Fall 2012 Math 152

Week in Review 3

courtesy: *Oksana Shatalov*

(covering Sections 7.3, 7.4& 7.5)

7.3: Volumes by Cylindrical Shells

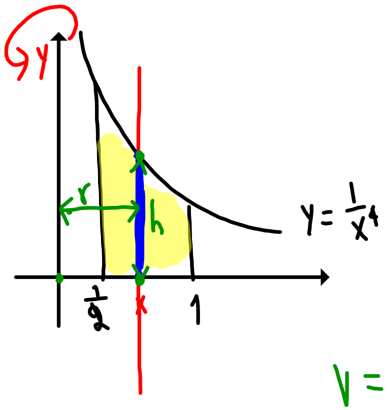
Key Points

- Area of cross sections: $A = 2\pi(\text{radius})(\text{height})$.
  - For rotation about a *vertical* axis we use $V = \int_a^b A(x)\,\mathrm{d}x = 2\pi \int_a^b r(x)h(x)\,\mathrm{d}x$.
    Note $h(x)$=(Top−Bottom).
  - For rotation about a *horizontal* axis we use $V = \int_c^d A(y)\,\mathrm{d}y = 2\pi \int_c^d r(y)h(y)\,\mathrm{d}y$.
    Note $h(y)$=(Right−Left)
- In many problems both washer method and shell method can be used. In some problems both take about the same amount of work, but sometimes one is definitely easier than the other.

1. *Find the volume generated by rotating the region bounded by the given curve(s).*

(a) $y = \frac{1}{x^4}$, $x = \frac{1}{2}$, $x = 1$, $y = 0$. Rotate about the y-axis.

Cylindrical shells :



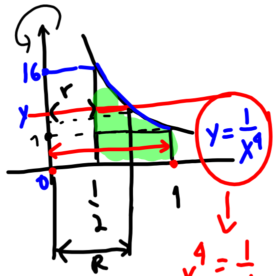
$$V = \int_{\frac{1}{2}}^{1} A(x)\,dx = 2\pi \int_{\frac{1}{2}}^{1} r(x)\,h(x)\,dx$$

$$r(x) = x$$

$$h(y) = \frac{1}{x^4} - 0 = \frac{1}{x^4}$$

$$V = 2\pi \int_{\frac{1}{2}}^{1} x \cdot \frac{1}{x^4}\,dx = 2\pi \int_{1/2}^{1} x^{-3}\,dx = \boxed{3\pi}$$

Washer:



$$x^4 = \frac{1}{y}$$

$$x = \frac{1}{\sqrt[4]{y}}$$

$$V = \int_0^{16} A(y)\,dy =$$

$$= \pi \int_0^{16} R(y)^2 - r(y)^2\,dy$$

$$r(y) = \frac{1}{2}$$

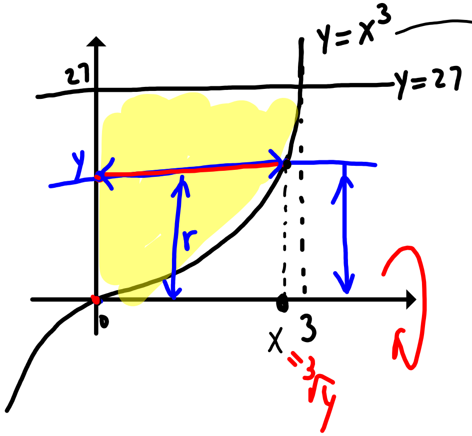
$$R(y) = \begin{cases} 1 & 0 \le y \le 1 \\ \frac{1}{\sqrt[4]{y}} & 1 \le y \le 16 \end{cases}$$

$$V = \pi \int_0^1 1^2 - \left(\frac{1}{2}\right)^2 dy + \pi \int_1^{16} \left(\frac{1}{\sqrt[4]{y}}\right)^2 - \left(\frac{1}{2}\right)^2 dy$$

1. *Find the volume generated by rotating the region bounded by the given curve(s).*

(b) $y = x^3$, $x = 0$, $y = 27$. Rotate about the x-axis.

