

Section 2.5: Continuity

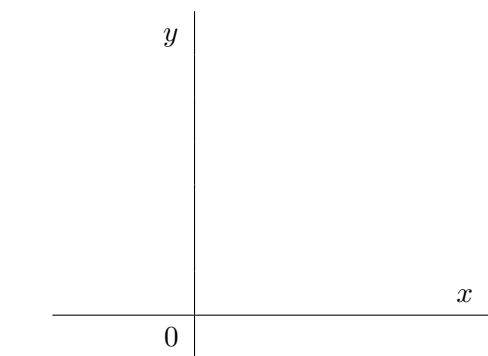
DEFINITION 1. A function $f(x)$ is **continuous** at $x = a$ if $\lim_{x \rightarrow a} f(x) = f(a)$. More implicitly: if f is continuous at a then

1. $f(a)$ is defined (i.e. a is in the domain of f);
2. $\lim_{x \rightarrow a} f(x)$ exists.
3. $\lim_{x \rightarrow a} f(x) = f(a)$.

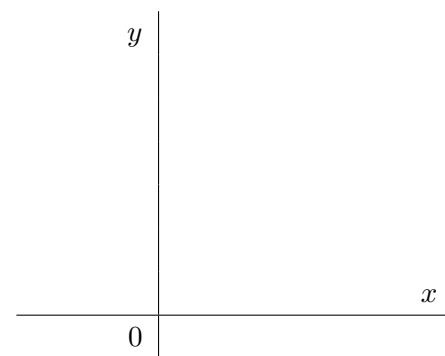
A function is said to be continuous on the interval $[a, b]$ if it is continuous at each point in the interval.

Geometrically, if f is continuous at any point in an interval then its graph has no break in it (i.e. can be drawn without removing your pen from the paper).

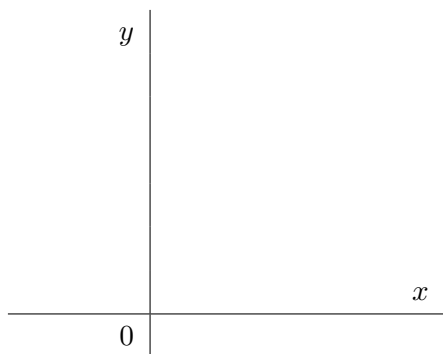
REASONS FOR BEING DISCONTINUOUS:



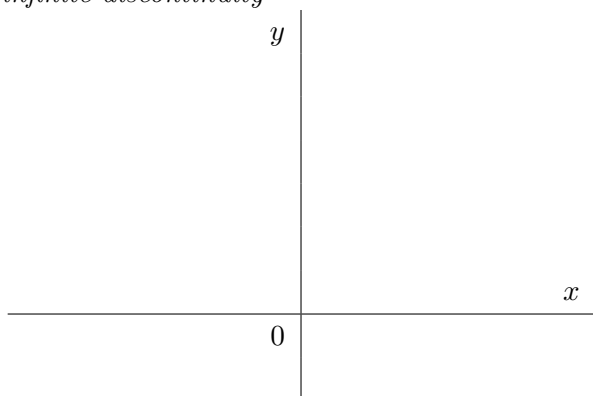
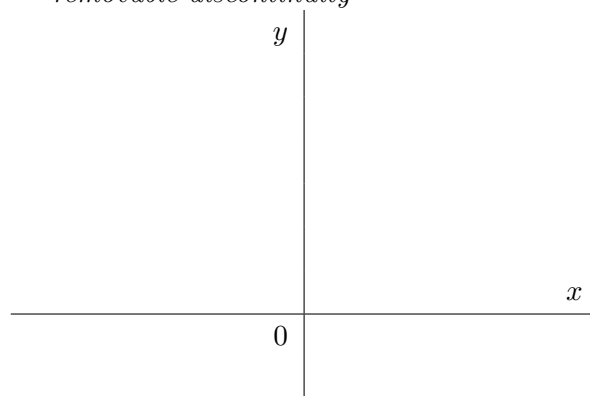
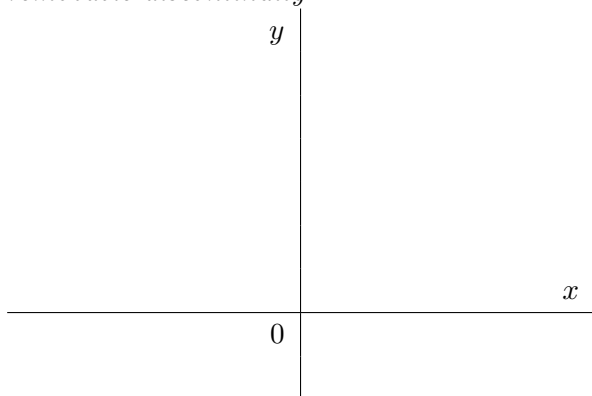
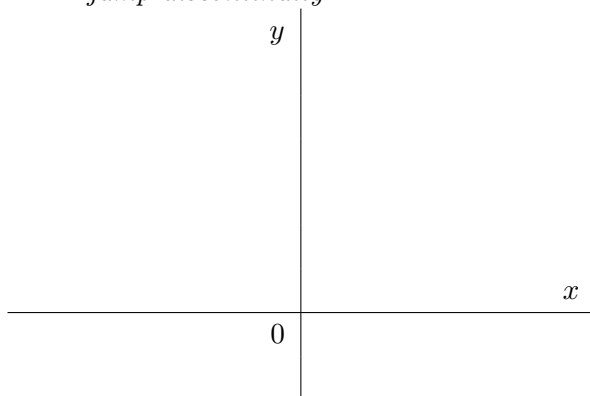
$f(a)$ is not defined
(i.e. a is not in the domain of f)



