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Fall 2012 Math 152

Week in Review 5

courtesy: *Dr. Oksana Shatalov*

(covering Sections 8.3,8.4&8.9)

8.3: Trigonometric Substitutions

integral with	substitution	identity
$a^2 - x^2$	$x = a\sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$a^2 + x^2$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$x^2 - a^2$	$x = a\sec\theta, \quad 0 \le \theta < \frac{\pi}{2} \quad \text{or} \quad \pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$
$ax^2 + bx + c$	complete squares and then do the correct substitution	

Evaluate the given integral:

(a) $\int \sqrt{1-3x^2}\,dx = \int \sqrt{1-(\sqrt{3}\,x)^2}\,dx$

$\sqrt{3}\,x = \sin\theta$

$x = \frac{\sin\theta}{\sqrt{3}}$

$dx = \frac{1}{\sqrt{3}}\cos\theta\, d\theta$

$\sqrt{1-3x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta}$

$\sqrt{1-3x^2} = \cos\theta$

$1-\sin^2\theta = \cos^2\theta$
$1+\tan^2\theta = \sec^2\theta$
$\sec^2\theta - 1 = \tan^2\theta$

$$\int \sqrt{1-3x^2}\,dx = \int \cos\theta \frac{1}{\sqrt{3}}\cos\theta\, d\theta = \frac{1}{\sqrt{3}}\int \cos^2\theta\, d\theta =$$

$$= \frac{1}{\sqrt{3}}\int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{2\sqrt{3}}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

Return to x

$\sqrt{3}\,x = \sin\theta$

$\theta = \arcsin(\sqrt{3}\,x)$

$\frac{\sin 2\theta}{2} = \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}} = \sqrt{3}x\sqrt{1-3x^2}$

$$\int \sqrt{1-3x^2}\,dx = \boxed{\frac{1}{2\sqrt{3}}\left(\arcsin(\sqrt{3}\,x) + \sqrt{3}\,x\sqrt{1-3x^2}\right) + C}$$

1. Evaluate the given integral:

(b) $\displaystyle\int \frac{1}{x^2\sqrt{25x^2-9}}\,dx$

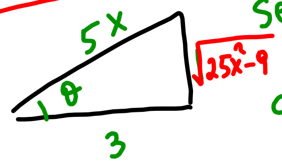
$$\sqrt{25x^2-9} = \sqrt{9\left(\frac{25x^2}{9}-1\right)} = 3\sqrt{\left(\frac{5x}{3}\right)^2-1} =$$

$$\boxed{\frac{5x}{3}=\sec\theta} \qquad \sqrt{25x^2-9} = 3\sqrt{\sec^2\theta-1} = 3\tan\theta$$

$$x = \frac{3\sec\theta}{5} \Rightarrow dx = \frac{3}{5}\sec\theta\tan\theta\,d\theta$$

$$\int \frac{\cancel{\frac{3}{5}}\,\cancel{\sec\theta}\,\cancel{\tan\theta}\,d\theta}{\left(\frac{3}{5}\right)^{\cancel{2}}\sec^{\cancel{2}}\theta\cdot 3\,\cancel{\tan\theta}} = \frac{5}{9}\int\frac{d\theta}{\sec\theta} = \frac{5}{9}\int\cos\theta\,d\theta$$

$$= \frac{5}{9}\sin\theta + C$$



$$\sec\theta = \frac{5x}{3} \qquad \cos\theta = \frac{3}{5x} \qquad \sin\theta = \frac{\sqrt{25x^2-9}}{5x}$$

$$= \frac{5}{9}\cdot\frac{\sqrt{25x^2-9}}{5x} + C = \boxed{\frac{\sqrt{25x^2-9}}{9x} + C}$$

