the distance the object has moved (or displacement).*

EXAMPLE 7. *Find the work done by a force of* $50lb$ *acting in the direction* $N30^oW$ *in moving an object* $10ft$ *due west.*

EXAMPLE 8. *A constant force* $\mathbf{F} = 25\mathbf{i} + 4\mathbf{j}$ *(the magnitude of* $\mathbf{F}$ *is measured in Newtons) is used to move an object from* $A(1,1)$ *to* $B(5,6)$*. Find the work done if the distance is measured in meters*

DEFINITION 9. *The* **orthogonal compliment of** $\mathbf{a} = \langle a_1, a_2 \rangle$ **is** $\mathbf{a}^{\perp} = \langle -a_2, a_1 \rangle$.

Note that $|\mathbf{a}| = \left|\mathbf{a}^{\perp}\right|$ and $\mathbf{a} \cdot \mathbf{a}^{\perp} =$

EXAMPLE 10. *Given* $\langle 4, -2 \rangle$, $\langle 2, -1 \rangle$, $\langle -2, 1 \rangle$ *and* $\mathbf{a} = \langle 1, 2 \rangle$*. Which of these vectors is*

- *orthogonal to* $\mathbf{a}$*?*

- *the orthogonal compliment of* $\mathbf{a}$*?*

Scalar and vector projections: For given two vectors $\mathbf{a}$ and $\mathbf{b}$ we determine the projection of $\mathbf{b}$ onto $\mathbf{a}$.

- The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is denotes by $\text{proj}_{\mathbf{a}}\mathbf{b}$ and can be found by the formula

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}.$$

- The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ (or the component of $\mathbf{b}$ along $\mathbf{a}$) is denotes by $\text{comp}_{\mathbf{a}}\mathbf{b}$ and can be found by the formula

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}.$$

EXAMPLE 11. *Given* $\mathbf{a} = \langle 4, 3\rangle$ *and* $\mathbf{b} = \langle 1, -1\rangle$. *Find:*

- $\mathbf{a} \cdot \mathbf{b} =$

- $|\mathbf{a}| =$

- $|\mathbf{b}| =$

- $\text{proj}_{\mathbf{b}}\mathbf{a} =$

- $\text{comp}_{\mathbf{a}}\mathbf{b} =$

EXAMPLE 12. *Find the distance from the point* $P(-2, 3)$ *to the line* $y = 3x + 5$.