10.7: Taylor and Maclaurin Series

Problem: Assume that a function $f(x)$ has a power series representation about $x = a$:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

and $f(x)$ has derivatives of every order. Find formula for c_n in terms of f.

Solution. We have

$f(x) =$

Putting $x = a$ in $f(x)$, we get $\boxed{}$

$f'(x) =$

Substituting $x = a$ we have $\boxed{}$

Similarly,

$f''(x) =$

then $x = a \Rightarrow f''(a) =$ and $\boxed{}$

$f'''(x) =$

then $x = a \Rightarrow f''(a) =$ and $\boxed{}$

Continuing in this manner, you can see the pattern:

$f^{(n)}(a) =$

The Taylor series for $f(x)$ about $x = a$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n =$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

Split the Taylor series as follows:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \underbrace{\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{T_N(x) \\ N-\text{th degree} \\ \textbf{Taylor polynomial}}} + \underbrace{\sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{R_N(x) \\ \textbf{Remainder}}}$$

THEOREM 1. *If* $\lim_{n\to\infty} R_N(x) = 0$ *when* $|x-a| < R$ *then*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \qquad |x-a| < R.$$

REMARK 2. In all examples that we will be looking at, we assume that $f(x)$ has a power series expansion, i.e.
$\lim_{n\to\infty} R_N(x) = 0$ for some R. (This means you don't need to show it.)

EXAMPLE 3. *Given that function f has power series expansion (i.e. Taylor series) centered at 4. Find this expansion and its radius of convergence if it is given that*

$$f^{(n)}(4) = \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)}{2^{3n-1} n}$$

EXAMPLE 4. *Find Taylor series for* $f(x) = e^{3x}$ *at* $x = 1$*. What is the associated radius of convergence?*

EXAMPLE 5. *Find Taylor series for* $f(x) = \ln x$ *at* $x = 1$*. What is the associated radius of convergence?*

EXAMPLE 6. *Find Taylor series for* $\ln(1+x)$ *centered at* $x = 0$. *What is the associated radius of convergence?*

The Maclaurin series is the Taylor series about $x = 0$ **(i.e. a=0):**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

EXAMPLE 7. *Find the Maclaurin series for* $f(x)$*:*

(a) $f(x) = e^x$

(b) $f(x) = e^{-x}$

(c) $f(x) = x^5 e^{-2x^2}$

EXAMPLE 8. *Find the Maclaurin series for $f(x)$:*

(a) $f(x) = \cos x$

(b) $f(x) = \sin x$

Known Mclaurin series and their intervals of convergence you must have memorized:

$\frac{1}{1-x}$	$=$	$\sum_{n=0}^{\infty} x^n$	$=$	$1 + x + x^2 + x^3 + \dots$		$(1,1)$
e^x	$=$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$=$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$		$(-\infty,\infty)$
$\cos x$	$=$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$=$	$1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$		$(-\infty,\infty)$
$\sin x$	$=$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$=$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$		$(-\infty,\infty)$
$\arctan x$	$=$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$=$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$		$[-1,1]$

EXAMPLE 9. *Find the sum of the series:*

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n!)}$

(b) $\sum_{n=0}^{\infty} \frac{2012^n}{n!}$

(c) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$

EXAMPLE 10. **(a)** *Determine Maclaurin Series for* $\displaystyle\int \frac{\sin x}{x}\,\mathrm{d}x$

(b) *Evaluate* $\displaystyle\int_0^1 \frac{\sin x}{x}\,\mathrm{d}x$ *correct to within an error of* 0.001.

EXAMPLE 11. *Use series to find the limit:* $\lim_{x\to 0} \dfrac{1-\cos x}{1+x-e^x}$