Cylindrical shells :



$$V = 2\pi \int_0^{27} r(y)\, h(y)\, dy$$

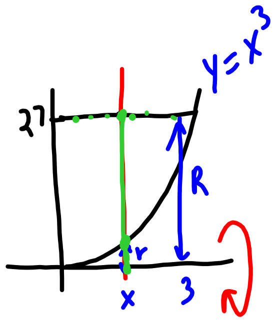
$$r(y) = y$$

$$h(y) = x = \sqrt[3]{y}$$

$$V = 2\pi \int_0^{27} y \cdot \sqrt[3]{y}\, dy$$

$$V = 2\pi \int_0^{27} y^{4/3}\, dy = \ldots = \boxed{\frac{13122\pi}{7}}$$

Washer:



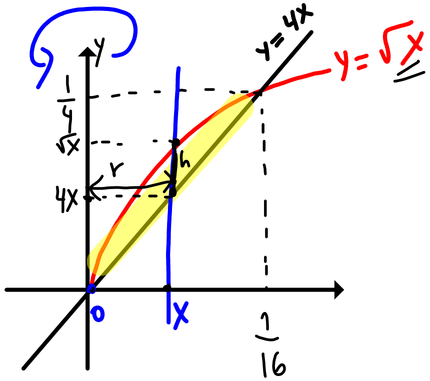
$$V = \pi \int_0^3 R(x)^2 - r(x)^2\, dx$$

$$= \pi \int_0^3 27^2 - (x^3)^2\, dx$$

1. *Find the volume generated by rotating the region bounded by the given curve(s).*

(c) $y = \sqrt{x}$, $y = 4x$. Rotate about the y-axis.

Cylindrical shells :



Find intersection points

$4x = \sqrt{x}$

$16x^2 = x$

$16x^2 - x = 0$

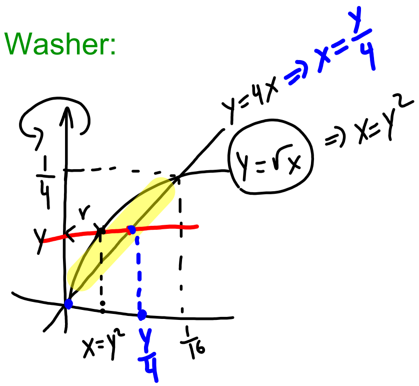
$x(16x-1) = 0 \rightarrow x = 0$, $x = \frac{1}{16}$

$$V = 2\pi \int_0^{\frac{1}{16}} r(x)\, h(x)\, dx$$

$r(x) = x \qquad h(x) = \sqrt{x} - 4x$

$$V = 2\pi \int_0^{1/16} x(\sqrt{x} - 4x)\, dx = 2\pi \int_0^{1/16} x^{3/2} - 4x^2\, dx = \ldots = \boxed{\frac{\pi}{1680}}$$

Washer:



$$V = \pi \int_0^{1/4} R(y)^2 - r(y)^2\, dy$$

$$V = \pi \int_0^{\frac{1}{4}} \left(\frac{y}{4}\right)^2 - \left(y^2\right)^2 dy$$