1. Evaluate the given integral: (c) $\displaystyle\int_0^{4/7} \frac{1}{(49x^2+16)^{3/2}}\,dx$

$$49x^2+16 = 16\left(\frac{49x^2}{16}+1\right) = 16\left(\left(\frac{7x}{4}\right)^2+1\right) = 16\sec^2\theta$$

$$\boxed{\frac{7x}{4} = \tan\theta} \Rightarrow x = \frac{4\tan\theta}{7} \Rightarrow$$

$$x=0 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$$

$$x=\frac{4}{7} \Rightarrow \tan\theta = \frac{7}{4}\cdot\frac{4}{7} = 1 \Rightarrow \theta = \frac{\pi}{4}$$

$$dx = \frac{4}{7}\sec^2\theta\, d\theta$$

$$\int_0^{\pi/4} \frac{\frac{4}{7}\sec^2\theta\, d\theta}{(16\sec^2\theta)^{3/2}} = \frac{\cancel{4}^{1}}{7}\int_0^{\pi/4} \frac{\cancel{\sec^2\theta}\, d\theta}{4^{\cancel{3}\,2}\sec^{\cancel{3}}\theta} = \frac{1}{7\cdot 4^2}\int_0^{\pi/4} \frac{d\theta}{\sec\theta}$$

$$= \frac{1}{112}\int_0^{\pi/4} \cos\theta\, d\theta = \frac{1}{112}\sin\theta\Big|_0^{\pi/4} = \boxed{\frac{1}{112\sqrt{2}}}$$

2. Use a trigonometric substitution to eliminate the root:

(a) $\sqrt{(x+1)^2 - 64} = \sqrt{64\left(\frac{(x+1)^2}{64} - 1\right)}$

$$= 8\sqrt{\left(\frac{x+1}{8}\right)^2 - 1}$$

$\frac{x+1}{8} = \sec\theta$

$$= 8\sqrt{\sec^2\theta - 1} = 8\sqrt{\tan^2\theta} = \boxed{8|\tan\theta|}$$

2. Use a trigonometric substitution to eliminate the root:

(b) $\sqrt{4(x-5)^2+1}$ $= \sqrt{(2(x-5))^2+1}$

$\boxed{2(x-5)=\tan\theta}$

$= \sqrt{\tan^2\theta+1} = \sqrt{\sec^2\theta} = \boxed{|\sec\theta|}$

3. (a) Use a trigonometric substitution to eliminate the root: $\sqrt{24-2x-x^2}$.

Complete squares:

$$\sqrt{24-2x-x^2} = \sqrt{-(x^2+2x-24)}$$

$$(a+b)^2 = a^2+2ab+b^2$$

$$= \sqrt{-(\underbrace{x^2+2\cdot x\cdot 1+1}_{(x+1)^2}-\underbrace{1-24}_{25})}$$

$$= \sqrt{-((x+1)^2-25)} = \sqrt{25-(x+1)^2}$$

$$= \sqrt{25\left(1-\frac{(x+1)^2}{25}\right)} = 5\sqrt{1-\left(\frac{x+1}{5}\right)^2}$$

$$\boxed{\frac{x+1}{5} = \sin\theta}$$

$$= 5\sqrt{1-\sin^2\theta} = \boxed{5\,|\cos\theta|}$$

(b) Evaluate the integral $\int \frac{(x+1)^2}{(24-x^2-2x)^{3/2}}\,dx$

$$\int \frac{(x+1)^2\,dx}{\left(\sqrt{24-x^2-2x}\right)^3}$$

Using (a): $\frac{x+1}{5} = \sin\theta$

$\sqrt{24-x^2-2x} = 5\cos\theta$

$x+1 = 5\sin\theta$

$dx = 5\cos\theta\,d\theta$

$$\int \frac{\cancel{5^2}\sin^2\theta\ \cancel{5\cos\theta}\,d\theta}{\cancel{5^3}\cos^{\cancel{3}\,2}\theta} = \int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$$

$$= \int (\sec^2\theta - 1)\,d\theta = \tan\theta - \theta + C$$

Return to x:

$x+1 = 5\sin\theta \Rightarrow \theta = \arcsin\frac{x+1}{5}$

$\sqrt{24-x^2-2x} = 5\cos\theta \Rightarrow \tan\theta = \frac{x+1}{\sqrt{24-x^2-2x}}$

Finally $\int = \boxed{\frac{x+1}{\sqrt{24-x^2-2x}} - \arcsin\frac{x+1}{5} + C}$

8.4: Integration Of Rational Functions By Partial Fractions

Key Points

- *Rational function:* $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

- Partial Fraction Decomposition only works on proper rational functions (degree of the numerator is strictly less than the degree of denominator.) (Otherwise (for improper rational functions), you must first do long division.)

