9.3: Arc Length

Description of a curve by parametric equations:

$$x = x(t), \quad y = y(t), \quad \alpha \le t \le \beta$$

EXAMPLE 1. *Parameterize the following curves:*

(a) $y = \sqrt{1 - x^2}$

(b) $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

(c) $y = \sin x$, $0 \le x \le \pi$

(d) $y^4 = x$

(e) $y = f(x)$, $a \le x \le b$

(f) $x = g(y)$, $c \le y \le d$

DEFINITION 2. *A curve $C : x = x(t), \quad y = y(t), \quad \alpha \le t \le \beta$, is called* **smooth** *if the derivatives $x'(t)$ and $y'(t)$ are continuous and not simultaneously zero on $[\alpha, \beta]$.*

The length of a smooth curve

$$C : x = x(t), \quad y = y(t), \quad \alpha \le t \le \beta,$$

assuming that C is traversed exactly once as t increases from α to β is

$$\int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2}\, \mathrm{d}t =: \int_C \mathrm{d}s,$$

where ds is called *differential of the arc length function* and

$$\mathrm{d}s = \sqrt{[x'(t)]^2 + [y'(t)]^2}\mathrm{d}t$$

EXAMPLE 3. *Find the length of the curve*

$$C : \;\; x = 3t - t^3, \quad y = 3t^2, \quad 0 \le t \le 2.$$

Particular cases for the differential of the arc length function:

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

- C is given by $y = f(x)$, $a \le x \le b$

- C is given by $x = g(y)$, $c \le y \le d$

EXAMPLE 4. *Find the arc length of the graph of* $y = \ln \cos x$ *from* $x = 0$ *to* $x = \pi/4$.

EXAMPLE 5. *Find the arc length of the curve* $(y-1)^3 = x^2$ *from* $x = 0$ *to* $x = 8$.

EXAMPLE 6. *A telephone wire hanging between two poles (at* $x = -100$ *and* $x = 100$*) takes the shape of a catenary with equation*

$$y = 75(e^{x/150} + e^{-x/150}).$$

Find the length of the wire.