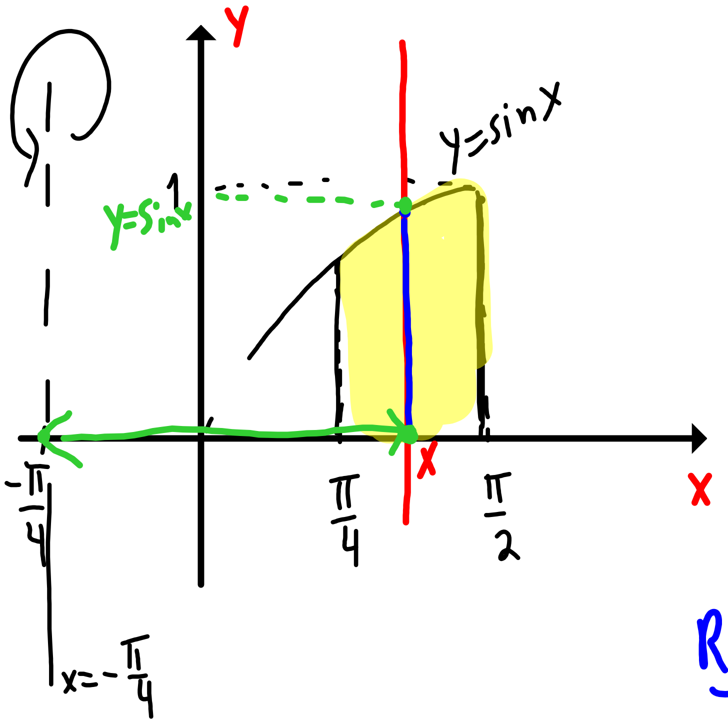
2. *Find the volume generated by rotating the region bounded by the given curve(s) about the specified lines.* Do not evaluate either integral. Just set up.

(a) $y = \sin x$, $x = \pi/4$, $x = \pi/2$, $y = 0$.

i. Rotate about the the line $x = -\pi/4$.

$$V = 2\pi \int_{\pi/4}^{\pi/2} r(x)\,h(x)\,dx$$

Cylindrical shells :



$$r(x) = x - \left(-\frac{\pi}{4}\right) = x + \frac{\pi}{4}$$

$$h(x) = \sin x$$

$$\boxed{V = 2\pi \int_{\pi/4}^{\pi/2} \left(x + \frac{\pi}{4}\right) \sin x \, dx}$$

Remark Use integration by parts to find $V = \pi\left(2 - \sqrt{2} + \frac{\pi}{\sqrt{2}}\right)$

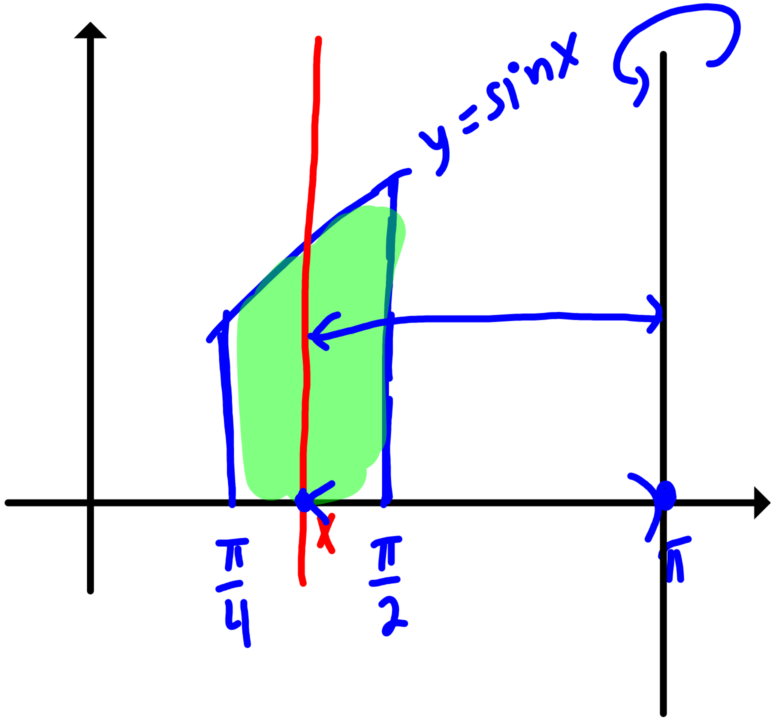
2. *Find the volume generated by rotating the region bounded by the given curve(s) about the specified lines.* Do not evaluate either integral. Just set up.