- You have to factor a denominator as much as possible and "break down" the fraction using individual denominators.

- For each **factor** in the denominator use the following table:

	Factor in denominator	Term in partial fraction decomposition
L.	linear factor $ax + b$	$\dfrac{A}{ax+b}$
R.L.	repeated linear factor $(ax+b)^2$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2}$
R.L.	repeated linear factor $(ax+b)^k$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_k}{(ax+b)^k}$
P.Q.	prime quadratic factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$	$\dfrac{Ax+B}{ax^2+bx+c}$
R.P.Q	repeated prime quadratic factor $(ax^2+bx+c)^2$, where $b^2 - 4ac < 0$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2}$
P.R.Q.	repeated prime quadratic factor $(ax^2+bx+c)^k$, where $b^2 - 4ac < 0$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

- Solve for numerators using roots of the denominator and/or matching powers of x. THEN INTEGRATE.

Examples

4. Write out the form of the partial fraction decomposition of the following rational functions. (Do not try to solve) Note that in (a)-(d) no need in long devision

(a) $$\frac{3x}{\underbrace{(x-1)}_{L}\underbrace{(3x+12)}_{L}} = \frac{A}{x-1} + \frac{B}{3x+12}$$

(b) $$\frac{5x^2}{(x-1)^2(x^2-1)} = \frac{5x^2}{(x-1)^2(x-1)(x+1)} = \frac{5x^2}{\underbrace{(x-1)^3}_{R.L.}\underbrace{(x+1)}_{L.}}$$

$$= \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3}{(x-1)^3} + \frac{B}{x+1}$$

(c) $$\frac{7}{x(x^3+x^2+x)} = \frac{7}{\underbrace{x^2}_{R.L.}\underbrace{(x^2+x+1)}_{P.Q.}} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{Bx+C}{x^2+x+1}$$

(d) $$\frac{x+5}{\underbrace{(x-3)}_{L}\underbrace{(x^2+25)^2}_{R.P.Q.}} = \frac{A}{x-3} + \frac{B_1x+C_1}{x^2+25} + \frac{B_2x+C_2}{(x^2+25)^2}$$

5. Compute the following integrals.

(a) $I = \int \frac{x^2+4x}{(x-1)(x-2)(x+3)}\,dx$

$\deg(\text{num}) < \deg(\text{denom.})$
$\quad\quad \| \quad\quad\quad\quad \|$
$\quad\quad 2 \quad\quad\quad\quad 3$

$$\frac{x^2+4x}{(x-1)(x-2)(x+3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+3}$$

$$\frac{x^2+4x}{(x-1)(x-2)(x+3)} = \frac{A(x-2)(x+3)+B(x-1)(x+3)+C(x-1)(x-2)}{(x-1)(x-2)(x+3)}$$

$$x^2+4x = A(x-2)(x+3)+B(x-1)(x+3)+C(x-1)(x-2)$$

$x=1 \Rightarrow 1+4 = A(-1)\cdot 4 \Rightarrow \boxed{A = -\frac{5}{4}}$

$x=2 \Rightarrow 4+8 = 0 + B\cdot 1\cdot 5 + 0 \Rightarrow \boxed{B = \frac{12}{5}}$

$x=-3 \Rightarrow 9-12 = 0+0+C(-4)\cdot(-5) \Rightarrow \boxed{C = -\frac{3}{20}}$

$$I = \int \frac{-5/4}{x-1} + \frac{12/5}{x-2} + \frac{-3/20}{x+3}\,dx$$

$$= -\frac{5}{4}\int\frac{dx}{x-1} + \frac{12}{5}\int\frac{dx}{x-2} - \frac{3}{20}\int\frac{dx}{x+3}$$

$$= -\frac{5}{4}\ln|x-1| + \frac{12}{5}\ln|x-2| - \frac{3}{20}\ln|x+3| + C$$

(b) $I = \int \frac{x^2-3x+7}{\underbrace{(x-1)}_{L}\underbrace{(x^2+1)}_{P.Q.}}\,dx$ $\quad$ $\underset{\substack{\parallel \\ 2}}{\deg(\text{num})} < \underset{\substack{\parallel \\ 3}}{\deg(\text{denom.})}$

$$\frac{x^2-3x+7}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$x^2-3x+7 = A(x^2+1) + (Bx+C)(x-1)$ for all x.