$f(a)$ is defined, but
the limit as $x \rightarrow a$ DNE



$f(a)$ is defined and $\lim_{x \rightarrow a} f(x)$ exists,
but $\lim_{x \rightarrow a} f(x) \neq f(a)$

Classification of discontinuities:*infinite discontinuity**removable discontinuity**removable discontinuity**jump discontinuity*

EXAMPLE 2. Explain why each function is discontinuous at the given point:

(a) $f(x) = \frac{2x}{x-3}, \quad x = 3$

(b) $f(x) = \begin{cases} \frac{x^2 - 2x + 1}{x - 1} & \text{if } x \neq 1 \\ 5 & \text{if } x = 1, \end{cases} \quad x = 1$

DEFINITION 3. A function f is **continuous from the right** at $x = a$ if

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

and f is **continuous from the left** at a if

$$\lim_{x \rightarrow a^-} f(x) = f(a).$$

REMARK 4. Functions **continuous on an interval** if it is continuous at every number in the interval. At the end point of the interval we understand continuous to mean continuous from the right or continuous from the left.

EXAMPLE 5. Find the interval(s) where $f(x) = \sqrt{9 - x^2}$ is continuous.

EXAMPLE 6. Find the constant c that makes g continuous on $(-\infty, \infty)$:

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx^2 - 1 & \text{if } x \geq 4 \end{cases}$$

EXAMPLE 7. For each of the following, find all discontinuities, classify them by using limits, give the continuity interval(s) for the corresponding function. If the discontinuity is removable, find a function g that agrees with the given function except of the discontinuity point and is continuous at that point.

(a) $f(x) = \frac{x^2 - 9}{x^4 - 81}$

(b) $f(x) = \frac{7}{x + 12}$

$$(c) f(x) = \begin{cases} x^2 + x & \text{if } x < 2 \\ 8 - x & \text{if } x > 2 \\ 4 & \text{if } x = 2 \end{cases}$$

Intermediate Value Theorem: If $f(x)$ is continuous on the closed interval $[a, b]$ and N is any number strictly between $f(a)$ and $f(b)$, then there is a number c , $a < c < b$, so that $f(c) = N$.

EXAMPLE 8. If $f(x) = x^5 - 2x^3 + x^2 + 2$, show there a number c so that $f(c) = 1$.

EXAMPLE 9. *Show that following equation has a solution (a root) between 1 and 2:*

$$3x^3 - 2x^2 - 2x - 5 = 0.$$