(a) $y = \sin x$, $x = \pi/4$, $x = \pi/2$, $y = 0$.

ii. Rotate about the the line $x = \pi$.

$$V = 2\pi \int_{\frac{\pi}{4}}^{\pi/2} r(x)\, h(x)\, dx$$

Cylindrical shells :



$h(x) = \sin x$ (as in (i))

$r(x) = \pi - x$

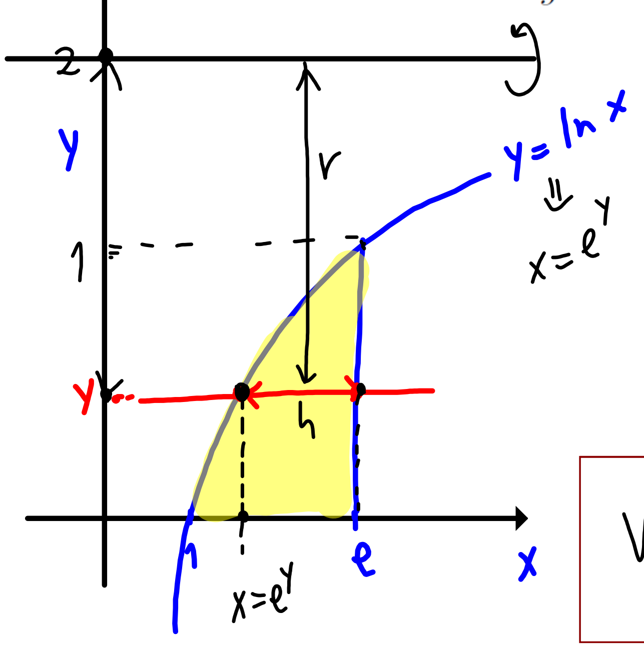
$$\boxed{V = 2\pi \int_{\frac{\pi}{4}}^{\pi/2} (\pi - x) \sin x \, dx}$$

Washer:

2. *Find the volume generated by rotating the region bounded by the given curve(s) about the specified lines.* Do not evaluate either integral. Just set up.

(b) $y = \ln x$, $x = e$, $y = 0$.

i. Rotate about the line $y = 2$.



Cylindrical shells :

$$V = 2\pi \int_0^1 r(y)\, h(y)\, dy$$

$$r(y) = 2 - y$$

$$h(y) = e - e^y$$

$$\boxed{V = 2\pi \int_0^1 (2-y)(e-e^y)\, dy}$$

Remark: $V = 6 - e$

Washer:

$$V = \pi \int_1^e 2^2 - (2 - \ln x)^2\, dx$$

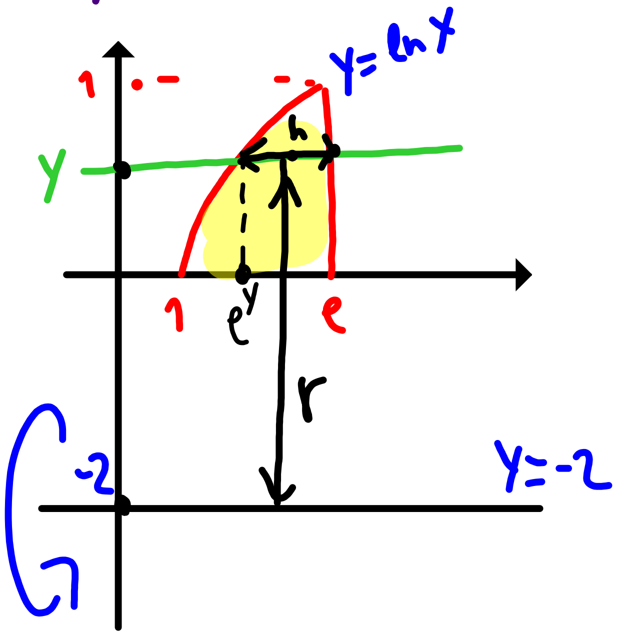
2. *Find the volume generated by rotating the region bounded by the given curve(s) about the specified lines.* Do not evaluate either integral. Just set up.