$x=1 \Rightarrow 1-3+7 = A\cdot 2 + 0 \Rightarrow \boxed{A=\frac{5}{2}}$

$1\cdot x^2-3x+7 = x^2(A+B) + x(-B+C) + A-C$

x^2: $1=A+B \Rightarrow B=1-A = 1-\frac{5}{2} = -\frac{3}{2}$

x^0: $7=A-C \Rightarrow C = A-7 = \frac{5}{2}-7 \Rightarrow C = -\frac{9}{2}$

$$I = A\int\frac{dx}{x-1} + \int\frac{(Bx+C)dx}{x^2+1} = \frac{5}{2}\int\frac{dx}{x-1} + \int\frac{-\frac{3}{2}x-\frac{9}{2}}{x^2+1}$$

$$= \frac{5}{2}\ln|x-1| - \frac{3}{2}\int\frac{x\,dx}{x^2+1} - \frac{9}{2}\int\frac{dx}{x^2+1}$$

$\boxed{u=x^2+1 \Rightarrow du = 2x\,dx}$

$$= \frac{5}{2}\ln|x-1| - \frac{3}{4}\int\frac{du}{u} - \frac{9}{2}\arctan x + C$$

$$= \frac{5}{2}\ln|x-1| - \frac{3}{4}\ln(x^2+1) - \frac{9}{2}\arctan x + C$$

(c) $I = \int \frac{x^4 - x^3 - 12x^2 + 10}{x^3 - 4x^2}\,dx$

$\underset{4}{\underset{\shortparallel}{\deg(\text{num})}} > \underset{3}{\underset{\shortparallel}{\deg(\text{denom.})}}$

Long division

$$\begin{array}{r|l} & x + 3 = Q \\ \hline D = x^3 - 4x^2 & x^4 - x^3 - 12x^2 + 10 = P \\ & -\left(x^4 - 4x^3\right) \\ \hline & 3x^3 - 12x^2 \\ & -\left(3x^3 - 12x^2\right) \\ \hline & 0 + 10 = R \end{array}$$

$\frac{P}{D} = Q + \frac{R}{D}$ $\qquad I = \int x + 3 + \frac{10}{x^3 - 4x^2}\,dx$

$$I = \frac{x^2}{2} + 3x + 10\underbrace{\int \frac{dx}{x^2(x-4)}}_{I_1}$$

$$\frac{1}{\underbrace{x^2}_{\text{R.L.}}\underbrace{(x-4)}_{L}} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-4}$$

$0 \cdot x^2 + 1 = A\,x(x-4) + B(x-4) + C x^2$

$x = 0 \Rightarrow 1 = 0 - 4B + 0 \Rightarrow \boxed{B = -\frac{1}{4}}$

$x = 4 \Rightarrow 1 = 0 + 0 + 16C \Rightarrow \boxed{C = \frac{1}{16}}$

Equate coeff. of x^2: $0 = A + C \Rightarrow A = -C \Rightarrow \boxed{A = -\frac{1}{16}}$

$$I_1 = \int \frac{-\frac{1}{16}}{x}\,dx + \int \frac{-\frac{1}{4}\,dx}{x^2} + \int \frac{\frac{1}{16}}{x-4}\,dx$$

$$= -\frac{1}{16}\ln|x| + \frac{1}{4x} + \frac{1}{16}\ln|x-4| + C$$

$$I = \frac{x^2}{2} + 3x + 10 I_1$$

$$= \frac{x^2}{2} + 3x + 10\left[-\frac{1}{16}\ln|x| + \frac{1}{4x} + \frac{1}{16}\ln|x-4|\right] + C$$

(d) $I = \int \frac{1}{\underbrace{(x^2+1)}_{\text{P.Q.}}\underbrace{(x^2+x+1)}_{\text{P.Q.}}}\,dx$

$$\frac{1}{(x^2+1)(x^2+x+1)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+x+1}$$

$$1 = (Ax+B)(x^2+x+1) + (Cx+D)(x^2+1)$$

$$1 = x^3(A+C) + x^2(A+B+D) + x(A+B+C) + B+D$$

Equate coefficients:

x^3 : $0 = A+C$ (1)

x^2 : $0 = A+B+D$ (2)

x : $0 = A+B+C$ (3)

x^0 : $1 = B+D$ (4)

