10.1: Sequences

A **sequence** ia a list of numbers written in a definite order.

General sequence terms are denotes as follows:

$$\begin{array}{ccc c} a_1 & - & first & term \\ a_2 & - & second & term \\ & \vdots & & \\ a_n & - & n^{th} & term \\ a_{n+1} & - & (n+1)^{th} & term \\ & \vdots & & \end{array}$$

Notice that, in general, $a_{n+1} \neq a_n + 1$.

A sequence can be defined as a function whose domain is the set of positive numbers:

NOTATION: $\{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\}$, $\{a_n\}$, $\{a_n\}_{n=1}^{\infty}$.

EXAMPLE 1. *Write down the first few terms of the following sequences:*

(a) $\left\{\dfrac{n+1}{n^2}\right\}_{n=1}^{\infty}$

(b) $\left\{\dfrac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$

(c) The Fibonacci sequence $\{f_n\}$ is defined recursively:

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}, \quad n \geq 3.$$

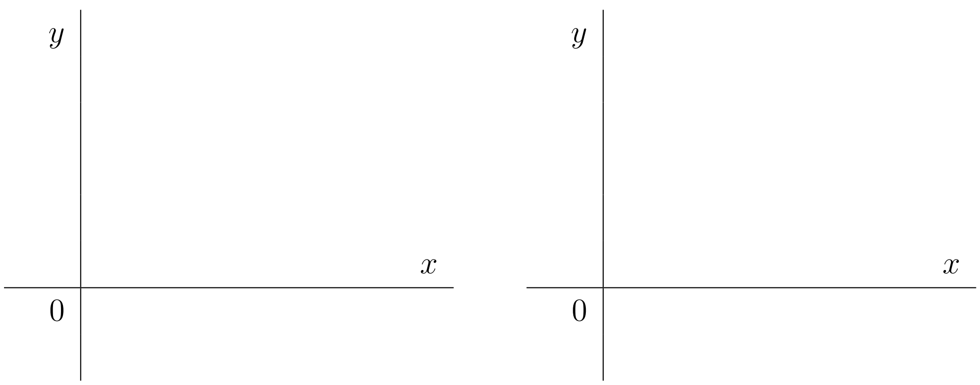
EXAMPLE 2. *Find a general formula for the sequence:*

(a) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \ldots$

(b) $-\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$

DEFINITION 3. *If* $\lim_{n\to\infty} a_n$ *exists then we say that the sequence* $\{a_n\}$ **converges** *(or is* **convergent.***) Otherwise, we say the sequence* $\{a_n\}$ **diverges** *(or is* **divergent.***)*

Graphs of two sequences with $\lim_{n\to\infty} a_n = L$.



EXAMPLE 4. *Determine if* $\{a_n\}_{n=1}^{\infty}$ *converges or diverges. If converges, find its limit.*

(a) $a_n = \dfrac{n+1}{2n+3}$

(b) $a_n = \dfrac{3n^2-1}{10n+5n^2}$

(c) $a_n = \arctan(2n)$

(d) $a_n = \ln(2n+4) - \ln n$

(e) $a_n = \cos \frac{\pi n}{2}$

(f) $a_n = \frac{3 + (-1)^n}{n^2}$

DEFINITION 5. *A sequence* $\{a_n\}$ *is* **bounded above** *if there is a number* M *s.t.*

$$a_n \leq M \quad \textit{for all } n.$$

A sequence $\{a_n\}$ *is* **bounded below** *if there is a number* m *s.t.*

$$m \leq a_n \quad \textit{for all } n.$$

If its bounded above and below, then a_n *is a* **bounded sequence**.

DEFINITION 6. *A sequence* $\{a_n\}$ *is* **increasing** *if*

$$a_n < a_{n+1} \quad \textit{for all } n.$$

A sequence $\{a_n\}$ *is* **decreasing** *if*

$$a_n > a_{n+1} \quad \textit{for all } n.$$

MONOTONIC SEQUENCE THEOREM. Every bounded, monotonic sequence is convergent.

EXAMPLE 7. *Determine whether a_n is increasing, decreasing or not monotonic.*

(a) $a_n = -n^2$

(b) $\left\{\dfrac{2}{n^2}\right\}_{n=5}^{\infty}$

(c) $\left\{(-1)^{n+1}\right\}_{n=1}^{\infty}$

(d) $a_n = \dfrac{\sqrt{n+1}}{5n+3},\ n = 0, 1, 2 \ldots$