(b) $y = \ln x$, $x = e$, $y = 0$.

ii. Rotate about the line $y = -2$.

$$V = 2\pi \int_0^1 r(y)\, h(y)\, dy$$

Cylindrical shells :



$$r(y) = y - (-2) = y + 2$$

$$h(y) = e - e^y$$

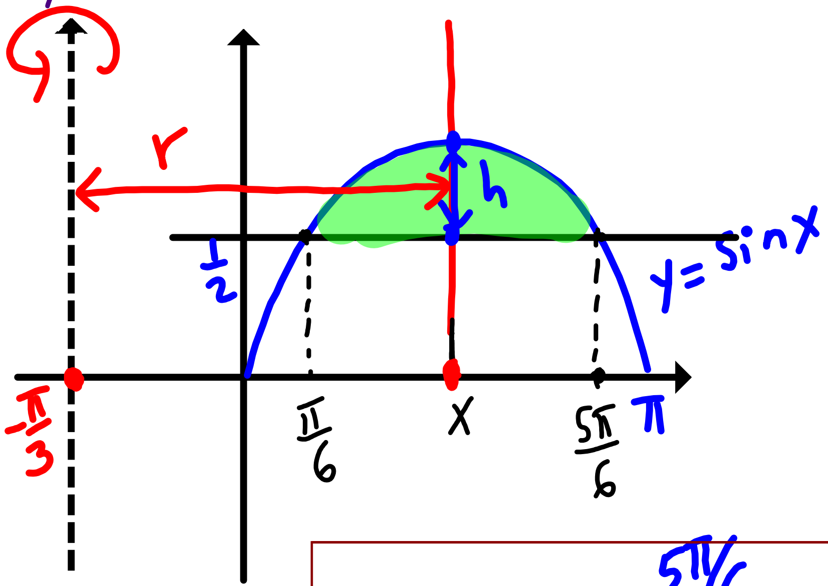
$$\boxed{V = 2\pi \int_0^1 (y+2)(e - e^y)\, dy}$$

Washer:

2. *Find the volume generated by rotating the region bounded by the given curve(s) about the specified lines.* Do not evaluate either integral. Just set up.

(c) $y = \sin x$ on the interval $[0, \pi]$, $y = \frac{1}{2}$. Rotate about the line $x = -\frac{\pi}{3}$.

Cylindrical shells :



$$V = 2\pi \int_{\pi/6}^{5\pi/6} r(x)\, h(x)\, dx$$

$$r(x) = x - \left(-\frac{\pi}{3}\right) = x + \frac{\pi}{3}$$

$$h(x) = \sin x - \frac{1}{2}$$

$$\boxed{V = 2\pi \int_{\pi/6}^{5\pi/6} \left(x + \frac{\pi}{3}\right)\left(\sin x - \frac{1}{2}\right) dx}$$

Washer:

Key Points

7.4: Work

- Work W done in moving an object under
  - a constant force F a distance d is $W = Fd$.
  - under a variable force $F(x)$ (x is displacement) from $x = a$ to $x = b$ is $W = \int_a^b F(x)\,dx$

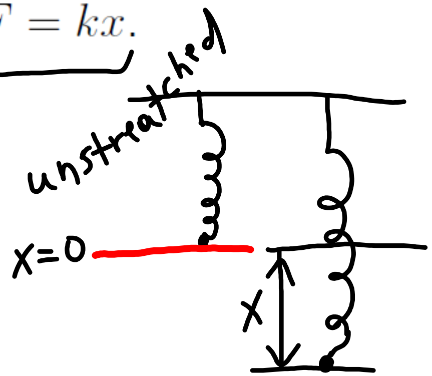
Examples

3. A force of $F(x) = x^4 - \sin(4\pi x) + 12$ (where x is in meters) acts on an object. What is the work required to move the object from x=3 to $x = 5$?

$$W = \int_3^5 x^4 - \sin(4\pi x) + 12\,dx$$

$$W = \frac{x^5}{5} + \frac{1}{4\pi}\cos(4\pi x) + 12x \Big|_3^5 = \ldots = \boxed{\frac{3002}{5}}$$

- Hooke's law: the force required to stretch a spring x units beyond its natural length is $F = kx$.



$$\boxed{F(x) = kx}$$

4. If the force required to stretch a spring 3ft beyond its natural length is 12lb, how much work is needed to stretch it 15 inches beyond its natural length?

$$F(3) = 12$$

$$3K = 12$$

$$\boxed{K = 4}$$

$$15'' = \frac{15}{12} = \frac{5}{4}\text{ ft}$$

$$W = \int_0^{5/4} 4x\,dx = 2x^2\Big|_0^{5/4} = \boxed{\frac{25}{8}\text{ ft-lb}}$$

5. A spring has a natural length of 2m. If a 10J work is required to keep it stretched to a length 2.5m, how much work is done in stretching the spring from 3m to 8m?