(2) & (4) $\Downarrow$ $0 = A+1 \Rightarrow A=-1$

(1) & (3) $\Downarrow$ $B=0$

(1) $\Rightarrow$ $C=1$

(4) $\Rightarrow$ $D=1$

$$I = \underbrace{\int \frac{-x\,dx}{x^2+1}}_{I_1} + \underbrace{\int \frac{(x+1)\,dx}{x^2+x+1}}_{I_2} = I_1 + I_2$$

Find I_1:

$$I_1 = -\int \frac{x\,dx}{x^2+1} \underset{\substack{u=x^2+1\\du=2x\,dx}}{\overset{\text{u-sub}}{=\!=\!=}} -\int \frac{du}{2u} = -\frac{1}{2}\ln|u|$$

$$\boxed{I_1 = -\frac{1}{2}\ln(x^2+1)}$$

continued on the next page

Find I_2:

$$I_2 = \int \frac{x+1}{x^2+x+1}dx$$

Complete squares:

$$x^2+x+1 = x^2+2\cdot\frac{1}{2}x+\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2+1$$

$$= \boxed{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \int \frac{(x+1)\,dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \int \frac{\left(\left(x+\frac{1}{2}\right)+\frac{1}{2}\right)dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \; \frac{u\text{-sub}}{\substack{u=x+\frac{1}{2}\\ du=dx}} \; \int \frac{\left(u+\frac{1}{2}\right)du}{u^2+\frac{3}{4}}$$

$$= \int \frac{u\,du}{u^2+\frac{3}{4}} + \frac{1}{2}\int \frac{du}{u^2+3/4} =$$

use $U = u^2+\frac{3}{4}$ for first integral and $\int \frac{dx}{x^2+1} = \arctan x$ for the second one

$$= \frac{1}{2}\int \frac{dU}{U} + \frac{1}{2}\int \frac{du}{\frac{3}{4}\left(\frac{4}{3}u^2+1\right)}$$

$$= \frac{1}{2}\ln U + \frac{2}{3}\int \frac{du}{\left(\frac{2u}{\sqrt{3}}\right)^2+1} =$$

$$= \frac{1}{2}\ln\left(u^2+\frac{3}{4}\right) + \frac{\cancel{2}}{3}\cdot\frac{\sqrt{3}}{\cancel{2}}\arctan\left(\frac{2u}{\sqrt{3}}\right)$$

Return to x: $x+\frac{1}{2}=u$

$$= \frac{1}{2}\ln\left|\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right| + \frac{\sqrt{3}}{3}\arctan\frac{2\left(x+\frac{1}{2}\right)}{\sqrt{3}} + C$$

$$= \boxed{\frac{1}{2}\ln(x^2+x+1) + \frac{\sqrt{3}}{3}\arctan\frac{2x+1}{\sqrt{3}} + C}$$

Finally: $I = I_1 + I_2$ (see previous page for I_1)

$$\boxed{I = -\frac{1}{2}\ln(x^2+1) + \frac{1}{2}\ln(x^2+x+1) + \frac{\sqrt{3}}{3}\arctan\frac{2x+1}{\sqrt{3}} + C}$$

8.9: Improper Integrals

Key Points

$f(x)$

- TYPE I: Infinite Interval and Continuous Integrand:
  $$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx \quad \text{(likewise with } -\infty\text{)}$$
- TYPE II: Discontinuous Integrand and Finite Interval:
  $f(x)$ is discontinuous at $x = a$: $$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx \text{ (likewise with } b^-\text{)}$$

Examples

6. Compute the following integrals or show they diverge:

   (a) $I = \displaystyle\int_e^\infty \frac{1}{x(\ln(x)^5)}\,dx$ cont. on $[e, \infty)$

$\Downarrow$ TYPE I

$$I = \lim_{t\to\infty}\int_e^t \frac{dx}{x(\ln x)^5}$$

Find antiderivative

$$\int \frac{dx}{x(\ln x)^5} \qquad u = \ln x \qquad du = \frac{dx}{x}$$

$$= \int \frac{du}{u^5} = -\frac{1}{4u^4} = -\frac{1}{4(\ln x)^4}$$

$$I = \lim_{t\to\infty} -\frac{1}{4(\ln x)^4}\Bigg|_e^t$$

$$I = -\frac{1}{4}\lim_{t\to\infty}\left(\frac{1}{(\ln t)^4} - \frac{1}{(\ln e)^4}\right) = -\frac{1}{4}(0-1) = \boxed{\frac{1}{4}}$$

