10.6: Representation of Functions as Power Series

Problem: Find the sum of the following series: $\sum_{n=0}^{\infty} x^n$

EXAMPLE 1. *Find a power series representation for $f(x)$ and determine the interval of convergence.*

(a) $f(x) = \dfrac{1}{5-x}$

(b) $f(x) = \dfrac{x}{5-x}$

(c) $f(x) = \dfrac{1}{1+x^2}$

(d) $f(x) = \dfrac{x^2}{16x^4 - 9}$

Differentiation and Integration of power series

THEOREM 2. *If the power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has radius of convergence* $R > 0$, *then the function* f *defined by*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ *and*

- $f'(x) = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$
- $\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$

The radii of convergence of the power series for $f'(x)$ *and* $\int f(x)\,\mathrm{d}x$ *are both* R.

EXAMPLE 3. *Find a power series representation for* $f(x)$ *and determine the radius of convergence.*

(a) $f(x) = \dfrac{1}{(5-x)^2}$

(b) $f(x) = \ln(2 + x)$

(c) $f(x) = \arctan(2x)$

EXAMPLE 4. *Use a power series decomposition to approximate the integral*

$$\int_0^{0.2} \frac{1}{1+x^4}\,\mathrm{d}x$$

to six decimal places.