Let us reformulate the problem:

If a 10J work is required to keep a spring 0.5 m beyond its natural length, how much work is done in stretching it from 1 m to 6 m beyond its natural length?

$$10 = W = \int_0^{0.5} kx\,dx$$

$$10 = k\frac{x^2}{2}\Big|_0^{1/2}$$

$$10 = \frac{k}{2}\cdot\frac{1}{4} \Rightarrow \boxed{k = 80}$$

$$W = \int_1^6 kx\,dx = 80\int_1^6 x\,dx =$$

$$= 80\,\frac{x^2}{2}\Big|_1^6 = \boxed{1400\text{ J}}$$

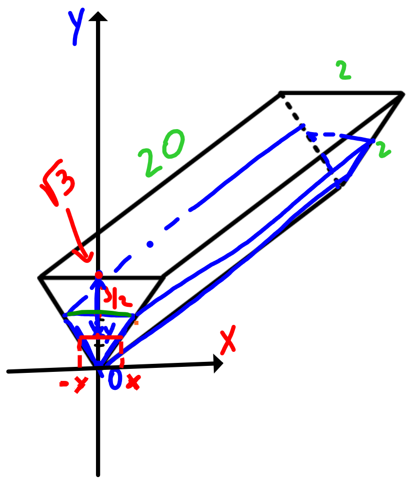
- "Water Pumping" problems require partition along vertical axis (called y here) and work is given by

$$W = \rho g \int_a^b (Area)(distance)\, \mathrm{d}y$$

where

  - $y = 0$ can be top, bottom, or center (circular ends);
  - ρg is weight density of liquid;
  - (Area) is cross sectional area of horizontal slice; $= A(y)$
  - (distance) is the distance the slice travels to reach the top. $= dist(y)$

6. A tank of water is 20 feet long and has a vertical cross section in the shape of an equilateral triangle with sides 2 feet long (point of the triangle points directly down). The tank is filled with water to a depth of 18 inches. Determine the amount of work needed to pump all of the water to the top of the tank. Assume that the density of water is 62.5lb/ft^3.

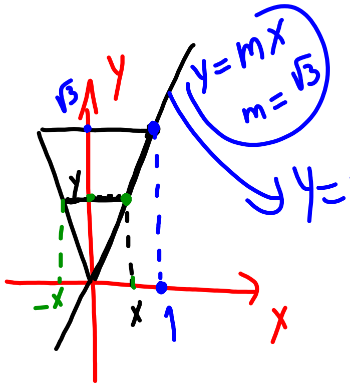


$18'' = \frac{18}{12} = \frac{3}{2}$ ft

$$W = \rho g \int_0^{3/2} A(y)\, \text{dist}(y)\, dy$$

$\text{dist}(y) = \sqrt{3} - y$

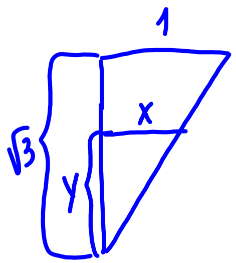
$A(y) = 20 \cdot 2x = 40 \cdot \frac{y}{\sqrt{3}}$