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(b) $I = \int_{-\infty}^{0} \underbrace{(1+x)e^x}_{\text{cont. function}}\, dx$

$$I = \lim_{t\to-\infty} \int_t^0 (1+x)e^x\, dx$$

$$= \lim_{t\to-\infty} x e^x \Big|_t^0 = \lim_{t\to-\infty} (0 - te^t)$$

$\int (1+x)e^x dx$

$u = 1+x$, $e^x dx = dv$

$du = dx$, $v = e^x$

$\int (1+x)e^x dx = (1+x)e^x - \int e^x dx =$

$= e^x + xe^x - e^x = xe^x$

L'Hospitals

$$= -\lim_{t\to-\infty} te^t = (\infty \cdot 0) = -\lim_{t\to-\infty} \frac{t}{e^{-t}} = -\lim_{t\to-\infty} \frac{1}{-e^{-t}}$$

$$= \lim_{t\to-\infty} e^t = 0$$

(c) $I = \int_{-\infty}^{\infty} \frac{5x^4}{\underbrace{(x^5+5)^3}_{\text{continuos}}}\,dx = \underbrace{\int_{-\infty}^{0} \frac{5x^4\,dx}{(x^5+5)^3}}_{I_1} + \underbrace{\int_{0}^{\infty} \frac{5x^4\,dx}{(x^5+5)^3}}_{I_2}$

$$I_1 = \lim_{t\to-\infty} \int_t^0 \frac{5x^4\,dx}{(x^5+5)^3}$$

$$= \lim_{t\to-\infty} -\frac{1}{2(x^5+5)^2}\Big|_t^0$$

$$= -\frac{1}{2}\lim_{t\to-\infty}\left[\frac{1}{5^2} - \underbrace{\frac{1}{(t^5+5)^2}}_{\downarrow\, 0}\right] = -\frac{1}{50}$$

Find antiderivative

$u = x^5 + 5 \Rightarrow du = 5x^4\,dx$

$$\int \frac{5x^4\,dx}{(x^5+5)^3} = \int \frac{du}{u^3} = -\frac{1}{2u^2} =$$

$$= -\frac{1}{2(x^5+5)^2}$$

$$I_2 = \lim_{t\to\infty} \int_0^t \frac{5x^4\,dx}{(x^5+5)^3} = \lim_{t\to\infty} -\frac{1}{2(x^5+5)^2}\Big|_0^t$$

$$= -\frac{1}{2}\lim_{t\to\infty}\left[\underbrace{\frac{1}{(t^5+5)^2}}_{\downarrow\, 0} - \frac{1}{25}\right] = \frac{1}{50}$$

$$I = -\frac{1}{50} + \frac{1}{50} = 0$$

I is convergent

(d) $I = \int_0^9 \frac{1}{\sqrt[3]{x-4}}\,dx$

$0 \quad 4 \quad 9$

not cont. on $[0,9]$ (Namely, at $x=4$)

TYPE II Improper integral

$$\int \frac{dx}{\sqrt[3]{x-4}} = \int (x-4)^{-\frac{1}{3}}dx = \frac{(x-4)^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} = \frac{3}{2}(x-4)^{2/3}$$

$$I = \underbrace{\int_0^4 \frac{dx}{\sqrt[3]{x-4}}}_{I_1} + \underbrace{\int_4^9 \frac{dx}{\sqrt[3]{x-4}}}_{I_2}$$

$$I_1 = \lim_{t\to 4^-} \int_0^t \frac{dx}{\sqrt[3]{x-4}} = \frac{3}{2}\lim_{t\to 4^-} (x-4)^{2/3}\Big|_0^t = \frac{3}{2}\lim_{t\to 4^-}\left[(t-4)^{2/3} - (-4)^{2/3}\right]$$

$$= \frac{3}{2}\left[0 - \sqrt[3]{16}\right] = -\frac{3}{2}\cdot 2\sqrt[3]{2} = -3\sqrt[3]{2}$$