$y = \sqrt{3}x \Rightarrow x = \frac{y}{\sqrt{3}}$

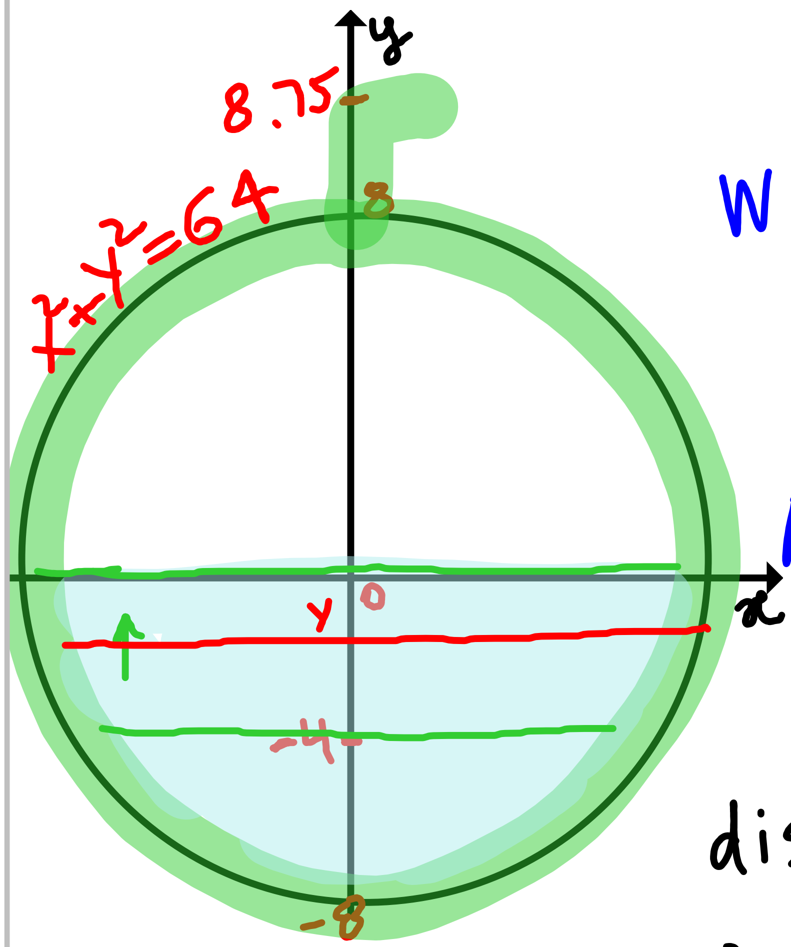
$$W = 62.5 \int_0^{3/2} \frac{40}{\sqrt{3}}\, y \left(\sqrt{3} - y\right) dy$$

$W = \ldots$ ft-lb



$\frac{1}{x} = \frac{\sqrt{3}}{y} \Rightarrow y = \sqrt{3}x$

7. A spherical tank with radius 8m is 1/2 full of water. The water pumped from a spout at the top of the tank that is 75cm high. Find the work needed to pump out the water (through the spout) until the water level is 4m from the bottom. (The density of water is $1000 kg/m^3$.)



$$W = \rho g \int_{-4}^{0} A(y)\, dist(y)\, dy$$

$$A(y) = \pi r(y)^2 = \pi x^2 = \pi(64-y^2)$$

$$dist(y) = 8.75 - y$$

$$W = 1000 \cdot 9.8\pi \int_{-4}^{0} (64-y^2)(8.75-y)\,dy = \ldots = \boxed{7.86 \cdot 10^6 J}$$

- "Rope/cable pulling" problems require partition along vertical axis (called y here). If $y = 0$ is at the top of rope/cable then a work required to pull s feet (or s meter) of a rope/cable that weighs ω lb/ft (or ω N/m) is given by

$$W = \int_0^s \omega y \, \mathrm{d}y$$

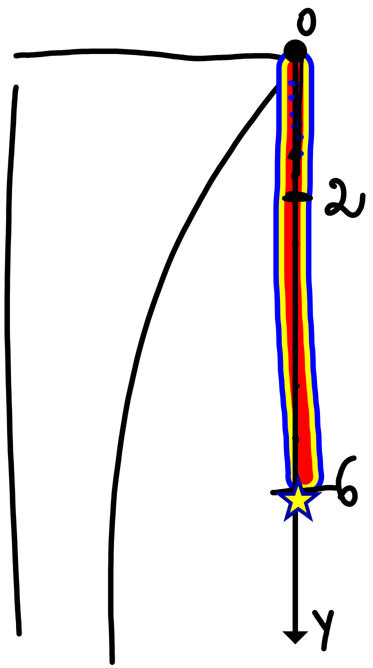
ω is weight density (weight per unit length) of rope/cable