$$I_2 = \lim_{t\to 4^+} \int_t^9 \frac{dx}{\sqrt[3]{x-4}} = \frac{3}{2}\lim_{t\to 4^+} (x-4)^{2/3}\Big|_t^9 =$$

$$= \frac{3}{2}\lim_{t\to 4^+}\Big(5^{2/3} - \underbrace{(t-4)^{2/3}}_{\downarrow\, 0}\Big) = \frac{3}{2}5^{2/3} = \frac{3}{2}\sqrt[3]{25}$$

$$I = -3\sqrt[3]{2} + \frac{3}{2}\sqrt[3]{25}$$

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- **Comparison Theorem:** *Let* $f(x)$ *and* $g(x)$ *be continuous and* $f(x) \geq g(x) \geq 0$ *for* $x \geq a$. *Then*
  - *if* $\int_a^\infty f(x)\,\mathrm{d}x$ *is convergent then* $\int_a^\infty g(x)\,\mathrm{d}x$ *is convergent.*

    (Note if $\int_a^\infty f(x)\,\mathrm{d}x$ is divergent, no conclusion can be drawn about $\int_a^\infty g(x)\,\mathrm{d}x$.)
  - *if* $\int_a^\infty g(x)\,\mathrm{d}x$ *is divergent then* $\int_a^\infty f(x)\,\mathrm{d}x$ *is divergent.*

    (Note if $\int_a^\infty g(x)\,\mathrm{d}x$ is convergent, no conclusion can be drawn about $\int_a^\infty f(x)\,\mathrm{d}x$.)

7. Determine whether the given integrals converge or diverge using the Comparison Theorem.

(a) $I = \int_0^\infty \frac{1}{x^{2012} + e^{2012x}}\,\mathrm{d}x$

$a, b > 0$

$a + b > a$

$\frac{1}{a+b} < \frac{1}{a}$

$$\int \frac{dx}{x^{2012} + e^{2012x}} < \int \frac{dx}{e^{2012x}}$$

$$\int_0^\infty \frac{dx}{e^{2012x}} = \lim_{t\to\infty} \int_0^t e^{-2012x}\,dx =$$

$$= \lim_{t\to\infty} -\frac{1}{2012} e^{-2012x} \Big|_0^t = -\frac{1}{2012} \lim_{t\to\infty} \left(\underset{\downarrow\, 0}{e^{-2012t}} - \underset{= 1}{e^0}\right)$$

$$= \frac{1}{2012} \quad \text{convergent}$$

Conclusion: By Comparison Theorem the given integral I is $\boxed{\text{convergent}}$

(b) $I = \int_5^\infty \frac{x^2}{x^{5/2} - x}\,dx$

First note that if a>b>0 then 0<a-b<a.
Hence,

$$\frac{1}{a-b} \geq \frac{1}{a}$$

$a > b > 0$

$0 < a - b \leq a$

$$\frac{1}{a-b} \geq \frac{1}{a}$$

In our case, we have $x^{5/2} > x > 0$ for all $x \geq 5$. Thus,

$$\frac{1}{x^{5/2} - x} \geq \frac{1}{x^{5/2}}$$

$$\frac{x^2}{x^{5/2} - x} \geq \frac{x^2}{x^{5/2}} \Rightarrow \frac{x^2}{x^{5/2} - x} \geq \frac{1}{x^{\frac{1}{2}}}$$

We know the following Fact:

FACT: If $a > 0$ then $\int_a^\infty \frac{1}{x^p}\,dx$ is convergent as $p > 1$ and divergent as $p \leq 1$.

Thus $\int_5^\infty \frac{dx}{x^{\frac{1}{2}}}$ diverges $\left(p = \frac{1}{2} \leq 1\right)$

By Comparison Theorem I $\boxed{\text{diverges}}$

(c) $I = \int_{10}^{\infty} \frac{\sin^4(7x)}{x^7}\, dx$

$|\sin x| \le 1$

$\sin^4 x \le 1$

$$\int_{10}^{\infty} \frac{\sin^4(7x)}{x^7}\, dx \le \underbrace{\int_{10}^{\infty} \frac{1}{x^7}\, dx}_{\text{conv. } p=7>1}$$ (see Fact)

By Comp. theorem I is also conv.