8. A heavy rope $40ft$ long, weighs $0.4lb/ft$ and hangs over the edge of a tall building. How much work is done in pulling the rope to the top of the building?

$$W = \int_0^{40} 0.4\,y\,dy = 0.4\,\frac{y^2}{2}\Big|_0^{40} = \boxed{320\text{ ft-lb}}$$

9. A uniform cable hanging over the edge of a tall building is $6m$ long and weight $20kg$. If a $25kg$ weight is attached to the end of the cable, how much work is required to pull $2m$ of the cable to the top? Use $g \approx 10m/s^2$ for the gravitational constant.

$$w = \frac{20\,kg \cdot 10\,m/s^2}{6m} = \frac{100}{3}\,\frac{N}{m}$$



$$W = W_1 + W_2 + W_3$$

needed to pull up top 2m of cable

$$\int_0^2 w\,y\,dy$$

needed to pull up bottom 4m of cable to the distance $y = 2\,m$

$$\int_2^6 w \cdot 2\,dy$$

needed to lift 25 kg to the distance $d = 2\,m$

$F \cdot d = mg \cdot d$

$25 \cdot 10 \cdot 2 = 500$

$$W = \int_0^2 \frac{100}{3}\,y\,dy + 4 \cdot 2 \cdot \frac{100}{3} + 500 = \boxed{\frac{2500}{3}\,J}$$

7.5: Average Value of a Function

Key Points

- The **average value of a function** $y = f(x)$ over the interval $[a, b]$:

$$\boxed{f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x.}$$

- MEAN VALUE THEOREM FOR INTEGRALS: *If f is continuous on $[a, b]$, then there exists a number c on $[a, b]$ s.t.*

$$\boxed{\int_a^b f(x)\,\mathrm{d}x = f(c)(b-a).}$$

The geometric interpretation of the Mean Value Theorem for Integrals: for *positive* functions f, there is a number c s.t. the rectangle with base $[a, b]$ and height $f(c)$ has the same area as the region under the graph of f from a to b.

10. Determine the average value of $f(x) = \frac{\sqrt{3}-1}{1+x^2}$ over the interval $[1, \sqrt{3}]$.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\cancel{\sqrt{3}-1}}\int_1^{\sqrt{3}} \frac{\cancel{\sqrt{3}-1}}{1+x^2}\,dx$$

$$= \int_1^{\sqrt{3}} \frac{dx}{1+x^2} = \arctan \Big|_1^{\sqrt{3}} = \arctan\sqrt{3} - \arctan 1 = \frac{\pi}{3} - \frac{\pi}{4} = \boxed{\frac{\pi}{12}}$$

11. Find the value(s) b such that the average value of $f(x) = 3x^2 - 2x - 3$ on the interval $[a, 0]$ is equal to 1.

$$f_{ave} = \frac{1}{0-a}\int_a^0 3x^2 - 2x - 3\,dx = 1$$

$$-\frac{1}{a}\left(x^3 - x^2 - 3x\right)\Big|_a^0 = 1$$

$$-\frac{1}{a}\left(0 - \left(a^3 - a^2 - 3a\right)\right) = 1$$

$$\frac{1}{a}\left(a^3 - a^2 - 3a\right) = 1$$

$$a^2 - a - 3 = 1$$

$$a^2 - a - 4 = 0$$

$$a_{1,2} = \frac{1 \pm \sqrt{1 + 4\cdot 4}}{2} = \frac{1 \pm \sqrt{17}}{2}$$

$a < 0$

$$a = \frac{1+\sqrt{17}}{2} > 0 \quad \text{OR} \quad a = \frac{1-\sqrt{17}}{2} < 0$$

Final answer: a=(1-√17)/2.